Sumudu transform Iterative Method for solving time-fractional Schrödinger Equations

*** R. K. Bairwa[a] and Karan Singh[b]**

[a] Assistant Professor , Department of Mathematics, University of Rajasthan, Jaipur, Rajasthan-302004, India.
[b] Research Scholar , Department of Mathematics, University of Rajasthan, Jaipur, Rajasthan-302004, India.
[*]Corresponding Author's Email : dr.rajendra.maths@gmail.com

Abstract: The purpose of this study is to solve the time-fractional Schrödinger equations under initial conditions by using a newly developed analytical method known as the Sumudu transform iterative method. The time-fractional derivative is defined in the Caputo sense, and the solutions are achieved in closed form, in terms of Mittag-Leffler functions. Furthermore, the obtained results show that the proposed method is simple to implement and computationally attractive.

Keywords: Caputo fractional derivative, Sumudu transform, Schrödinger equations, iterative method, and fractional differential equations

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1. Introduction

The fractional Schrodinger equation (SE) is a generalization of the standard SE, which naturally arises from fractional quantum mechanics. The equation was discovered and developed in [16] by extending the Feynman path integral from Brownian-like quantum mechanical paths to Levy-like quantum mechanical paths [13-15]. The fractional SE includes a time derivative of fractional order $0 < \alpha \leq 1$ instead of the first-order time derivative in the standard SE. Thus, according to modern terminology, the fractional SE is a fractional partial differential equation. The physical theory underlying the solution concept for time-fractional SE is significant and fascinating, and it is a focus of active physical, mathematical, and engineering research [5,9,24,27,30].Several analytical and numerical methods have been reported for the solutions of linear and nonlinear fractional partial differential equations, including the adomian decomposition method (ADM) [22], the homotopy perturbation method (HPM) [19], the residual power series method (RPSM) [1], the homotopy analysis method (HAM) [12], the homotopy analysis transform method (HATM) [23], the iterative Laplace transform method (ILTM) [25], the modified Variational iteration method (MVIM) [18], the generalized difference transform method (GDTM) [20] etc. Recently, Wang and Liu developed the Sumudu transform iterative method [26] by combining the Sumudu transform with an iterative technique to find approximate analytical solutions to time-fractional Cauchy reactions-diffusion equations. The Sumudu transform iterative method was used to successfully solve fractional Fokker-Planck equations [3].

In this study, we examine the linear time-fractional Schrödinger equations of the following form [23, 25]

$$D_t^{\alpha} u(x,t) + i u_{xx}(x,t) = 0, \quad u(x,0) = g(x), \quad i = \sqrt{-1}, \tag{1}$$

and the nonlinear time-fractional Schrödinger equations are

$$i D_t^{\alpha} u(x,t) + u_{xx}(x,t) \pm \lambda |u(x,t)|^2 u(x,t) = 0, \quad u(x,0) = g(x), \; i = \sqrt{-1}, \tag{2}$$

and

$$i D_t^{\alpha} u(x,t) + u_{xx}(x,t) \pm \lambda |u(x,t)|^{2r} u(x,t) = 0, \; u(x,0) = g(x), \; r \geq 1, \; i = \sqrt{-1}, \; 0 \leq \lambda \in R \tag{3}$$

(where $0 < \alpha \leq 1$) with a cubic and power law nonlinearities respectively, where λ is a constant and $u(x,t)$ is a complex function. The fractional derivatives are considered in the Caputo sense.
The main aim of this article is to expand the application of Sumudu transform iterative technique for constructing analytical solutions to Schrodinger equations with time-fractional derivatives.

2. Preliminaries

In this section, we provide some fundamental definitions, notations, and properties of fractional calculus and Sumudu transform theory, which will be used later in this paper.

