# Stability and mild of solutions forintegro-differential impulsive equations with infinite interval in a Banach space

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**Abstract:** In this work, we establish several results about the stability, uniquenessand existence of mild solutions for integro-differential impulsive equations with infiniteinterval in a Banach space; we assume that the linear part generates a strongly continuous semi group on a general Banach space. The arguments are based upon Banach and Leray-Schauder alternative fixed point theorems.

**Keywords:** Impulsive Differential Equations, fixed point, existence and uniqueness, semi group of linear operators, integro-differential equation, Stability, Banachspace, infinite interval.

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## 1. Introduction

The theory of impulsive differential equations is a new and important branch of differential equations. The first paper in this theory is related to A. D. Mishkis and V. D.Milman in 1960 and 1963 [9]. An impulsive differential equation is described by three components: a continuous-time differential equation, which governs the state of thesystem between impulses; an impulse equation, which models an impulsive jump defined by a jump function at the instant an impulse occurs; and a jump criterion, which defines a set of jump events in which the impulse equation is active. There are many good monographs on the impulsive differential equations [1]- [11]. Many phenomena in theoretical physics, radio physics, pharmacokinetics, mathematical economy, chemical technology, electric technology, metallurgy, ecology, industrial robotics, biotechnology processes, chemistry, engineering, control theory, medicine and so on. Adequate mathematical models of such processes are systems of differential equations with impulses.

In this paper, we consider the following integro-differential impulsive differential equations with infinite interval:

$$\begin{cases} y'(t) = Ay(t) + \int_{-\infty}^{t} B(t,s)y(s)ds, \ t \in J := [\sigma,\infty) \setminus \{t_1, t_2, \dots\}, \\ \Delta y_{t=t_k} = I_k\big(y(t_k^-)\big), \ k = 1, \dots, \\ y(t) = \varphi(t), \ t \in (-\infty, \sigma] \end{cases} \quad (1.1)$$

where$\sigma \in \mathbb{R}$, $\varphi \in C_\sigma = C([-\infty,\sigma],E)$, $B(t,s): J \times (-\infty,\sigma] \to E$ is given function, A is the infinitesimal generator of a $C_0$- semi group $\{S(t)\}_{t\geq 0}$,$I_k \in C(E,E), k = 1, \dots, m, \sigma = t_0 < t_1 < \cdots < t_m < \cdots$,

$\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-), y(t_k^+) = \lim_{h\to 0^+} y(t_k + h)\,, y(t_k^-) = \lim_{h\to 0^+} y(t_k - h)$represent the left and right limits of $y(t)\ at\ \ t = t_k$ and E is a Banach space with the norm $\|.\|$.

If

$\mathrm{PC} = \{\mathrm{y}: (\sigma,\infty) \to \mathrm{E}: \mathrm{y}(\mathrm{t}_k^-), \mathrm{y}(\mathrm{t}_k^+)\ \text{exist with}\, \mathrm{y}(\mathrm{t}_k) = \mathrm{y}(\mathrm{t}_k^-), \mathrm{y}_k \in \mathrm{C}(\mathrm{J}_k,\mathrm{E}), \mathrm{k} = 0, \dots\}$, which is a Banch space with the norm

$$\|\mathrm{y}\|_{\mathrm{PC}} := \max\{\|\mathrm{y}_k\|_\infty : \mathrm{k} = 0, \dots\},$$

Where $y_k$ is the restriction of y to$J_0 = [t_0, t_1]\ and\, J_k = (t_k, t_{k+1}], k = 1, \dots$

Set

$$\Omega = \{y: (-\infty, \infty) \to E: y \in PC \cap C_\sigma\}.$$

With the norm

$$\|\mathbf{y}\|_{\Omega} = \mathbf{max}\{\|\mathbf{y}\|_{\mathbf{PC}}, \|\boldsymbol{\varphi}\|_{\mathbf{C}_{\sigma}}\}$$

and

$$\|\varphi\|_{C_\sigma} = sup\{\|\varphi(t)\|, t \in (-\infty, \sigma]\}.$$

This paper is organized as follows: In Section 2, we will recall briefly some basic definitions, some fixed point theorems and preliminary facts which will be needed in the following sections.

In Section 3, we give one of our main existence results for solutions of (1.1), with the proof based on Banach fixed point theorem.

