ON SOME BOUNDS OF THE MINIMUM EDGE DOMINATING ENERGY OF A GRAPH

A. Sharmila a, S. Lavanya b

aResearch Scholar, Bharathiar University Coimbatore - 641 046, Tamil Nadu, INDIA
bDepartment of Mathematics, Justice Basheer Ahmed Sayeed College For Women, Chennai - 600018, Tamil Nadu, INDIA
sharmi.beermohamed@gmail.com

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Abstract: Let G be a simple graph of order n with vertex set V = {v1, v2, ..., vn} and edge set E = {e1, e2, ..., em}. A subset of E is called an edge dominating set of G if every edge of E is adjacent to some edge in . Any edge dominating set with minimum cardinality is called a minimum edge dominating set [2]. Let be a minimum edge dominating set of a graph G. The minimum edge dominating matrix of G is the m x m matrix defined by

\[ D'(k_D) = \begin{cases} 1 & \text{if } e_i \text{ and } e_j \text{ are incident} \\ d_i' & \text{if } i = j \text{ and } e_i \in D' \text{ otherwise} \\ 0 & \text{otherwise} \end{cases} \]

The characteristic polynomial of \( D'(k_D) \) is denoted by

\[ f_n(G, \rho) = \det (\rho I - (G)) \]

The minimum edge dominating eigenvalues of a graph G are the eigenvalues of \( f_n(G, \rho) \). Minimum edge dominating energy of G is defined as

\[ E'(G) = \sum_{i=1}^{m} |1| \] [12]

In this paper we have computed the Minimum Edge Dominating Energy of a graph. Its properties and bounds are discussed. All graphs considered here are simple, finite and undirected.

Key Words: Edge Adjacency Matrix, Edge Energy, Edge Dominating set, Minimum Edge Dominating Eigen values, Minimum Edge Dominating Energy

1. Introduction

Euler’s work on Konigsberg bridge problem in 1736 paved the way to a new branch of Mathematics called Graph theory. In the year 1978, Ivan Gutman [5] introduced the concept of energy of a graph. The various upper and lower bounds for energy of a graph have been found [4, 6].

Recently the interest in graph energy has increased and various energies have been introduced and their properties were discussed. Adiga. C, Bayad. A, Gutman J, Srinivas .S, A, has introduced a new energy Minimum covering energy of a graph and its properties were discussed [1]. Recently Rajesh Kanna. M, R, Dharmendra. B, N, Sridhara .G introduced the minimum dominating energy of a graph which depends on the minimum dominating set [11]. The concept of edge domination was introduced by Mitchell and Hedetniemi [10]. Meenakshi. S, Lavanya. S has introduced a new energy Minimum Dom Strong Dominating Energy and its properties and bounds were found [9].

Motivated by these papers, we have introduced the Minimum Edge Dominating Energy of a graph [12]. In this paper we are concerned with finite, simple and undirected graphs. In this paper we have computed the Minimum Edge Dominating Energy of a graph. Its properties and bounds are discussed.

2. PRELIMINARIES

Definition: 2.1

The adjacency matrix \( A(G) \) of a graph \( G(V, E) \) with a vertex set \( V = \{v_1, v_2, \ldots, v_n\} \) and an edge set \( E = \{e_1, e_2, \ldots, e_m\} \) is an n x n matrix

\[ A = (a_{ij}) = \begin{cases} 1 & \text{if } v_i \text{ is adjacent to } v_j \\ 0, \text{ otherwise} \end{cases} \]

A is a real symmetric matrix.