Definition 1. In Caputo's sense, the fractional derivative of a function $u(x,t)$ is defined as [6]

$$D_t^{\alpha}u(x,t)=\frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\xi)^{m-\alpha-1}u^{(m)}(x,\xi)\,d\xi,\quad m-1<\alpha\le m,\, m\in N, \\ = I_t^{m-\alpha}D^m u(x,t). \tag{4}$$

Here $D^m \equiv \frac{d^m}{dt^m}$ and I_t^{α} stands for the Riemann-Liouville fractional integral operator of order $\alpha>0$, defined as [17]

$$I_t^{\alpha}u(x,t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-\xi)^{\alpha-1}u(x,\xi)\,d\xi,\ \ \xi>0. \tag{5}$$

Definition 2. The Sumudu transform is defined over the set of functions

$$\left\{ f(t)\,|\,\exists\, M,\rho_1>0,\ \rho_2>0\,,\ |\,f(t)\,|<Me^{|t|/\rho_j}\ \ if\ \ t\in(-1)^j\times\left[\,0,\infty\right)\right\}$$

by the following formula [4, 28]

$$S\,[\,f(t)]=F(\omega)=\int_0^{\infty}e^{-t}f(\omega t)\,dt,\ \ \omega\in(-\rho_1,\rho_2). \tag{6}$$

Definition 3. The Sumudu transform of Caputo fractional derivative is defined in following manner [8, 26]

$$S\left[D_t^{\alpha}u(x,t)\right]=\omega^{-\alpha}S[u(x,t)]-\sum_{k=o}^{m-1}\omega^{-\alpha+k}\,u^{(k)}(x,0),\, m-1<\alpha\le m,\ m\in N, \tag{7}$$

where $u^{(k)}(x,0)$ is the k-order derivative of $u(x,t)$ with respect to t at $t=0$.

Definition 4. The Mittag-Leffler function, a generalization of the exponential function, is defined as follows [17, 21]

$$E_{\alpha}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n\alpha+1)},\ \alpha\in C,\ \mathrm{Re}(\alpha)>0. \tag{8}$$

A further generalization of equation (8) is as follows [29]

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)},\ \ \alpha,\beta\in C,\ R(\alpha)>0, R(\beta)>0\,, \tag{9}$$

where $\Gamma(.)$ is the well-known Gamma function.

3. Basic Idea of Sumudu Transform Iterative Method

To demonstrate the fundamental concept of this method [26], we consider the following general fractional partial differential equation with initial conditions of the form

$$D_t^{\alpha}u(x,t)+Ru(x,t)+N\,u(x,t)=g(x,t)\,,\ \ m-1<\alpha\le m,\ \ \ m\in N, \tag{10}$$

$$u^{(k)}(x,0)=h_k(x),\ \ k=0,1,2,...,m-1\,, \tag{11}$$

where $D_t^{\alpha}u(x,t)$ is the Caputo fractional derivative of order $\alpha,\ m-1<\alpha\le m,\ m\in N$, defined by equation (4), R is a linear operator and may include other fractional derivatives of order less than α, N is a non-linear operator which may include other fractional derivatives of order less than α and $g(x,t)$ is a known function.

Applying the Sumudu transform on both sides of equation (11), we have

$$S\left[D_t^{\alpha}u(x,t)\right]+S[Ru\,(x,t)+Nu(x,t)]=S[g(x,t)]. \tag{12}$$

By using the equation (7), we get

$$S[u(x,t)]=\sum_{k=0}^{m-1}\omega^k u^{(k)}(x,0)+\omega^{\alpha}S[g(x,t)]-\omega^{\alpha}S\,[Ru(x,t)+N\,u(x,t)]. \tag{13}$$

On taking inverse Sumudu transform on equation (13), we have

$$u(x,t)=S^{-1}\left[\omega^{\alpha}\left(\sum_{k=0}^{m-1}\omega^{-\alpha+k}u^{(k)}(x,0)+S[g(x,t)]\right)\right]-S^{-1}\left[\omega^{\alpha}S\,[Ru(x,t)+Nu(x,t)]\right]. \tag{14}$$