In Section 4, we give other existence results for solutions of (1.1), with the proof based on Leray-Schauder alternative fixed point theorem.

In Section 5, the study the stability of the system (1.1).

**2. Preliminaries**

In what follows we introduce definitions, notations, and preliminary facts which are used in the sequel.

**Definition 2.2[10]**A one-parameter family S(t) of bounded linear operators on aBanach space E is a$C_0$-semigroup (or strongly continuous) on E if

(i) S(t)∘S(s) = S(t + s); for t; s ≥ 0; (semi group property),

(ii) S(0) = I, (the identity on E),

(iii) the map $t \to S(t)x$ is strongly continuous for each x ∈ E , i.e:

$$\lim_{t\to 0} S(t)x = x, \forall x \in E.$$

A semi group of bounded linear operators S(t), is uniformly continuous if

$$\lim_{t\to 0}\|S(t) - I\| = 0.$$

Here I denotes the identity operators in E.

**Theorem 2.3[10]**If S(t) is a $C_0$-semigroup, then there exist $\omega \geq 0$ and M ≥ 1 such that

$\|S(t)\|_{B(E)} \leq M\exp(\omega t)$ for $0 \leq t \leq \infty$.

**Definition 2.4[10**] Let S(t) be a semi group of class $C_0$ defined on E. The infinitesimal generator A of S(t) is the linear operator defined by

$$A(x) = \lim_{h\to 0}\frac{S(h)x - x}{h}, \text{for } x \in D(A),$$

where$D(A) = \left\{x \in D(A)/\lim_{h\to 0}\frac{S(h)x-x}{h} \text{ exists in } E\right\}$.

**Proposition 2.5[10]**The infinitesimal generator A is a closed, linear and densely defined operator in E. If $x \in D(A)$, then S(t)(x) is a $C^1$-map and

$$\frac{d}{dt}S(t)x = A(S(t)x) = S(t)(A(x)) \text{ on } [0, \infty).$$

**3. Uniqueness of mild solutions**

This section is devoted to the existence results for problem (1.1). Before starting and proving this result, we give the definition of its mild solution.

**Definition 3.1.** We say that a function $y \in \Omega$ is a mild solution of problem (1.1) if

$y(t) = \varphi(t), t \in (-\infty, \sigma]$ and

$$y(t) = S(t-\sigma)\varphi(\sigma) + \int_\sigma^t S(t-s)\left(\int_{-\infty}^s B(s,r)y(r)dr\right)ds + \sum_{\sigma<t_k<t} S(t-t_k)I_k\big(y(t_k)\big), \qquad t \in J.$$

Let us introduce the following hypotheses:

**($A_1$)** A is the generator of a strongly continuous semi group $S(t), t \in J$ which is compact for $t > 0$ in the Banach space E.

Let $M \geq 1$ be such that

$$\|S(t)\| \leq M \text{ for all } t \in J.$$

**($A_2$)** For each $t \in J$, $B(t,s)$ is measurable on $[\sigma, t]$ and $B(t) = \sup_{s\in(-\infty,\sigma]} \|B(t,s)\|$

is bounded and $\int_\sigma^t B(s)\,ds < \infty$.

**($A_3$)** There exist constants $c_k \geq 0$ such that

$$\|I_k(y) - I_k(\overline{y})\| \leq c_k\|y - \overline{y}\|, \text{ for each } k = 1,\ldots,m, \forall\, y, \overline{y} \in E.$$

We will let $\tilde{l}(t) = MM_{k+1}B(t)$, $L(t) = \int_\sigma^t \tilde{l}\,(s)$, $\sigma \leq t \leq t_{k+1} \leq M_{k+1}$ and $\|.\|_{B_*}$ denote the Bielecki-type norm on $\Omega$ defined by

$$\| y \|_{B_*} = \sup_{\sigma\leq s\leq t} e^{-\tau L(s)} \| y(s) \|, t \in J.$$

**Theorem 3.2** Assume that conditions ($A_1$)- ($A_3$) hold, if $\theta < 1$; where

$$\theta = \frac{1}{\tau} + M\sum_{k=1}^{\infty} c_k < 1, \qquad (3.1)$$

then, the problem (1.1) has a unique mild solution..

