New Tail Probability Type Concentration Inequalities and Complete Convergence for WOD Random Variables

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Article History: *Do not touch during review process(xxxx)*

Abstract: Let $\{\boldsymbol{Z_n}, \boldsymbol{n} \geq \boldsymbol{1}\}$ be sequence of Widely Orthant Dependent random variables (WOD). The goal of this paper is to obtain some concentration inequalities for unbounded Widely Orthant Dependent (WOD) random variables. Then we will use this inequality for establishing the almost complete convergence for a sequence of widely dependent random variables (WOD).

Keywords: WOD sequence, exponential inequalities, complete convergence, concentration inequalities

1. Introduction

The topic of this article is the study of random variables of widely orthant dependent. Concentration inequalities quantify such statements, typically by bounding the probability that such a variable differs from its expected value (or from its median) by more than a certain amount. Concentration inequalities has been a topic of intensive research in the last decades in a variety of areas because of their importance in numerous applications. Among the areas of applications, without trying to be exhaustive, we mention statistics, learning theory, discrete mathematics, statistical mechanics, random matrix theory, information theory, and high-dimensional geometry. The concentration inequality plays an important role in various proofs of limit theorems. In particular, it provides a measure of the complete convergence for partial sums. Firstly, we will recall the definitions of widely upper orthant dependent and widely lower orthant dependent. For the random variables $\{Z_n, n \geq 1\}$, if there exists a finite real sequence $\{g_U(n), n \geq 1\}$ satisfying for each $n \geq 1$, $\mathbb{P}(Z_1 > z_1, Z_2 > z_2, \dots, Z_n > z_n) \leq g_U(n) \prod_{i=1}^{n} \mathbb{P}(Z_i > z_i) \dots (1.1)$, then we say that the $\{Z_n, n \geq 1\}$ are widely upper orthant dependent (WUOD, in short). If there exists a finite real sequence $\{g_L(n), n \geq 1\}$ satisfying for each $n \geq 1$ and for all $z_i \in]-\infty; +\infty[$, $1 \leq i \leq n$, $\mathbb{P}(Z_1 \leq z_1, Z_2 \leq z_2, \dots, Z_n \leq z_n) \leq g_L(n) \prod_{i=1}^{n} \mathbb{P}(Z_i \leq z_i) \dots (1.2)$, then we say that the $\{Z_n, n \geq 1\}$ are widely lower orthant dependent (WLOD, in short) .

Further, a sequence of random variables $\{Z_n, n \geq 1\}$ is said to be widely orthant dependent (WOD) if it is both WUOD and WLOD. The sequences $\{g_U(n), n \geq 1\}$ and $\{g_L(n), n \geq 1\}$ are said dominating coefficients of $\{Z_n, n \geq 1\}$. Denote $g(n) = \max(g_U(n), g_L(n))$.

Clearly, we have $g_U(n) \geq 1, g_L(n) \geq 1, n \geq 2$ and $g_U(1) = g_L(1) = 1$. **(Wang & Wang & Gao 2013)** and **(Wang & Li & Gao 2011)** give some examples of WNOD random variables with various dominating coefficients. These examples show that WNOD random variables contain some common negatively dependent random variables, some positively dependent random variables and some others.

In the case $g_L(n) = g_U(n) = M$ for some constant M, we say that $\{Z_n, n \geq 1\}$ are negatively extended-dependent (END). The concept of general extended negative dependence (END, in short) random variables was proposed by **(Liu 2009)**.

Clearly, in the case $M = 1$ the notion of END random variables reduces to the notion of NOD random variables. As mentioned in **(Liu 2009)**, the END structure is substantially more comprehensive than the NOD structure in that it can reflect not only a negative dependence structure but also a positive one, to some extent. **(Liu 2009)** also provided some interesting examples to illustrate that the END indeed allows a wide range of dependence structures. **(Joag-Dev & Proschan 1983)** pointed out that negatively associated (NA) random variables must be NOD and NOD is not necessarily NA, thus NA random variables are END. So it is very significant to study convergence properties of this wider END class.

Since the paper of **(Ebrahimi & Ghosh 1981)** appeared, the convergence properties of NOD random sequences were studied in different aspects.