Furthermore, we employ the iterative method proposed by Daftardar-Gejji and Jafari [7], which represents a solution in an infinite series of components as

$$u(x,t)=\sum_{i=0}^{\infty}u_i(x,t). \tag{15}$$

As R is a linear operator, so we have

$$R\left(\sum_{i=0}^{\infty}u_i(x,t)\right)=\sum_{i=0}^{\infty}R\left[u_i(x,t)\right], \tag{16}$$

and the non-linear operator N is decomposed as follows

$$N\left(\sum_{i=0}^{\infty}u_i(x,t)\right)=N\left[u_0(x,t)\right]+\sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i}u_j(x,t)\right)-N\left(\sum_{j=0}^{i-1}u_j(x,t)\right)\right\}. \tag{17}$$

Substituting the results given by equations from (15) to (17) in the equation (14), we get

$$\begin{aligned}\sum_{i=0}^{\infty}u_i(x,t)&=S^{-1}\left[\omega^{\alpha}\left(\sum_{k=0}^{m-1}\omega^{-\alpha+k}u^{(k)}(x,0)+S[g(x,t)]\right)\right]\\&-S^{-1}\left[\omega^{\alpha}S\left[\sum_{i=0}^{\infty}R\left[u_i(x,t)\right]+N\left[u_0(x,t)\right]+\sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i}u_j(x,t)\right)-N\left(\sum_{j=0}^{i-1}u_j(x,t)\right)\right\}\right]\right].\end{aligned} \tag{18}$$

We have defined the recurrence formulae as

$$\left.\begin{aligned}u_0(x,t)&=S^{-1}\left[\omega^{\alpha}\left(\sum_{k=0}^{m-1}\omega^{-\alpha+k}u^{(k)}(x,0)+S\left(g(x,t)\right)\right)\right]\\u_1(x,t)&=-S^{-1}\left[\omega^{\alpha}S\left[R\left(u_0(x,t)\right)+N\left(u_0(x,t)\right)\right]\right]\\u_{m+1}(x,t)&=-S^{-1}\left[\omega^{\alpha}S\left[R\left(u_m(x,t)\right)-\left\{N\left(\sum_{j=0}^{m}u_j(x,t)\right)-N\left(\sum_{j=0}^{m-1}u_j(x,t)\right)\right\}\right]\right]\ ,m\geq 1\end{aligned}\right\}. \tag{19}$$

Therefore, the approximate analytical solution of equations (10) and (11) in truncated series form is given by

$$u(x,t)\cong\lim_{N\to\infty}\sum_{m=0}^{N}u_m(x,t). \tag{20}$$

In general, the solutions in the above series converge quickly. The classical approach to convergence of this type of series has been presented by Bhalekar and Daftardar-Gejji [5] and Daftardar-Gejji and Jafari [7].

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4. Implementation of the Method

In this section, we implement the above-mentioned reliable approach to handle linear and nonlinear Schrödinger equations with time-fractional derivatives in a realistic and efficient manner.

Example 4.1. We consider the following linear Schrödinger equation [23, 25]

$$D_t^{\alpha}u+iu_{xx}=0,\quad 0<\alpha\leq 1, \tag{21}$$

with the initial condition

$$u(x,0)=\mathrm{e}^{3ix}. \tag{22}$$

Taking the Sumudu transform on the both sides of equation (21), and making use of the result given by equation (22), we have

$$S\left[u\left(x,t\right)\right]=\mathrm{e}^{3ix}-\omega^{\alpha}S\left[i\frac{\partial^2 u}{\partial x^2}\right]. \tag{23}$$

Operating with the inverse Sumudu transform on both sides of equation (23) gives

$$u\left(x,t\right)=\mathrm{e}^{3ix}-S^{-1}\left[\omega^{\alpha}S\left(i\frac{\partial^2 u}{\partial x^2}\right)\right]. \tag{24}$$