Definition: 2.2
The Eigen values \( \lambda_1, \lambda_2, \ldots, \lambda_n \) of A, assumed in non increasing order, are the Eigen values of the graph G. As A is real symmetric, the Eigen values of G are real with sum equal to zero. The Energy \( E(G) \) of G is defined to be the sum of the absolute values of the Eigen values of G.

\[
i.e., \quad E(G) = \sum_{i=1}^{n} |\lambda_i| \quad [5].
\]

**Definition: 2.3**

Let G be a simple graph of order \( n \) with vertex set \( V = \{v_1, v_2, \ldots, v_n\} \) and edge set \( E = \{e_1, e_2, \ldots, e_m\} \). A subset of \( E \) is called an Edge Dominating set of G if every edge of \( E \) is adjacent to some edge in \( E' \). Any edge dominating set with minimum cardinality is called a Minimum Edge Dominating Set [10]. Let \( E' \) be a Minimum Edge Dominating Set of a graph G. The Minimum Edge Dominating Matrix of G is the \( m \times m \) matrix defined by

\[
\text{Adj} D'(G) = \left( d'_i \right), \quad \text{where} \quad d'_i = \begin{cases} 1 & \text{if } e_i \text{ and } e_j \text{ are adjacent} \\ 1 & \text{if } i = j \text{ and } e_i \in E' \\ 0 & \text{otherwise}. \end{cases}
\]

The characteristic polynomial of \( D'(G) \) is denoted by

\[
f_m(G, \rho) = \det (\rho I - D'(G)).
\]

The Minimum Edge Dominating Eigen values of a graph G are the eigen values \( \rho_1, \rho_2, \ldots, \rho_m \) of \( D'(G) \). The Minimum Edge Dominating Energy of G is defined as

\[
E_D(G) = \sum_{i=1}^{m} |\rho_i| [12].
\]

**Example: 1**

Consider the above graph G.

(i) Let the Minimum Edge Dominating set be \( E' = \{e_1, e_3\} \).

Then the Minimum Edge Dominating adjacency matrix is

\[
(G) = \begin{bmatrix}
1 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0
\end{bmatrix}
\]

The characteristic equation is \( \rho^6 - 2\rho^5 - 7\rho^4 + 7\rho^3 + 13\rho^2 - 1 = 0 \).

The Minimum Edge Dominating eigen values are

\( \rho_1 \approx -1.8363, \rho_2 \approx -1.1157, \rho_3 \approx -0.3132, \rho_4 \approx 0.2642, \rho_5 \approx 1.9050, \rho_6 \approx 3.0962 \).

The Minimum Edge Dominating Energy, \( E_D(G) \approx 8.5306 \).

(ii) If we take another Minimum Edge Dominating set \( E' = \{e_2, e_3\} \).
The characteristic equation is 
\[ \rho^6 - 2\rho^5 - 7\rho^4 + 6\rho^3 + 13\rho^2 - 6\rho - 3 = 0 \]

The Minimum Edge Dominating Eigen values are 
\[ \rho_1 \approx -1.7321, \rho_2 \approx -1, \rho_3 \approx -0.6751, \rho_4 \approx 0.4608, \rho_5 \approx 1.7321, \rho_6 \approx 3.2143 \]

The Minimum Edge Dominating Energy, \( E_D(G) \approx 8.8144 \).

This example illustrates the fact that the Minimum Edge Dominating Energy of a graph \( G \) depends on the choice of the Minimum Edge Dominating Set. 

i.e. The Minimum Edge Dominating Energy is not a graph invariant.

3. PROPERTIES OF MINIMUM EDGE DOMINATING ENERGY:

**Theorem: 3.1**

Let \( G \) be a simple graph of order \( n \) and size \( m \), let \( \rho \) be the Minimum Edge Dominating Set and let \( f_m(G, \rho) = c_0\rho^m + c_1\rho^{m-1} + c_2\rho^{m-2} + \ldots + c_m \) be the characteristic polynomial of the Minimum Edge Dominating Matrix of the graph \( G \). Then

\[ c_2 = \frac{1}{2} \left| D_i \right| \sum_{i=1}^{m} \deg_i. \]

**Proof:**

The sum of the determinants of all 2 x 2 principal sub matrices of \( G = (-1)^2 \).