A sequence of random variables $\{U_n, n \geq 1\}$ is said to converge completely to a constant a if for any $\varepsilon > 0$

$$\sum_{n=1}^{\infty} \mathbb{P}(|U_n - a| > \varepsilon) < \infty$$

By the Borel–Cantelli lemma, this implies that $U_n \longrightarrow a$ almost surly. And so complete convergence is a stronger concept than almost surly convergence. The concept of complete convergence was given first by **(Robbins & Hsu 1947)**. In generally we can refer to **(Baum &Katz 1965)**, **(Chow 1988)**, **(Wang et al 2011)**, **(Qiu et al 2014)**, **(Wang et al 2014)** and **(Wang & Wu & Rosalsky 2017)**.

2. Preliminary Lemmas

Lemma 2.1 $\{Z_n, n \geq 1\}$ are nonnegative and WUOD with dominating coefficient $\{g_U(n), n \geq 1\}$, then for each $n \geq 1$

$$\mathbb{E} \prod_{i=1}^{n} Z_i \leq g_U(n) \prod_{i=1}^{n} \mathbb{E}(Z_i).$$

In particular, if $\{Z_n, n \geq 1\}$ are WUOD with dominating coefficients $(g_U(n), n \geq 1)$ then for each $n \geq 1$ and $s \geq 0$,

$$\mathbb{E} \exp\{s \sum_{i=1}^{n} Z_i\} \leq g_U(n) \prod_{i=1}^{n} \mathbb{E}(e^{sZ_i})$$

By **lemma 2.1** we can get the following corollary immediately.

Corollary 2.1 (Berkane & Benaissa 2019)

Let $\{Z_n, n \geq 1\}$ be sequence of WOD random variables. Then for each $n \geq 1$ and $s \in \mathbb{R}$

$$\mathbb{E} \exp\{s \sum_{i=1}^{n} Z_i\} \leq g(n) \prod_{i=1}^{n} \mathbb{E}(e^{sZ_i})$$

Lemma 2.2 (Azzedine, Zeblah & Benaissa.S 2021)

For all $z \in \mathbb{R}$

$$\exp\{z\} \leq 1 + z + |z|^{1+\beta} \exp\{2|z|\}$$

Where $0 < \beta \leq 1$

Lemma 2.3 Let $Z_1, \dots, Z_n$ be Widely Orthant Dependent random variables with $\mathbb{E}(Z_i) = 0$ for $1 \leq i \leq n$, If there exists a sequence of positive numbers c such that $\mathbb{P}(|Z_i| \leq c) = 1$, Set $S_n = \sum_{i=1}^{n} Z_i$ and $K_\beta = \sum_{i=1}^{n} \mathbb{E}|Z_i|^{1+\beta}$. Then for any $t > 0$ and for any $\xi > 0$,

$$\mathbb{P}(|S_n| > n\xi) \leq 2g(n)e^1 \left[K_\beta^{\frac{\xi}{(1+\beta+2c)}} \right]^n \quad (1)$$

Proof For $t > 0$ and $\mathbb{E}(Z_i) = 0$, $i \geq 1$ and the fact that $1 + z \leq e^z$, therefore, by **Markov's inequality** and **Lemma 2.1** it follows that

$$\begin{aligned}
\mathbb{P}(S_n > n\xi) = \mathbb{P}(tS_n > tn\xi) = \mathbb{P}(e^{tS_n} > e^{tn\xi}) \\
&\leq e^{-tn\xi}\mathbb{E}(e^{tS_n}) \\
&\leq g(n)e^{-tn\xi} \prod_{i=1}^{n} \mathbb{E}(e^{tZ_i}) \\
&\leq g(n)e^{-tn\xi} \prod_{i=1}^{n} \left[1 + t^{1+\beta}\mathbb{E}\left(|Z_i|^{1+\beta}e^{2t|Z_i|}\right)\right] \\
&\leq g(n)e^{-tn\xi} \prod_{i=1}^{n} \left[1 + t^{1+\beta}\left(\mathbb{E}\left(|Z_i|^{1+\beta}e^{2tc}\right)\right)\right] \\
&\leq g(n)e^{-tn\xi} \prod_{i=1}^{n} \left[1 + e^{t(1+\beta+2c)}\mathbb{E}\left(|Z_i|^{1+\beta}\right)\right]
\end{aligned}$$

$$\leq g(n)e^{-tn\xi}\prod_{i=1}^{n} e^{\,e^{t(1+\beta+2c)}\mathbb{E}(|Z_i|^{1+\beta})} \qquad (\text{By } 1+z \leq e^z)$$

$$\leq g(n)e^{-tn\xi+e^{t(1+\beta+2c)}K_\beta} \qquad (2)$$

The right term is optimal for $t = \frac{1}{(1+\beta+2c)}\log[\frac{n\xi}{K_\beta(1+\beta+2c)}]$