Substituting the results from equations (15) to (17) in the equation (24) and applying the equation (19), we determine the components of the solution as follows

$$u_0\left(x,t\right)=u\left(x,0\right)=\mathrm{e}^{3ix}, \tag{25}$$

$$u_1\left(x,t\right)=-S^{-1}\left[\omega^{\alpha}S\left(i\frac{\partial^2 u_0}{\partial x^2}\right)\right]=\frac{(9i)\mathrm{e}^{3ix}\ t^{\alpha}}{\Gamma(\alpha+1)}, \tag{26}$$

$$u_2\left(x,t\right)=-S^{-1}\left[\omega^{\alpha}S\left(i\frac{\partial^2\left(u_0+u_1\right)}{\partial x^2}\right)\right]+S^{-1}\left[\omega^{\alpha}S\left(i\frac{\partial^2 u_0}{\partial x^2}\right)\right]=\frac{(9i)^2\ \mathrm{e}^{3ix}\ t^{2\alpha}}{\Gamma(2\alpha+1)}, \tag{27}$$

$$u_3\left(x,t\right)=-S^{-1}\left[\omega^{\alpha}S\left(i\frac{\partial^2\left(u_0+u_1+u_2\right)}{\partial x^2}\right)\right]+S^{-1}\left[\omega^{\alpha}S\left(i\frac{\partial^2\left(u_0+u_1\right)}{\partial x^2}\right)\right]=\frac{(9i)^3\ \mathrm{e}^{3ix}\ t^{3\alpha}}{\Gamma(3\alpha+1)}, \tag{28}$$

and so on. The other components can be found accordingly.
Therefore, the approximate analytical solution in the series form can be obtained as

$$u(x,t) \cong \lim_{N\to\infty}\sum_{m=0}^{N} u_m(x,t) = u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)+,...,$$

$$u(x,t) = e^{3ix}\left[1+\frac{(9i)\,t^{\alpha}}{\Gamma(\alpha+1)}+\frac{(9i)^2\,t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{(9i)^3\,t^{3\alpha}}{\Gamma(3\alpha+1)}+,...,\right]$$

$$= e^{3ix}\sum_{n=0}^{\infty}\frac{(9it^{\alpha})^n}{\Gamma(n\alpha+1)},$$

thus, the exact solution can be given as

$$u(x,t) = e^{3ix}\,E_{\alpha}\left[(9i)t^{\alpha}\right]. \tag{29}$$

The same result was obtained by Mohyud-Din *et al.*[18] using MVIM, Saba *et al.* [23] using HTAM, Sharma and Bairwa [25] using ILTM and Kamran *et al.*[11] using HPM.
If we put $\alpha = 1$, in equation (29), we have

$$u(x,t) = e^{3i(x+3t)}. \tag{30}$$

Which is the exactly the same solution obtained by earlier Mousa *et al.* [19] using HPM method.

Example 4.2. We consider the following linear Schrödinger equation [23, 25]

$$D_t^{\alpha}u + iu_{xx} = 0,\ \ 0<\alpha\le 1, \tag{31}$$

with the initial condition

$$u(x,0) = 1+2\cosh(2x). \tag{32}$$

Taking the Sumudu transform on the both sides of equation (31), and making use of the result given by equation (32), we have

$$S\left[u(x,t)\right] = 1+2\cosh(2x)-\omega^{\alpha}S\left[i\frac{\partial^2 u}{\partial x^2}\right]. \tag{33}$$

Operating with the inverse Sumudu transform on both sides of equation (33) gives

$$u(x,t) = 1+2\cosh(2x)-S^{-1}\left[\omega^{\alpha}S\left(i\frac{\partial^2 u}{\partial x^2}\right)\right]. \tag{34}$$