**Proof:** Transform the problem (1.1) into a fixed point problem. Consider the operator

$N : \Omega \to \Omega$ defined by:

$$N(y)(t) = y(t) = S(t-\sigma)\varphi(\sigma) + \int_\sigma^t S(t-s)\left(\int_{-\infty}^s B(s,r)y(r)dr\right)ds + \sum_{\sigma<t_k<t} S(t-t_k)I_k\big(y(t_k)\big), t \in J$$

and $N(y)(t) = \varphi(t),\ t \in (-\infty, \sigma]$ .

Note that a fixed point of N is a mild solution of (1.1). We will show that N is a contraction. Indeed, consider $y, \overline{y} \in \Omega$. Thus, for $t \in J$, we have:

$$\|N(y) - N(\overline{y})\| \leq \|\int_\sigma^t S(t-s)\left(\int_{-\infty}^s B(s,r)(y(r)-\overline{y}(r))dr\right)ds\|$$
$$+ \sum_{\sigma<t_k<t} \|S(t-t_k)\|\|I_k\big(y(t_k)\big) - I_k\big(\overline{y}(t_k)\big)\|$$

$$\leq \| \int_{\sigma}^{t} S(t-s)\left(\int_{-\infty}^{s} \sup_{r\in(-\infty,\sigma]} \|B(s,r)\| y(r)-\bar{y}(r)\| dr\right) ds\|$$

$$+\sum_{\sigma<t_k<t} \|S(t-t_k)\| I_k\big(y(t_k)\big)-I_k\big(\bar{y}(t_k)\big)\|$$

$$\leq M\int_{\sigma}^{t} \|B(s)\| \int_{-\infty}^{s} \|y(r)-\bar{y}(r)\| dr\, ds + M\sum_{\sigma<t_k<t} \|y(t_k)-\bar{y}(t_k)\|,$$

for$s \in (-\infty;\ t]$, we have:

$$\int_{-\infty}^{s} \|y(r)-\bar{y}(r)\| dr = \int_{-\infty}^{\sigma} \|y(r)-\bar{y}(r)\| dr + \int_{\sigma}^{t} \|y(r)-\bar{y}(r)\| dr$$

$$= \int_{-\infty}^{s} \|\varphi(r)-\varphi(r)\| dr + \int_{-\sigma}^{t} \|y(r)-\bar{y}(r)\|\, dr$$

$= \int_{-\sigma}^{t} \|y(r)-\bar{y}(r)\| dr.$

Then

$$\|N(y)(t)-\ N(\bar{y})(t)\| \leq M\int_{\sigma}^{t} \|B(s)\| \int_{\sigma}^{t} \|y(r)-\bar{y}(r)\| drds$$

$$+M\sum_{k=1}^{\infty} c_k \|y(t_k)-\bar{y}(t_k)\|$$

$$\leq M\int_{\sigma}^{t} \|B(s)\|\|y(s)-\bar{y}(s)\| \int_{\sigma}^{t} drds$$

$$+M\sum_{k}^{\infty} c_k \|y(t_k)-\bar{y}(t_k)\|$$

$$\leq M\int_{\sigma}^{t} \|B(s)\|\|y(s)-\bar{y}(s)\|(t-\sigma)ds$$

$$+M\sum_{k=1}^{\infty} c_k \|y(t_k)-\bar{y}(t_k)\|$$

$$\leq M\int_{\sigma}^{t} M_{k+1} \|B(s)\|\|y(s)-\bar{y}(s)\| ds$$

$$+M\sum_{k=1}^{\infty} c_k \|y(t_k)-\bar{y}(t_k)\|$$

$$\leq M\int_{\sigma}^{t} M_{k+1} \|B(s)\| e^{-\tau L(s)} e^{\tau L(s)} \|y(s)-\bar{y}(s)\| ds$$

$$+M\sum_{k=1}^{\infty} c_k \|y(t_k)-\bar{y}(t_k)\|$$

$$\leq MM_{k+1}\int_{\sigma}^{t} \|B(s)\| \sup_{\sigma\leq s\leq t} e^{-\tau L(s)} \|y(s)-\bar{y}(s)\| e^{\tau L(s)} ds$$