We obtain

$$\mathbb{P}(S_n > n\xi) \leq g(n)e^{-\frac{n\xi}{(1+\beta+2c)}\left[\log\left\{\frac{n\xi}{K_\beta(1+\beta+2c)}\right\}-1\right]}$$

$$\mathbb{P}(S_n > n\xi) \leq g(n)e^{-\frac{n\xi}{(1+\beta+2c)}\left[\log\left\{\frac{n\xi}{(1+\beta+2c)}\right\}-\log(K_\beta)-1\right]}$$

Using $1-\frac{1}{u} \leq \log(u)$ for $u > 0$, we obtain

$$\mathbb{P}(S_n > n\xi) = g(n)e^{1+\frac{n\xi}{(1+\beta+2c)}\log(K_\beta)} = g(n)e^1\left[K_\beta^{\frac{\xi}{(1+\beta+2c)}}\right]^n$$

And we have

$$\mathbb{P}(|S_n| > n\xi) = \mathbb{P}(S_n > n\xi) + \mathbb{P}(-S_n < n\xi) \qquad (3)$$

Since $\{-Z_n, n \geq 1\}$, is also WOD we obtain by (1.2) that

$$\mathbb{P}(S_n > n\xi) = \mathbb{P}(-S_n < n\xi) \leq g(n)e^1\left[K_\beta^{\frac{\xi}{(1+\beta+2c)}}\right]^n \qquad (4)$$

By (2) and (3) the result (1) follows

3. Main Results and Proofs

Theorem 3.1 let $Z_1, \dots, Z_n$ be Widely Orthant Dependent random variables such that $\mathbb{E}(Z_i) = 0$ if for $1 \leq i \leq n$, If there exists a sequence of positive numbers c such that $\mathbb{P}(|Z_i| \leq c) = 1$, Set $S_n = \sum_{i=1}^{n} Z_i$, there exists a finite sequence $g(n)$ such that

$$\mathbb{E}(e^{tS_n}) \leq g(n)e^{K_\beta e^{t(1+\beta+2c)}} \qquad (5)$$

Proof From condition $\mathbb{E}(Z_i) = 0$ and using the **Lemma2.2** and $1+z \leq e^z$, we have

$$\mathbb{E}(e^{tZ_i}) \leq 1 + t^{1+\beta}\mathbb{E}\left(|Z_i|^{1+\beta}e^{2t|Z_i|}\right)$$

$$\leq 1 + t^{1+\beta}\,\mathbb{E}\left(|Z_i|^{1+\beta}e^{2tc}\right)$$

$$\leq 1 + e^{t(1+\beta+2c)}\,\mathbb{E}\left(|Z_i|^{1+\beta}\right)$$

$$\leq e^{e^{t(1+\beta+2c)}\mathbb{E}(|Z_i|^{1+\beta})} \qquad (\text{By } 1+x \leq e^x) \qquad (6)$$

For any $t > 0$. by **Corollary 2.1** and (6)

$$\mathbb{E}\exp\{t\sum_{i=1}^{n} Z_i\} \leq g(n)\prod_{i=1}^{n}\mathbb{E}(e^{tZ_i})$$

$$\leq g(n)e^{K_\beta e^{t(1+\beta+2c)}} \qquad (7)$$

Theorem 3.2 Let $Z_1, \dots, Z_n$ be Widely Orthant Dependent random variables with $\mathbb{E}(Z_i) = 0$ and $\mathbb{E}(Z_i^2) < \infty$, if there exists a positive constant c ,such that $\mathbb{P}(|Z_i| \leq c) = 1$, $i \geq 1$ and $K_\beta = \sum_{i=1}^{n}\mathbb{E}|Z_i|^{1+\beta}$ and for any $\xi > e^1(1+\beta+2c)$ and $n \geq 1$,

$$\mathbb{P}\left({S_n}/{K_\beta} > \xi\right) \leq g(n)e^{\frac{K_\beta\xi}{1+\beta+2c}}\left(\frac{1+\beta+2c}{\xi}\right)^{\frac{K_\beta\xi}{1+\beta+2c}} \qquad (8)$$

Proof For $t > 0$, then from **Markov's inequality** it follows that

$$\mathbb{P}\left({S_n}/{K_\beta} > \xi \right) = \mathbb{P}\left(e^{tS_n} > e^{t\xi K_\beta}\right) \le e^{-t\xi K_\beta}\mathbb{E}\left(\prod_{i=1}^{n} e^{tZ_i}\right),$$

$$\le g(n)e^{-t\xi K_\beta + K_\beta e^{t(1+\beta+2c)}} \tag{9}$$

By taking $t = \frac{1}{(1+\beta+2c)}\log\left[\frac{\xi}{1+\beta+2c}\right]$ we obtain (8) from (9)