Substituting the results from equations (15) to (17) in the equation (34) and applying the equation (19), we determine the components of the solution as follows

$$u_0(x,t) = u(x,0) = 1+2\cosh(2x), \tag{35}$$

$$u_1(x,t) = -S^{-1}\left[\omega^{\alpha}S\left(i\frac{\partial^2 u_0}{\partial x^2}\right)\right] = \frac{(-4i)\left[2\cosh(2x)\right]t^{\alpha}}{\Gamma(\alpha+1)}, \tag{36}$$

$$u_2(x,t) = -S^{-1}\left[\omega^{\alpha}S\left(i\frac{\partial^2(u_0+u_1)}{\partial x^2}\right)\right]+S^{-1}\left[\omega^{\alpha}S\left(i\frac{\partial^2 u_0}{\partial x^2}\right)\right]$$

$$= \frac{(-4i)^2\left[2\cosh(2x)\right]t^{2\alpha}}{\Gamma(2\alpha+1)}, \tag{37}$$

$$u_3(x,t) = -S^{-1}\left[\omega^{\alpha}S\left(i\frac{\partial^2(u_0+u_1+u_2)}{\partial x^2}\right)\right]+S^{-1}\left[\omega^{\alpha}S\left(i\frac{\partial^2(u_0+u_1)}{\partial x^2}\right)\right]$$

$$= \frac{(-4i)^3\left[2\cosh(2x)\right]t^{3\alpha}}{\Gamma(3\alpha+1)}, \tag{38}$$

and so on. The other components can be found accordingly.
Therefore, the approximate analytical solution in the series form can be obtained as

$$u(x,t) \cong \lim_{N\to\infty}\sum_{m=0}^{N} u_m(x,t) = u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)+,...,$$

$$u(x,t)=1+2\cosh(2x)\left[1+\frac{(-4i)\,t^{\alpha}}{\Gamma(\alpha+1)}+\frac{(-4i)^{2}\,t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{(-4i)^{3}\,t^{3\alpha}}{\Gamma(3\alpha+1)}+,...,\right]$$

$$=1+2\cosh(2x)\sum_{n=0}^{\infty}\frac{\left(-4it^{\alpha}\right)^{n}}{\Gamma(n\alpha+1)},$$

thus, the exact solution can be given as

$$u(x,t)=1+2\cosh(2x)E_{\alpha}\left[(-4i)t^{\alpha}\right]. \tag{39}$$

The same result was obtained by Mohyud-Din *et al.*[18] using MVIM,Sharma and Bairwa [25] using ILTM, Saba *et al.* [23] using HTAM and Kamran *et al.*[11] using HPM.
If we put $\alpha=1$, in equation (39), we have

$$u(x,t)=1+2\cosh(2x)\mathrm{e}^{-4it}. \tag{40}$$

Which is the exactly the same solution obtained by earlierMousa*et al.*[19] using HPM method.

Example 4.3. We consider the following nonlinear Schrödinger equation [23, 25]

$$iD_t^{\alpha}u+u_{xx}+2|u|^{2}u=0,\quad 0<\alpha\le 1, \tag{41}$$

with the initial condition

$$u(x,0)=e^{ix}. \tag{42}$$

Taking the Sumudu transform on the both sides of equation (41), and making use of the result given by equation (42), we have

$$S\left[u(x,t)\right]=e^{ix}+i\,\omega^{\alpha}S\left[u_{xx}+2|u|^{2}u\right]. \tag{43}$$

Operating with the inverse Sumudu transform on both sides of equation (43) gives

$$u(x,t)=e^{ix}+S^{-1}\left[i\,\omega^{\alpha}S\left(\frac{\partial^{2}u}{\partial x^{2}}+2|u|^{2}u\right)\right]. \tag{44}$$

