ON SOME BOUNDS OF THE MINIMUM EDGE DOMINATING ENERGY OF A GRAPH

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Abstract: Let G be a simple graph of order n with vertex set V= {v_1, v_2, ..., v_n} and edge set E = {e_1, e_2, ..., e_m}. A subset of E is called an edge dominating set of G if every edge of E - is adjacent to some edge in .Any edge dominating set with minimum cardinality is called a minimum edge dominating set [2]. Let be a minimum edge dominating set of a graph G. The minimum edge dominating matrix of G is the m x m matrix defined by

$$D'(G) = (d' \text{, where } (d' = \begin{cases} 1 & if\, e_i and e_j \text{ are incide} \\ 1 & if\ i = j \text{ and } e_i \in \\ 0 & otherw \end{cases}$$

The characteristic polynomial of $D'($ is denoted by

$f_n(G, \rho) = \det(\rho I - (G))$.

The minimum edge dominating eigen values of a graph G are the eigen values of (G). Minimum edge dominating energy of G is defined as

$$(G) = \sum_{i=1}^{m} | \quad [12]$$

In this paper we have computed the Minimum Edge Dominating Energy of a graph. Its properties and bounds are discussed. All graphs considered here are simple, finite and undirected.

Key Words: Edge Adjacency Matrix, Edge Energy, Edge Dominating set, Minimum Edge Dominating Eigen values, Minimum Edge Dominating Energy

1.Introduction

Euler's work on Konigsberg bridge problem in 1736 paved the way to a new branch of Mathematics called Graph theory. In the year 1978, Ivan Gutman [5] introduced the concept of energy of a graph. The various upper and lower bounds for energy of a graph have been found [4, 6].

Recently the interest in graph energy has increased and various energies have been introduced and their properties were discussed. Adiga. C, Bayad. A, Gutman .I, Srinivas .S. A, has introduced a new energy Minimum covering energy of a graph and its properties were dicussed [1]. Recently Rajesh Kanna. M. R, Dharmendra. B. N, Sridhara .G introduced the minimum dominating energy of a graph which depends on the minimum dominating set [11]. The concept of edge domination was introduced by Mitchell and Hedetniemi [10]. Meenakshi. S, Lavanya. S has introduced a new energy Minimum Dom Strong Dominating Energy and its properties and bounds were found [9].

Motivated by these papers, we have introduced the Minimum Edge Dominating Energy of a graph [12]. In this paper we are concerned with finite, simple and undirected graphs. In this paper we have computed the Minimum Edge Dominating Energy of a graph. Its properties and bounds are discussed.

2. PRELIMINARIES

Definition: 2.1

The **adjacency matrix** A(G) of a graph G(V, E) with a vertex set $V = \{v_1, v_2, \ldots,$ and an edge set E = {e_1, e_2 ,….,e_m} is an n x n matrix

$$A = (a_{ij}) = \begin{cases} 1\ if\ v_i is\ adjacent\ to \\ \quad 0\ , otherwise \end{cases}$$

A is a real symmetric matrix.

Definition: 2.2

The Eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A, assumed in non increasing order, are the Eigen values of the graph G. As A is real symmetric, the Eigen values of G are real with sum equal to zero. The **Energy** E (G) of G is defined to be the sum of the absolute values of the Eigen values of G.

i.e., $E(G) = \sum_{i=1}^{n} |\lambda_i|$ [5].

Definition: 2.3

Let G be a simple graph of order n with vertex set V={$v_1, v_2, \ldots, v_n$} and edge set E = {$e_1, e_2, \ldots, e_m$}. A subset of E is called an Edge Dominating set of G if every edge of E - is adjacent to some edge in . Any edge dominating set with minimum cardinality is called a Minimum Edge Dominating Set [10]. Let be a Minimum Edge Dominating Set of a graph G. The Minimum Edge Dominating Matrix of G is the m x m matrix defined by

$$D'(G) = (d'_{ij}), \text{ where } (d'_{ij}) = \begin{cases} 1 & \text{if } e_i \text{ and } e_j \text{ are adjacent} \\ 1 & \text{if } i = j \text{ and } e_i \in D' \\ 0 & \text{otherwise} \end{cases}$$

The characteristic polynomial of $D'(G)$ is denoted by

$f_m(G, \rho) = \det(\rho I - (G))$.