Theorem 3.3 Let $Z_1, \dots, Z_n$ be Widely Orthant Dependent random variables with $\mathbb{E}\left({Z_i}^2\right) < \infty$ .If there exists a positive constant, such that $\mathbb{P}(|Z_i| \le c) = 1$ for $i \ge 1$ and $K_\beta = \sum_{i=1}^{n} \mathbb{E}|Z_i|^{1+\beta}$ and $W_\beta = \sum_{i=1}^{n}|\mathbb{E}Z_i|^{1+\beta}$ then for any $\xi > 0$ we have

$$\mathbb{P}(|S_n - \mathbb{E}S_n| > n\xi) \le 2g(n)e^1\left[\left[2^\beta\left(K_\beta + W_\beta\right)\right]^{\frac{\xi}{1+\beta+4c}}\right]^n \tag{10}$$

Proof By **Markov's inequality** and **Corollary 2.1** and **Lemma** *2.2*

$$\mathbb{P}(S_n - \mathbb{E}S_n > n\xi) \le e^{-tn\xi}\mathbb{E}\left(e^{t(S_n - \mathbb{E}S_n)}\right)$$

$$\le g(n)e^{-tn\xi}\prod_{i=1}^{n}\mathbb{E}\left(e^{t(Z_i - \mathbb{E}Z_i)}\right) \tag{11}$$

On the other hand we have

$$\mathbb{E}\left(e^{t(Z_i - \mathbb{E}Z_i)}\right) \le \left[1 + t\mathbb{E}(Z_i - \mathbb{E}Z_i) + t^{1+\beta}\mathbb{E}\left(|Z_i - \mathbb{E}Z_i|^{1+\beta}e^{2t|Z_i - \mathbb{E}Z_i|}\right)\right]$$

$$\le \left[1 + t^{1+\beta}e^{4tc}\left(\mathbb{E}\left(|Z_i - \mathbb{E}Z_i|^{1+\beta}\right)\right)\right]$$

$$\le 1 + 2^\beta t^{1+\beta}e^{4tc}\left(\mathbb{E}|Z_i|^{1+\beta} + |\mathbb{E}Z_i|^{1+\beta}\right)$$

$$\le e^{2^\beta t^{1+\beta}e^{4tc}\left(\mathbb{E}|Z_i|^{1+\beta} + |\mathbb{E}Z_i|^{1+\beta}\right)} \qquad (\text{By } 1 + z \le e^z)$$

We compensate for this result in the right-hand side of (11) we get

$$\mathbb{P}(S_n - \mathbb{E}S_n > n\xi) \le g(n)e^{-tn\xi}\prod_{i=1}^{n} e^{2^\beta t^{1+\beta}e^{4tc}\left(\mathbb{E}|Z_i|^{1+\beta} + |\mathbb{E}Z_i|^{1+\beta}\right)}$$

$$\le g(n)e^{-tn\xi + 2^\beta t^{1+\beta}e^{4tc}\left(\sum_{i=1}^{n}\mathbb{E}|Z_i|^{1+\beta} + \sum_{i=1}^{n}|\mathbb{E}Z_i|^{1+\beta}\right)}$$

$$\le g(n)e^{-tn\xi + 2^\beta e^{t(1+\beta+4c)}\left(K_\beta + W_\beta\right)},$$

by taking $t = \frac{1}{1+\beta+4c}\log\left(\frac{n\xi}{2^\beta(K_\beta+W_\beta)(1+\beta+4c)}\right)$ we obtain

$$\mathbb{P}(S_n - \mathbb{E}S_n > n\xi) \le g(n)e^{-\frac{n\xi}{1+\beta+4c}\log\left(\frac{n\xi}{2^\beta\left(K_\beta+W_\beta\right)(1+\beta+4c)}\right) + \frac{n\xi}{1+\beta+4c}}$$

$$\le g(n)e^{-\frac{n\xi}{1+\beta+4c}\left[\log\left(\frac{n\xi}{2^\beta\left(K_\beta+W_\beta\right)(1+\beta+4c)}\right) - 1\right]}$$

$$\le g(n)e^{-\frac{n\xi}{1+\beta+4c}\left[\log\left(\frac{n\xi}{(1+\beta+4c)}\right) + \log\left(\frac{1}{2^\beta\left(K_\beta+W_\beta\right)}\right) - 1\right]}$$