$$+M\sum_{k=1}^{\infty} c_k \|y(t_k)-\bar{y}(t_k)\|$$

$$\leq \frac{1}{\tau}\|y-\bar{y}\|_{B_*} \int_{\sigma}^{t} \tilde{l}(s) e^{\tau L(s)} \quad ds$$

$$+M\sum_{k=1}^{\infty} c_k \|y(t_k)-\bar{y}(t_k)\|$$

$$\leq \frac{1}{\tau} \|y-\bar{y}\|_{B_*} \int_{\sigma}^{t} (e^{\tau L(s)})' \quad ds$$

$$+M \sum_{k=1}^{\infty} c_k \, \|y(t_k) - \bar{y}(t_k)\|$$

$$\leq \frac{1}{\tau} \|y-\bar{y}\|_{B_*} (e^{\tau L(t)} - e^{\tau L(\sigma)}) \quad + M \sum_{k=1}^{\infty} c_k \, \|y(t_k) - \bar{y}(t_k)\|$$

$$\leq \frac{1}{\tau} \|y-\bar{y}\|_{B_*} e^{\tau L(t)} \quad + M \sum_{k=1}^{\infty} c_k \, \|y(t_k) - \bar{y}(t_k)\|.$$

Thus

$$e^{-\tau L(t)} \|N(y)(t) - N(\bar{y})(t)\| \leq \frac{1}{\tau} \|y-\bar{y}\|_{B_*} + M \sum_{k=1}^{\infty} c_k \, e^{-\tau L(t)} \|y(t_k) - \bar{y}(t_k)\|.$$

Therefore

$$\|N(y) - N(\bar{y})\|_{B_*} \leq \left( \frac{1}{\tau} + M \sum_{k=1}^{\infty} c_k \right) \|y-\bar{y}\|_{B_*}.$$

Since $\theta < 1$, N is a contraction. By the Banach fixed point theorem, we conclude thatN has a unique fixed point in $\Omega$ and the problem (1.1) has a unique mild solution.

## 4. Existence of mild solutions

In this section, we prove existence results, for the problem (1.1) by using the Leray-Schauder alternative fixed point theorem. Let us introduce the following hypotheses:

**$(H_1)$** There exist positive constants $c_k$, $k = 1; ...,$ such that:

$$\|I_k(y)\| < c_k^*, for\ all\ y \in \Omega \ and \ \sum_{k=1}^{\infty} c_k^* < \infty.$$

**$(H_2)$** There exists a constant $K_0 > 0$, such that:

$$\int_{\sigma}^{t} \left( \|B(s)\| \int_{-\infty}^{s} \|\varphi(r)\| dr \right) ds \leq K_0 \,.$$

Throughout this section, we assume $S(t)$; $t \in$ Jwhich is compact for $t > 0$ in the Banach space E.

**Theorem 4.1.**Assume that conditions **$(A_1)$, $(A_2)$,$(H_1)$** and **$(H_2)$**hold, then the problem (1.1) has at least one mild solution.

**Proof:**Transform the problem (1.1) into a fixed point problem. Consider the operator N defined in proof of Theorem 3.1. In order to apply the Leray-Schauder alternative fixed point theorem, we first show that N is completely continuous. The proof will be given in several steps.

**Step 1:**N is continuous.

Let $\{y_n\}$ a sequence such that $y_n \to y$ in $\Omega$ . Then

$$\|N(y_n)(t) - N(y)(t)\|$$
$$\leq \left\| \int_{\sigma}^{t} S(t-s) \left( \int_{-\infty}^{s} B(s,r) \left(y_n(r) - y(r)\right) dr \right) ds \right\|$$
$$+ \sum_{\sigma < t_k < t} \|S(t-t_k)\| \left\| I_k\left(y_n(t_k)\right) - I_k(y(t_k)) \right\|$$

$$\leq \left\| \int_{\sigma}^{t} S(t-s)\left( \int_{-\infty}^{s} \sup_{r\in(-\infty,\sigma]} \|B(s,r)\| \, \|y_n(r)-y(r)\| dr \right) ds \right\| + \sum_{\sigma<t_k<t} \|S(t-t_k)\| \left\| I_k(y_n(t_k)) - I_k(y(t_k)) \right\|$$