The Minimum Edge Dominating Eigen values of a graph G are the eigen values $\rho_1, \rho_2, \ldots, \rho_m$ of (G). **Minimum Edge Dominating Energy** of G is defined as

$$E(G) = \sum_{i=1}^{m} |\rho_i| \text{ [12].}$$

Example: 1

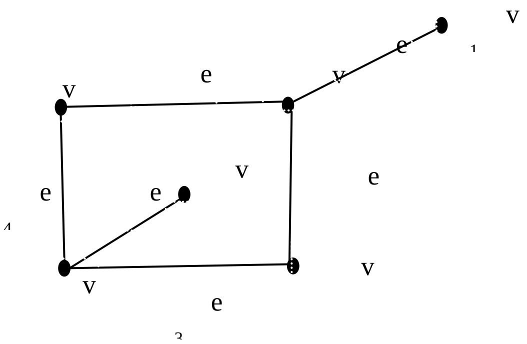

Figure 1

Consider the above graph G.

(i) Let the Minimum Edge Dominating set be = {e_1, e_3}.
Then the Minimum Edge Dominating adjacency matrix is

$$(G) = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

The characteristic equation is $\rho^6 - 2\rho^5 - 7\rho^4 + 7\rho^3 + 13\rho^2 - 0\rho - 1 = 0$.

The Minimum Edge Dominating eigen values are
$\rho_1 \approx -1.8363$, $\rho_2 \approx -1.1157$, $\rho_3 \approx -0.3132$, $\rho_4 \approx 0.2642$, $\rho_5 \approx 1.9050$, $\rho_6 \approx 3.0962$.
The Minimum Edge Dominating Energy, $E_{D'}(G) \approx 8.5306$.

(ii) If we take another Minimum Edge Dominating set = {e_2, e_3}.

$$(G) = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

The characteristic equation is $\rho^6 - 2\rho^5 - 7\rho^4 + 6\rho^3 + 13\rho^2 - 0\rho - 3 = 0$

The Minimum Edge Dominating Eigen values are

$\rho_1 \approx -1.7321$, $\rho_2 \approx -1$, $\rho_3 \approx -0.6751$, $\rho_4 \approx 0.4608$, $\rho_5 \approx 1.7321$, $\rho_6 \approx 3.2143$

The Minimum Edge Dominating Energy, $E_{D'}(G) \approx 8.8144$.

This example illustrates the fact that the Minimum Edge Dominating Energy of a graph G depends on the choice of the Minimum Edge Dominating Set.

i.e. The Minimum Edge Dominating Energy is not a graph invariant.

3. PROPERTIES OF MINIMUM EDGE DOMINATING ENERGY:

Theorem: 3.1

Let G be a simple graph of order n and size m, let be the Minimum Edge Dominating Set and let $f_m(G, \rho) = c_0\rho^m + c_1\rho^{m-1} + c_2\rho^{m-2} + \ldots\ldots + c_m$ be the characteristic polynomial of the Minimum Edge Dominating Matrix of the graph G. Then

$$c_2 = \left(\begin{matrix} |D' \\ 2 \end{matrix} - \sum_{i=1}^{m} \left(\begin{matrix} \deg \\ 2 \end{matrix} \right. \right. .$$

Proof:

The sum of the determinants of all 2 x 2 principal sub matrices of (G) = $(-1)^2 c_2$.

$$\text{Therefore, } c_2 = \sum_{1 \le i < j \le m} \begin{vmatrix} d'_{ii} & d' \\ d'_{ji} & d' \end{vmatrix}$$

$$= \sum_{1 \le i < j \le} d'_{ii} \, c - d'_{ij} \, c)$$

$$= \sum_{1 \le i < j \le m} d'_{ii} \, c - \sum_{1 \le i < j \le m} c$$

$$= \left(\begin{matrix} |D \\ 2 \end{matrix} - \sum_{i=1}^{m} \left(\begin{matrix} \deg \\ 2 \end{matrix} \right. \right. .$$

Theorem: 3.2

Let G = (V, E) be a simple graph of order n and size m. Let $\rho_1, \rho_2, \rho_1, \ldots\ldots, \rho_m$ be the eigen values of (G).

Then $\sum_{i=1}^{m} l = \; |L + \; 2\left[\sum_{i=1}^{m} \left(\begin{matrix} \deg v_i \\ 2 \end{matrix} \right. \right.$.

Proof:

The sum of the squares of the eigen values of $D'($ is the trace of $[D'(G)$.

$$\text{Therefore, } \sum_{i=1}^{m} l = \sum_{i=1}^{m} \sum_{j=1}^{m} d'_{ij} c$$

$$= \sum_{i=1}^{m} c + \sum_{i \neq j} d'_{ij} c$$

$$= \sum_{i=1}^{m} d + 2 \sum_{i<j} c$$

$$= |D + 2\left[\sum_{i=1}^{m} \left(\begin{matrix} \deg v \\ 2 \end{matrix} \right. \right. .$$

4. BOUNDS FOR MINIMUM EDGE DOMINATING ENERGY

In this section we find some bounds for E(G) of a graph.