$1 - \frac{1}{u} \le \log(u)$, for $u > 0$, we obtain

$$\mathbb{P}(S_n - \mathbb{E}S_n > n\xi) \le g(n)e^{-\frac{n\xi}{1+\beta+4c}\left[1 - \frac{(1+\beta+4c)}{n\xi} + \log\left(\frac{1}{2^\beta\left(K_\beta+W_\beta\right)}\right) - 1\right]}$$

$$\mathbb{P}(S_n - \mathbb{E}S_n > n\xi) \le g(n)e^1 e^{-\frac{n\xi}{1+\beta+4c}\log\left(\frac{1}{2^\beta\left(K_\beta+W_\beta\right)}\right)}$$

$$\mathbb{P}(S_n - \mathbb{E}S_n > n\xi) \leq g(n)e^1\left[\left[2^\beta\left(K_\beta + W_\beta\right)\right]^{\frac{\xi}{1+\beta+4c}}\right]^n$$

And

$$\mathbb{P}(|S_n - \mathbb{E}S_n| > n\xi) \leq 2g(n)e^1\left[\left[2^\beta\left(K_\beta + W_\beta\right)\right]^{\frac{\xi}{1+\beta+4c}}\right]^n$$

Corollary 3.1 Let $\{Z_n, n \geq 1\}$ be a sequence of identically distributed WOD random variables. Assume that there exists a positive integer n_0 for each $n \geq n_0$ .If there exists a positive constant c, such that $\mathbb{P}(|Z_i| \leq c) = 1$, for $i \geq 1$ and $0 < \beta \leq 1$, Then for any $t > 0$ and then for any $\xi > 0$ such that $\xi > 0$ we have

$$\mathbb{P}(|S_n - \mathbb{E}S_n| > n\xi) \leq 2g(n)e^1\left[\left[2^\beta\left(K_\beta + W_\beta\right)\right]^{\frac{\xi}{1+\beta+4c}}\right]^n \tag{12}$$

Theorem 3.4 let $\{Z_n, n \geq 1\}$ be a sequence WOD random variables with $\mathbb{E}(Z_i) = 0$, and $\mathbb{P}(|Z_i| \leq c) = 1$, for $i \geq 1$ where c is positive constant and $0 < \beta \leq 1$. Then for any $\Delta > 0$

$$\sum_{i=1}^{\infty} \mathbb{P}(|S_n| > n^\Delta \xi) < \infty \tag{13}$$

Proof Let $K_\beta < 1$. For any $\xi > 0$, from **Lemma 2.3** we have

$$\sum_{i=1}^{\infty} \mathbb{P}(|S_n| > n^\Delta \xi) \leq 2g(n)e^1 \sum_{i=1}^{\infty}\left[K_\beta^{\frac{\xi}{(1+\beta+2c)}}\right]^{n^\Delta}$$

$$= 2\,g(n)e^1 \sum_{i=1}^{\infty} [K_1]^{n^\Delta} < \infty, \tag{14}$$

Where K_1 is a positive number not depending on n. by suppose that $0 \leq K_1 < 1$ Hence the proof is complete.

Theorem 3.5 Let $\{Z_n, n \geq 1\}$ be a sequence of identically distributed WOD random variables. Assume that there exists a positive integer n_0 such that, $\mathbb{P}(|Z_i| \leq c) = 1$, for $i \geq 1$, for each $n \geq n_0$, where c is positive constant and $0 < \beta \leq 1$. Then for any $\Delta > 0$

$$\sum_{i=1}^{\infty} \mathbb{P}(|S_n - \mathbb{E}S_n| > n^\Delta \xi) < \infty \tag{15}$$

Proof We have from **Theorem 3.3,** for any $\xi > 0$, there exists

$$\sum_{i=1}^{\infty} \mathbb{P}(|S_n - \mathbb{E}S_n| > n^\Delta \xi) \leq 2g(n)e^1 \sum_{i=1}^{\infty}\left[\left[2^\beta\left(K_\beta + W_\beta\right)\right]^{\frac{\xi}{1+\beta+4c}}\right]^{n^\Delta}$$

$$= 2g(n)e^1 \sum_{i=1}^{\infty} [K_2]^{n^\Delta} < \infty$$

Where K_2 is a positive number not depending on n. By suppose that $0 \leq K_2 < 1$ Hence the proof is complete. After this result we get (15)

4.Conclusion

In this work we study the complete convergence of the WOD random variables, using the exponential inequalities and some concentration inequalities for probability of orthant dependent variables.

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