$$\leq \left\| \int_{\sigma}^{t} \|S(t-s)\| B(s) \left( \int_{-\infty}^{s} \|y_n(r)-y(r)\| dr \right) ds \right\| + \sum_{\sigma<t_k<t} \|S(t-t_k)\| \left\| I_k(y_n(t_k)) - I_k(y(t_k)) \right\|$$

$$\leq \left\| \int_{\sigma}^{t} \|S(t-s)\| B(s) \left( \int_{-\infty}^{\sigma} \|y_n(r)-y(r)\| dr + \int_{\sigma}^{s} \|y_n(r)-y(r)\| dr \right) ds \right\| + \sum_{\sigma<t_k<t} \|S(t-t_k)\| \left\| I_k(y_n(t_k)) - I_k(y(t_k)) \right\|$$

$$\leq M \left\| \int_{\sigma}^{t} B(s) \left( \int_{-\infty}^{t} \|\varphi(r)-\varphi(r)\| \, dr + \int_{\sigma}^{s} \|y_n(r)-y(r)\| dr \right) ds \right\| + \sum_{\sigma<t_k<t} M \left\| I_k(y_n(t_k)) - I_k(y(t_k)) \right\|$$

$$\leq M \left\| \int_{\sigma}^{t} B(s) \left( \int_{\sigma}^{s} \|y_n(r)-y(r)\| dr \right) ds \right\| + \sum_{k=1}^{\infty} M \left\| I_k(y_n(t_k)) - I_k(y(t_k)) \right\| \to 0, as\ n \to \infty.$$

Thus N is continuous.

**Step 2:** N maps bounded sets into bounded sets in Ω.

Indeed, it is enough to show that for any q > 0 there exists a positive constant $\ell$ such that for each $y \in B_q = \{u \in \Omega : \|u\|_{PC} \leq q\}$, we have $\|N(y)\|_{PC} \leq \ell$. Then we have for each $t \leq M_{k+1}$,

$$\|N(y)(t)\| \leq \left\| \int_{\sigma}^{t} S(t-s) \left( \int_{-\infty}^{s} B(s,r) y(r) dr \right) ds \right\| + \sum_{\sigma<t_k<t} \|S(t-t_k)\| \, \|I_k(y(t_k))\|$$

$$\leq \left\| \int_{\sigma}^{t} S(t-s) \left( \int_{-\infty}^{s} \sup_{r\in(-\infty,\sigma]} \|B(s,r)\| \, \|y(r)\| dr \right) ds \right\| + \sum_{\sigma<t_k<t} \|S(t-t_k)\| \|I_k(y(t_k))\|$$

$$\leq \int_{\sigma}^{t} \|S(t-s)\| \|B(s)\| \left( \int_{-\infty}^{s} \|y(r)\| dr \right) ds + \sum_{\sigma<t_k<t} \|S(t-t_k)\| \|I_k(y(t_k))\|$$

$$\leq M \int_{\sigma}^{t} \left( \|B(s)\| \left( \int_{-\infty}^{\sigma} \|\varphi(r)\| \, dr + \int_{\sigma}^{t} \|y(r)\| \, dr \right) \right) ds + \sum_{\sigma<t_k<t} M \left\| I_k(y(t_k)) \right\|$$

$$\leq M \int_{\sigma}^{t} \|B(s)\| \left( \int_{-\infty}^{\sigma} \|\varphi(r)\| \, dr \right) ds + M \int_{\sigma}^{t} \|B(s)\| \left( \int_{\sigma}^{t} \|y(r)\| \, dr \right) ds + \sum_{k=1}^{\infty} \|I_k(y(t_k))\|$$

$$\leq M \int_{\sigma}^{t} (\|B(s)\| \int_{-\infty}^{s} \|\varphi(r)\| \, dr) \, ds + M \int_{\sigma}^{t} \|B(s)\| \, (\int_{\sigma}^{t} \|y(r)\| \, dr) ds + M \sum_{k=1}^{\infty} c_k$$

$$\leq MK_0 + M \sum_{k=1}^{\infty} c_k^* + M \int_{\sigma}^{t} \|B(s)\| \, (\int_{\sigma}^{t} q dr) ds$$

$$\leq MK_0 + M\sum_{k=1}^{\infty} c_k^* + qMM_{k+1}\int_{\sigma}^{t}\|B(s)\|\,ds := \ell.$$