Theorem: 4.1

Let G be a simple graph with n vertices and m edges. If E(G) is the Minimum Edge Dominating Energy of the graph, then

$$\sqrt{\left|D'\right| + 2\left[\sum_{i=1}^{m}\binom{\deg v}{2}\right.} \leq E(G) \leq \sqrt{m\left[\left|D'\right| + 2\left[\sum_{i=1}^{m}\binom{\deg v_i}{2}\right.\right.}$$

Proof:

Consider the Cauchy-Schwartz inequality

$$\left(\sum_{i=1}^{n} a_i b \right. \quad \left(\sum_{i=1}^{n} a_i^{\,2}\right)\left(\sum_{i=1}^{n} b_i\right.$$

If = 1, = | |, i = 1,….,m

Then, $\left(\sum_{i=1}^{m} |\rho_i\right. \quad \left(\sum_{i=1}^{m} 1 \quad \left(\sum_{i=1}^{m} |\rho_i\right.\right.$

$$(E_{D'}(G \quad m \quad \left[\left|D'\right| + 2\left[\sum_{i=1}^{m}\binom{\deg v_i}{2}\right.\right.$$ [Theorem: 3.2]

$$\Rightarrow \quad E(G) \leq \sqrt{m\left[\left|D'\right| + 2\left[\sum_{i=1}^{m}\binom{\deg v_i}{2}\right.\right.}$$

Therefore, the upper bound holds. For the lower bound, since

$$\left(\sum_{i=1}^{m} |\rho_i| \geq \sum_{i=1}^{m}\right.$$

$$\Rightarrow \quad (E_{D'}(G) \quad |D + \quad 2\left[\sum_{i=1}^{m}\binom{\deg v}{2}\right.$$

$$\Rightarrow \quad E(G) \quad \sqrt{\left|D'\right| + 2\left[\sum_{i=1}^{m}\binom{\deg v}{2}\right.}$$

Therefore, $$\sqrt{\left|D'\right| + 2\left[\sum_{i=1}^{m}\binom{\deg v}{2}\right.} \leq E(G) \leq \sqrt{m\left[\left|D'\right| + 2\left[\sum_{i=1}^{m}\binom{\deg v_i}{2}\right.\right.}$$

Similar to Mc Clellands [8] bounds for energy of a graph, bounds for E(G) are given in the following theorem.

Theorem: 4.2

Let G be a simple graph with n vertices and m edges. If E(G) is the Minimum Edge Dominating Energy of the graph and let P $' = \det(D'(G$, then

$$E(G) \quad \sqrt{\left[\left|D'\right| + 2\left[\sum_{i=1}^{m}\binom{\deg v_i}{2}\right]\right] + m(m-1)P}$$

Proof:

From the relation between the arithmetic mean and geometric mean, we have

$$\frac{1}{m(m-1)}\sum_{i \neq j} |\rho_i| \, |\rho_j| \quad \left[\prod_{i \neq j} |\rho_i|\,|\rho_j|\right]^{\frac{1}{m(m}}$$

$$= \left[\prod_{i=1}^{m} |\rho_i|^{2(m-1)}\right]^{\frac{1}{m(m}}$$

$$= \left[\prod_{i=1}^{m} \rho_i\right.$$

$$= \det(D'(G)$$

$$= P$$

Therefore, $\sum_{i \neq j} |\rho_i| \, |\rho_j| \geq m(m-1)P$ …………………….. (1)

Now consider, $(E_{D'}(G) = \left(\sum_{i=1}^{m} |\rho_i\right.$

$$= \sum_{i=1}^{m} |\rho_i|^2 + \sum_{i \neq j} |\rho_i|\,|\rho_j|$$

$$\geq \sum_{i=1}^{m} |\rho_i|^2 + m(m-1)P$$ [From (1)]

$$= \left[\left|D'\right| + 2\left[\sum_{i=1}^{m}\binom{deg\ v_i}{2}\right.\right. + \quad m(m-1)P$$

[Theorem: 3.1]

$$\therefore \quad E(G) \quad \sqrt{\left[\left|D'\right| + 2\left[\sum_{i=1}^{m}\binom{\deg v_i}{2}\right]\right] + \mathrm{m}\,(\mathrm{m}-1)\mathrm{I}}$$

Theorem: 4.3

If $\rho_1($ is the largest Minimum Edge Dominating Eigen value of (G), then

$$\rho_1(G) \geq \frac{|D'| + 2\left[\sum_{i=1}^{m}\binom{\deg}{2}\right.}{m}.$$

Proof:

Let X be any non zero vector. Then by [3], we have

$$\rho_1(G) = \max_{X\neq 0}\left\{\frac{X'D}{X'.}\right.$$

Therefore,

$$\rho_1(G) \geq \frac{J'D'J}{J'J} = \frac{|D'| + 2\left[\sum_{i=1}^{m}\binom{\deg}{2}\right.}{m},$$

where J = [1,1,1,…..1]' is a unit column matrix of order m x 1.