Therefore, \( c_2 = \sum_{1 \leq i < j \leq m} \left| D_{ij} \right| = \sum_{1 \leq i < j \leq m} \left| d_{ij} \right| = \sum_{1 \leq i < j \leq m} \left| d_{ij} \right| = \left( \sum_{i=1}^{m} \deg_i \right). \]

**Theorem: 3.2**

Let \( G = (V, E) \) be a simple graph of order \( n \) and size \( m \). Let \( \rho_1, \rho_2, \rho_3, \ldots, \rho_m \) be the eigen values of \( G \).

Then \( \sum_{i=1}^{m} \deg_i = \left| D'_+ \right| \sum_{i=1}^{m} \deg_i \).

**Proof:**

The sum of the squares of the eigen values of \( D'_+ \) is the trace of \( D'_+(G) \).

Therefore, \( \sum_{i=1}^{m} \deg_i = \sum_{i=1}^{m} \sum_{j=1}^{m} d_{ij} = \sum_{i=1}^{m} \deg_i = \left| D'_+ \right| \sum_{i=1}^{m} \deg_i \).

4. BOUNDS FOR MINIMUM EDGE DOMINATING ENERGY

In this section we find some bounds for \( E(G) \) of a graph.

**Theorem: 4.1**
Let \( G \) be a simple graph with \( n \) vertices and \( m \) edges. If \( \ell_i(G) \) is the Minimum Edge Dominating Energy of the graph, then

\[
\sqrt{\left| D' \right| + 2 \sum_{i=1}^{m} \left( \frac{\deg v_i}{2} \right)} \leq \ell_i(G).
\]

**Proof:**

Consider the Cauchy-Schwartz inequality:

\[
\left( \sum_{i=1}^{n} a_i \right) \left( \sum_{i=1}^{n} b_i \right) \leq \left( \sum_{i=1}^{n} |a_i|^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^{n} |b_i|^2 \right)^{\frac{1}{2}}.
\]

If \( \rho_i = 1 \), then

\[
\sum_{i=1}^{m} \rho_i \leq \sum_{i=1}^{m} 1 = m.
\]

Then,

\[
|D'| + 2 \sum_{i=1}^{m} \left( \frac{\deg v_i}{2} \right) \leq \sqrt{m \left| D' \right| + 2 \sum_{i=1}^{m} \left( \frac{\deg v_i}{2} \right)}.
\]

Therefore, the upper bound holds. For the lower bound, since

\[
\sum_{i=1}^{m} 1 = m,
\]

we have

\[
\ell_i(G) \leq \sqrt{m \left| D' \right| + 2 \sum_{i=1}^{m} \left( \frac{\deg v_i}{2} \right)}.
\]

**Theorem: 4.2**

Let \( G \) be a simple graph with \( n \) vertices and \( m \) edges. If \( \ell_i(G) \) is the Minimum Edge Dominating Energy of the graph and let \( P \) be a det \((D'(G))\), then

\[
\ell_i(G) \leq \sqrt{\left| D' \right| + 2 \sum_{i=1}^{m} \left( \frac{\deg v_i}{2} \right)} + m(m-1)I.
\]

**Proof:**

From the relation between the arithmetic mean and geometric mean, we have

\[
\frac{1}{m(m-1)} \sum_{i \neq j} |\rho_i| |\rho_j| \leq \left( \prod_{i \neq j} |\rho_i| |\rho_j| \right)^{\frac{1}{m(m-1)}}
\]

\[
= \left( \prod_{i=1}^{n} |\rho_i|^2 \right)^{\frac{1}{m(m-1)}}
\]

\[
= \left( \prod_{i=1}^{n} |\rho_i| \right)^{\frac{1}{m-1}}
\]

\[
= \text{det}(D'(G))
\]

\[
= I
\]

Therefore,

\[
\sum_{i \neq j} |\rho_i| |\rho_j| \geq m(m-1)I
\]

\[\text{(1)}\]