**Step 3:** N maps bounded sets into equicontinuous sets of Ω.

Let $l_1, l_2 \in J,\ l_1 < l_2$ and $B_q$ be a bounded set of Ω as in **Step 2**. Let $y \in B_q$ then for each $t \in J$, we have:

$$\|N(y)(l_2) - N(y)(l_1)\| \leq \|(S(l_2) - S(l_1))\varphi(\sigma)\|$$
$$+\int_{\sigma}^{l_1-\varepsilon}\left(\|(S(l_2 - s) - S(l_1 - s))\|\left\|\int_{-\infty}^{s} B(s,r)y(r)\right\|dr\right)ds$$
$$+\int_{l_1-\varepsilon}^{l_1}\left(\|(S(l_2 - s) - S(l_1 - s))\|\int_{-\infty}^{s}\|B(s,r)y(r)\|\,dr\right)ds$$
$$+\int_{l_1}^{l_2}\left(\|S(l_2 - s)\|\int_{-\infty}^{s}\|B(s,r)y(r)\|\,dr\right)ds$$
$$+\sum_{\sigma<t_k<l_1}\|S(l_2 - t_k) - S(l_1 - t_k)\|\,\|I_k(y(t_k))\|$$
$$+\sum_{l_1<t_k<l_2}\|S(l_2 - t_k)\|\,\|I_k(y(t_k))\|.$$

The right-hand side tends to zero as $l_2 - l_1 \to 0$, and $\varepsilon$ sufficiently small, since S(t) is a strongly continuous operator and the compactness of S(t) for t > 0 implies the continuity in the uniform operator topology. As a consequence of **Steps 1-3**, and the Arzelà-Ascoli theorem, we can conclude that $N: \Omega \to \Omega$ is continuous and completely continuous.

**Step 4:** (A priori bounds on solutions)

Let y be a possible solution of the equation $y = \lambda N(y)$ with $y_\sigma = \varphi$ for some $\lambda \in (0,1)$. Then

$$\|y(t)\| \leq \|S(t-\sigma)\varphi(\sigma)\| + \left\|\int_{\sigma}^{t} S(t-s)\left(\int_{-\infty}^{s} B(s,r)\,y(r)dr\right)ds\right\| + \sum_{\sigma<t_k<t}\|S(t-t_k)\|\,\|I_k(y(t_k))\|$$

$$\leq M\|\varphi(\sigma)\| + \int_{\sigma}^{t}\|S(t-s)\|\left(\int_{-\infty}^{s}\sup_{r\in(-\infty,\sigma]}\|B(s,r)\|\|y(r)\|dr\right)ds + \sum_{\sigma<t_k<t}\|S(t-t_k)\|\,\|I_k(y(t_k))\|$$

$$\leq M\|\varphi(\sigma)\| + \int_{\sigma}^{t}\|S(t-s)\|\,\|B(s)\|\left(\int_{-\infty}^{s}\|y(r)\|dr\right)ds + \sum_{\sigma<t_k<t}\|S(t-t_k)\|\,\|I_k(y(t_k))\|$$

$$\leq M\|\varphi(\sigma)\| + M\int_{\sigma}^{t}\left(\|B(s)\|\left(\int_{-\infty}^{\sigma}\|\varphi(r)\|dr + \int_{\sigma}^{t}\|y(r)\|dr\right)\right)ds + \sum_{\sigma<t_k<t} M\,\|I_k(y(t_k))\|$$

$$\leq M\|\varphi(\sigma)\| + M\int_{\sigma}^{t}\left(\|B(s)\|\int_{-\infty}^{s}\|\varphi(r)\|dr\right)ds + M\int_{\sigma}^{t}\left(\|B(s)\|\int_{-\infty}^{s}\|y(r)\|dr\right)ds$$
$$+\sum_{\sigma<t_k<t} M\,\|I_k(y(t_k))\|$$
$$\leq M\left(\|\varphi(\sigma)\| + K_0 + \sum_{k=1}^{\infty} c_k^*\right) + M\,M_{k+1}\int_{\sigma}^{t}\|B(s)\|\|y(s)\|ds.$$

We consider the function μ(t) defined by:

$$\mu(t) = \sup\{\|y(s)\| : \sigma \le s \le t\}, t \in J.$$

Let $t^* \in (-\infty, t]$ be such that $\mu(t) = \|y(t^*)\|$. This implies that, for each $t \in [\sigma, \infty)$,

$$\|y(t)\| \le c + MM_{k+1} \int_\sigma^t \|B(s)\| \|\mu(s)\| ds,$$

where $c = M(\|\varphi(\sigma)\| + K_0 + \sum_{k=1}^{\infty} c_k^*)$.