Substituting the results from equations (15) to (17) in the equation (44) and applying the equation (19), we determine the components of the solution as follows

$$u_0(x,t)=u(x,0)=e^{ix}, \tag{45}$$

$$u_1(x,t)=S^{-1}\left[i\,\omega^{\alpha}S\left(\frac{\partial^{2}u_0}{\partial x^{2}}+2|u_0|^{2}u_0\right)\right]=\frac{it^{\alpha}e^{ix}}{\Gamma(\alpha+1)}, \tag{46}$$

$$u_2(x,t)=S^{-1}\left[i\,\omega^{\alpha}S\left(\frac{\partial^{2}(u_0+u_1)}{\partial x^{2}}+2\left|(u_0+u_1)\right|^{2}(u_0+u_1)\right)\right]-S^{-1}\left[i\,\omega^{\alpha}S\left(\frac{\partial^{2}u_0}{\partial x^{2}}+2|u_0|^{2}u_0\right)\right]$$

$$=\frac{\left(it^{\alpha}\right)^{2}e^{ix}}{\Gamma(2\alpha+1)}, \tag{47}$$

$$u_3(x,t)=S^{-1}\left[i\,\omega^{\alpha}S\left(\frac{\partial^{2}(u_0+u_1+u_2)}{\partial x^{2}}+2\left|(u_0+u_1+u_2)\right|^{2}(u_0+u_1+u_2)\right)\right]$$

$$-S^{-1}\left[i\,\omega^{\alpha}S\left(\frac{\partial^{2}(u_0+u_1)}{\partial x^{2}}+2\left|(u_0+u_1)\right|^{2}(u_0+u_1)\right)\right]$$

$$=\frac{\left(it^{\alpha}\right)^{3}e^{ix}}{\Gamma(3\alpha+1)}, \tag{48}$$

and so on. The other components can be found accordingly.
Therefore, the approximate analytical solution in the series form can be obtained as

$$u(x,t)\cong\lim_{N\to\infty}\sum_{m=0}^{N}u_m(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)+,...,$$

$$u(x,t)=e^{ix}\left[1+\frac{(it^{\alpha})}{\Gamma(\alpha+1)}+\frac{(it^{\alpha})^{2}}{\Gamma(2\alpha+1)}+\frac{(it^{\alpha})^{3}}{\Gamma(3\alpha+1)}+,...,\right]$$

$$=e^{ix}\sum_{n=0}^{\infty}\frac{(it^{\alpha})^{n}}{\Gamma(n\alpha+1)},$$

thus, the exact solution can be given as

$$u(x,t)=e^{ix}E_{\alpha}\left[it^{\alpha}\right]. \tag{49}$$

The same result was obtained by Odibat *et al.*[20] using GDTM, Saba *et al.* [23] using HTAM, Sharma and Bairwa [25] using ILTM and Kamran *et al.* [11] using HPM.

If we put $\alpha=1$, in equation (49), we have

$$u(x,t)=\mathrm{e}^{i(x+t)}. \tag{50}$$

Which is the exactly the same solution obtained by earlier Mousa *et al.* [19] using HPM method.

Example 4.4. We consider the following nonlinear Schrödinger equation [23, 25]

$$iD_{t}^{\alpha}u+u_{xx}-2|u|^{2}u=0,\quad 0<\alpha\leq 1, \tag{51}$$

with the initial condition

$$u(x,0)=e^{ix}. \tag{52}$$

Taking the Sumudu transform on the both sides of equation (51), and making use of the result given by equation (52), we have

$$S\left[u(x,t)\right]=e^{ix}+i\,\omega^{\alpha}S\left[u_{xx}-2|u|^{2}u\right]. \tag{53}$$

Operating with the inverse Sumudu transform on both sides of equation (53) gives

$$u(x,t)=e^{ix}+S^{-1}\left[i\,\omega^{\alpha}S\left(\frac{\partial^{2}u}{\partial x^{2}}-2|u|^{2}u\right)\right]. \tag{54}$$