Now consider,

\[
(D'(G)) \leq \sum_{i \neq j} |\rho_i| |\rho_j| \leq \sum_{i=1}^{m} |\rho_i|^2 \sum_{i=1}^{m} \left( \frac{\deg v_i}{2} \right) + m(m-1)I
\]

\[\text{[From (1)]}\]

\[
= \left| D' \right| + \sum_{i=1}^{m} \left( \frac{\deg v_i}{2} \right) + m(m-1)I
\]

\[\text{[Theorem: 3.1]}\]
Theorem: 4.3
If $\rho_1(G)$ is the largest Minimum Edge Dominating Eigen value of $(G)$, then
$$\rho_1(G) \geq \frac{|D'| + 2 \left[ \sum_{i=1}^{m} \left( \frac{\deg v_i}{2} \right) \right]}{m}.$$  

Proof:
Let $X$ be any non zero vector. Then by [3], we have
$$\rho_1(G) = \max_{X \neq 0} \left\{ \frac{XX^T I}{X^T r} \right\}.$$  

Therefore,
$$\rho_1(G) \geq \frac{|D'|}{m} + 2 \left[ \sum_{i=1}^{m} \left( \frac{\deg v_i}{2} \right) \right],$$  

where $I = [1,1,1,\ldots,1]'$ is a unit column matrix of order $m \times 1$.

Similar to Koolen and Moulton’s [7] upper bound for energy of a graph, upper bound for $E(G)$, is given in the following theorem.

Theorem: 4.4
If $G$ is a simple graph with $n$ vertices and $m$ edges and
$$E_{Dr}(G) \geq \frac{|D'| + 2 \left[ \sum_{i=1}^{m} \left( \frac{\deg v_i}{2} \right) \right]}{m},$$

then
$$\sqrt{(m-1) \left[ |D'| + 2 \left[ \sum_{i=1}^{m} \left( \frac{\deg v_i}{2} \right) \right] - \left( \frac{|D'| + 2 \left[ \sum_{i=1}^{m} \left( \frac{\deg v_i}{2} \right) \right]}{m} \right)^2 \right]} \geq \rho_1.$$  

Proof:
Consider the Cauchy-Schwartz inequality
$$\left( \sum_{i=2}^{n} a_i b_i \right)^2 \leq \left( \sum_{i=2}^{n} a_i^2 \right) \left( \sum_{i=2}^{n} b_i^2 \right).$$

Then,
$$\Rightarrow \left[ E_{Dr}(G) - \rho \right]^2 \leq (m-1) \left[ |D'| + 2 \left[ \sum_{i=1}^{m} \left( \frac{\deg v_i}{2} \right) \right] - \rho \right].$$

Let $f(x) = \sqrt{(m-1) \left[ |D'| + 2 \left[ \sum_{i=1}^{m} \left( \frac{\deg v_i}{2} \right) \right] - x^2 \right]}$.

For decreasing function $f(x)$, we have
$$f'(0) = 1 - \frac{(m-1) \left[ |D'| + 2 \left[ \sum_{i=1}^{m} \left( \frac{\deg v_i}{2} \right) \right] - x^2 \right]}{2 \sqrt{(m-1) \left[ |D'| + 2 \left[ \sum_{i=1}^{m} \left( \frac{\deg v_i}{2} \right) \right] - x^2 \right]}} \leq 0.$$  

Since
$$\sqrt{(m-1) \left[ |D'| + 2 \left[ \sum_{i=1}^{m} \left( \frac{\deg v_i}{2} \right) \right] \right]} \leq \frac{|D'| + 2 \left[ \sum_{i=1}^{m} \left( \frac{\deg v_i}{2} \right) \right]}{m},$$

We have
$$f(\rho_1) \leq f\left( \frac{|D'| + 2 \left[ \sum_{i=1}^{m} \left( \frac{\deg v_i}{2} \right) \right]}{m} \right).$$  

\[\text{Theorem: 3.1}\]
5. CONCLUSION:
In this paper we have found the Minimum Edge Dominating energy of a graph. The various upper and lower bounds for the Minimum Edge Dominating Energy of a graph have been found. Analogues works can be carried by us for other graphs also.

REFERENCES