Let us take the right-hand side of the above inequality as $v$ (t). Then we have $\mu(t) \le v$ (t) for all $t \in J$. From the definition of $v$, we get $v(\sigma) = c$ and $v'(t) = M\, M_{k+1}\|B(t)\|\mu(t) \le M\, M_{k+1}\|B(s)\| v(t)$, This implies, for each $t \in J$,

$$\int_c^{v(t)} \frac{du}{u} \le MM_{k+1} \int_\sigma^t B(s) ds$$

then

$$v(t) \le c\, exp\left( MM_{k+1} \int_\sigma^t B(s) ds \right).$$

Thus, there exists a constant $K^* > 0$ such that:

$$\mu(t) \le v(t^*) \le c \exp\left( MM_{k+1} \int_\sigma^t B(s) ds \right) \le K^*.$$

We have:

$$\|y\|_\Omega \le \max\{\|\varphi\|_{C_\sigma}, K^*\} = K_1.$$

Set

$$U = \{y \in \Omega : \|y\|_{PC} \le K_1 < K_1 + 1\}.$$

We see that $N: \overline{U} \to \Omega$ is continuous and completely continuous. From this choice of $U$, there is no $y \in \partial U$ such that $y = \lambda N(t)$, for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type [184], we deduce that N has a fixed point $y \in U$. Hence, N has a fixed point y that is a solution to problem (1.1).

## 5. Stability result

This section is devoted to the study of the stability of the solution for problem (1.1). Before starting and proving this result, we give the definition of its mild solution.

**Definition 5.1.** The trivial mild solution of system (1.1) is said to be stable if for every $\varepsilon > 0$ and $t_0 \in R$, there exists $\delta(t_0, \varepsilon)$ such that $\varphi_1, \varphi_2 \in C_\sigma$ two initial values and the solution $y_i(t, t_0, \varphi_i)$ of (1.1) where the initial value $\varphi_i, i = 1,2$ with $\| \varphi_1 - \varphi_2 \|_{C_\sigma} \le \delta$, then $\| y_1(t, t_0, \varphi_1) - y_2(t, t_0, \varphi_2) \| \le \varepsilon$ for all $t \ge t_0$.

For the next theorem we replace the condition $(H_2)$ by:
There exists a constant $K_0 > 0$, such that:

$$\int_\sigma^t \left( \|B(s)\| \int_{-\infty}^s \|\varphi(r)\| dr \right) ds \le K_0 \|\varphi\|_{c_\sigma}. \qquad (5.1)$$

**Theorem 5.1** Assume that conditions $(\mathbf{A_1})$, $(\mathbf{A_2})$, and (5.1) hold.
If

$$\lambda = \left( 1 - M \sum_{k=1}^{\infty} C_k \right) > 0,$$

then the zero solution of the problem (1.1) is stable.

**Proof:**

Let $y_i(t,t_0,\varphi_i)$ be a solution of the system (1.1) where the initial value $\varphi_i$, $i = 1,2$, for $t \in J$, we have

$$y_1(t,\sigma,\varphi_1) = S(t-\sigma)\varphi_1(\sigma) + \int_\sigma^t S(t-s)\left(\int_{-\infty}^s B(s,r)y_1(r,\sigma,\varphi_1)\,dr\right)d + \sum_{\sigma<t_k<t} S(t-t_k)I_k(y_1(t_k,\sigma,\varphi_1)),$$

$$y_2(t,\sigma,\varphi_2) = S(t-\sigma)\varphi_2(\sigma) + \int_\sigma^t S(t-s)\left(\int_{-\infty}^s B(s,r)y_2(r,\sigma,\varphi_2)\,dr\right)d + \sum_{\sigma<t_k<t} S(t-t_k)I_k(y_2(t_k,\sigma,\varphi_2)).$$