Similar to Koolen and Moulton's [7] upper bound for energy of a graph, upper bound for E(G), is given in the following theorem.

Theorem: 4.4

If G is a simple graph with n vertices and m edges and $|D_+ \quad 2\left[\sum_{i=1}^{m}\binom{\deg v_i}{2}\right] \geq$ then

$$E_{D'}(G)$$

$$\frac{|D'| + 2\left[\sum_{i=1}^{m}\binom{\deg v_i}{2}\right]}{m} +$$

$$\sqrt{(m-1)\left[|D'| + 2\left[\sum_{i=1}^{m}\binom{\deg v_i}{2}\right] - \left(\frac{|D'| + 2\left[\sum_{i=1}^{m}\binom{\deg v_i}{2}\right]}{m}\right)^2\right]}$$

Proof:

Consider the Cauchy-Schwartz inequality

$$\left(\sum_{i=2}^{n} a_i\, b \qquad \left(\sum_{i=2}^{n} a_i^{\,2}\right)\left(\sum_{i=2}^{n} b_i\right.\right.$$

If = 1, = | |, i = 2,…., m

Then, $\left(\sum_{i=2}^{m}|\rho_i \qquad \left(\sum_{i=2}^{m} 1 \qquad \left(\sum_{i=2}^{m}|\rho_i\right.\right.\right.$

$$\Rightarrow \quad [E_{D'}(G) - \rho_1]^2 \leq (m-1)\left[|D'| + 2\left[\sum_{i=1}^{m}\binom{\deg v_i}{2}\right] - \rho\right.$$

$$\Rightarrow \quad E_{D'}(G) \leq \rho_1 + \sqrt{(m-1)\left[|D'| + 2\left[\sum_{i=1}^{m}\binom{\deg v_i}{2}\right] - \rho\right.}$$

Let $f(=\quad x + \sqrt{(m-1)\left[|D'| + 2\left[\sum_{i=1}^{m}\binom{\deg v_i}{2}\right] - x\right.}$

For decreasing function

$$f'(\quad 0 \Rightarrow 1 - \frac{(m-1)\,x}{\sqrt{(m-1)\left[|D'| + 2\left[\sum_{i=1}^{m}\binom{\deg v_i}{2}\right] - x^2\right]}} \leq$$

$$\Rightarrow \quad x \geq \sqrt{\frac{|D'| + 2\left[\sum_{i=1}^{m}\binom{\deg}{2}\right.}{m}}$$

Since $|D_+ \quad 2\left[\sum_{i=1}^{m}\binom{\deg v_i}{2}\right] \geq$,

We have $\sqrt{\frac{|D'| + 2\left[\sum_{i=1}^{m}\binom{\deg v_i}{2}\right]}{m}} \leq \frac{|D'| + 2\left[\sum_{i=1}^{m}\binom{\deg v_i}{2}\right]}{m} \leq$ [Theorem: 3.1]

$$\therefore \quad f(\rho_1) \leq f\left(\frac{|D'| + 2\left[\sum_{i=1}^{m}\binom{\deg v_i}{2}\right.}{m}\right.$$

$$\Rightarrow \quad E_{D'}(G) \leq f(\rho_1) \leq f\left(\frac{|D'| + 2\left[\sum_{i=1}^{m}\binom{\deg v_i}{2}\right.}{m}\right.$$

$$\Rightarrow \quad E_{D'}(G) \leq f\left(\frac{|D'| + 2\left[\sum_{i=1}^{m}\binom{\deg v_i}{2}\right.}{m}\right.$$

$$\therefore E_{D'}(G)$$

$$\frac{|D'| + 2\left[\sum_{i=1}^{m}\binom{\deg v_i}{2}\right]}{m} + \sqrt{(m-1)\left[\left|D'\right| + 2\left[\sum_{i=1}^{m}\binom{\deg v_i}{2}\right] - \left(\frac{|D'| + 2\left[\sum_{i=1}^{m}\binom{\deg v_i}{2}\right]}{m}\right)\right.}$$

5. CONCLUSION:

In this paper we have found the Minimum Edge Dominating energy of a graph. The various upper and lower bounds for the Minimum Edge Dominating Energy of a graph have been found. Analogues works can be carried by us for other graphs also.

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