We have

$$y_1(t,\sigma,\varphi_1) - y_2(t;\ \sigma;\ \varphi_2) = S(t-\sigma)\big(\varphi_1(\sigma)-\varphi_2(\sigma)\big)$$
$$+\int_\sigma^t S(t-s)\left[\int_{-\infty}^s B(s,r)\big(y_1(r,\sigma,\varphi_1)-y_2(r,\sigma,\varphi_2)\big)\,dr\right]ds$$
$$+\textstyle\sum_{\sigma<t_k<t} S(t-t_k)(I_k(y_1(t_k,\sigma,\varphi_1)) - I_k(y_2(t_k,\sigma,\varphi_2)))).$$

Then

$$\| y_1(t,\sigma,\varphi_1) - y_2(t,\sigma,\varphi_2) \| \le \| S(t-\sigma)\big(\varphi_1(\sigma)-\varphi_2(\sigma)\big) \|$$
$$+M\int_\sigma^t B(s)\left(\int_{-\infty}^\sigma \| \varphi_1(r)-\varphi_2(r) \|\,dr\right)ds$$
$$+M\int_\sigma^t B(s)\left(\int_\sigma^s \| y_1(r,\sigma,\varphi_1)-y_2(r,\sigma,\varphi_2) \|\,dr\right)ds$$
$$+\sum_{\sigma<t_k<t} c_k \| y_1(t_k,\sigma,\varphi_1) - y_2(t_k,\sigma,\varphi_2) \|$$
$$\le M \| \varphi_1(\sigma)-\varphi_2(\sigma) \| + MK_0 \| \varphi_1-\varphi_2 \|_{c_\sigma}$$
$$+\int_\sigma^t B(s)\int_\sigma^s \| y_1(r,\sigma,\varphi_1)-y_2(r,\sigma,\varphi_2) \|\,dr\,ds$$
$$+\sum_{\sigma<t_k<t} c_k \| y_1(t_k,\sigma,\varphi_1) - y_2(t_k,\sigma,\varphi_2) \|.$$

Thus

$$\lambda \| y_1(t,\sigma,\varphi_1) - y_2(t,\sigma,\varphi_2) \| \le M(1+K_0) \| \varphi_1-\varphi_2 \|_{c_\sigma}$$
$$+M\int_\sigma^t B(s)\int_\sigma^s \| y_1(r,\sigma,\varphi_1)-y_2(r,\sigma,\varphi_2) \|\,dr\,ds$$
$$\le M(1+K_0) \| \varphi_1-\varphi_2 \|_{c_\sigma}$$
$$+MM_k\int_\sigma^t B(s) \| y_1(s,\sigma,\varphi_1)-y_2(s,\sigma,\varphi_2) \|\,ds.$$

Hence

$$\| y_1(t,\sigma,\varphi_1) - y_2(t;\ \sigma;\ \varphi_2) \| \le \frac{(1+K_0)}{\lambda} \| \varphi_1-\varphi_2 \|_{c_\sigma}$$
$$+\tfrac{MM_k}{\lambda}\textstyle\int_\sigma^t B(s) \| y_1(s,\sigma,\varphi_1)-y_2(s,\sigma,\varphi_2) \|\,ds$$
$$\le c_* \| \varphi_1(\sigma)-\varphi_2(\sigma) \| + c_{**}\int_\sigma^t B(s) \| y_1(r,\sigma,\varphi_1)-y_2(r,\sigma,\varphi_2) \|\,ds,$$

with

$c_* = \frac{1+K_0}{\lambda}$ $and$ $c_{**} = \frac{MM_k}{\lambda}$.

In the same way in **Step 4**, we find

$$\| y_1(t,\sigma,\varphi_1) - y_2(t;\ \sigma;\ \varphi_2) \| \le c_* \| \varphi_1-\varphi_2 \|_{c_\sigma}\ exp\left(c_{**}\int_\sigma^t B(s)ds\right).$$

Let $\varepsilon > 0$ and $\delta = \frac{\varepsilon}{c_* exp\left(c_{**}\int_\sigma^t B(s)ds.\right)}$; then from the above inequality, we obtain

$\| y_1(t,\sigma,\varphi_1) - y_2(t;\ \sigma;\ \varphi_2) \| \le \varepsilon$with $\| \varphi_1-\varphi_2 \|_{c_\sigma} \le \delta$, i.e., the zero solution is stable

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