Substituting the results from equations (15) to (17) in the equation (54) and applying the equation (19), we determine the components of the solution as follows

$$u_{0}(x,t)=u(x,0)=e^{ix}, \tag{55}$$

$$u_{1}(x,t)=S^{-1}\left[i\,\omega^{\alpha}S\left(\frac{\partial^{2}u_{0}}{\partial x^{2}}-2|u_{0}|^{2}u_{0}\right)\right]=\frac{(-3i)t^{\alpha}e^{ix}}{\Gamma(\alpha+1)}, \tag{56}$$

$$u_{2}(x,t)=S^{-1}\left[i\,\omega^{\alpha}S\left(\frac{\partial^{2}(u_{0}+u_{1})}{\partial x^{2}}-2|(u_{0}+u_{1})|^{2}(u_{0}+u_{1})\right)\right]$$

$$-S^{-1}\left[i\,\omega^{\alpha}S\left(\frac{\partial^{2}u_{0}}{\partial x^{2}}-2|u_{0}|^{2}u_{0}\right)\right]$$

$$=\frac{(-3i)^{2}t^{2\alpha}e^{ix}}{\Gamma(2\alpha+1)}, \tag{57}$$

$$u_{3}(x,t)=S^{-1}\left[i\,\omega^{\alpha}S\left(\frac{\partial^{2}(u_{0}+u_{1}+u_{2})}{\partial x^{2}}-2|(u_{0}+u_{1}+u_{2})|^{2}(u_{0}+u_{1}+u_{2})\right)\right]$$

$$-S^{-1}\left[i\,\omega^{\alpha}S\left(\frac{\partial^{2}(u_{0}+u_{1})}{\partial x^{2}}-2|(u_{0}+u_{1})|^{2}(u_{0}+u_{1})\right)\right]$$

$$=\frac{(-3i)^{3}t^{3\alpha}e^{ix}}{\Gamma(3\alpha+1)}, \tag{58}$$

and so on. The other components can be found accordingly.

Therefore, the approximate analytical solution in the series form can be obtained as

$$u(x,t) \cong \lim_{N\to\infty}\sum_{m=0}^{N} u_m(x,t) = u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)+,...,$$

$$u(x,t)=e^{ix}\left[1+\frac{\left(-3it^{\alpha}\right)}{\Gamma(\alpha+1)}+\frac{\left(-3it^{\alpha}\right)^2}{\Gamma(2\alpha+1)}+\frac{\left(-3it^{\alpha}\right)^3}{\Gamma(3\alpha+1)}+,...,\right]$$

$$=e^{ix}\sum_{n=0}^{\infty}\frac{\left(-3it^{\alpha}\right)^n}{\Gamma(n\alpha+1)},$$

thus, the exact solution can be given as

$$u(x,t)=e^{ix}E_{\alpha}\left[-3it^{\alpha}\right]. \tag{59}$$

The same result was obtained by Sharma and Bairwa [25] using ILTM, Mohyud-Din *et.al*. [18] using MVIM , Saba *et al.*[23] using HTAM and Kamran *et al.*[11] using HPM.

If we put $\alpha=1$, in equation (59), we have

$$u(x,t)=\mathrm{e}^{i(x-3t)}. \tag{60}$$

Which is the exactly the same solution obtained by earlier Mousa *et al.* [19] using HPM method.

Example 4.5. We consider the following nonlinear Schrödinger equation [23, 25]

$$iD_t^{\alpha}u+u_{xx}+2|u|^{2r}u=0,\ 0<\alpha\le 1, \tag{61}$$

with the initial condition

$$u(x,0)=\left(2(r+1)\sec h^2(2rx)\right)^{\frac{1}{2r}}\ ,\ r\ge 1. \tag{62}$$

Taking the Sumudu transform on the both sides of equation (62), and making use of the result given by equation (63), we have

$$S\left[u(x,t)\right]=\left(2(r+1)\sec h^2(2rx)\right)^{\frac{1}{2r}}+i\,\omega^{\alpha}S\left[u_{xx}+2|u|^{2r}u\right]. \tag{63}$$

Operating with the inverse Sumudu transform on both sides of equation (63) gives

$$u(x,t)=\left(2(r+1)\sec h^2(2rx)\right)^{\frac{1}{2r}}+S^{-1}\left[i\,\omega^{\alpha}S\left(\frac{\partial^2 u}{\partial x^2}+2|u|^{2r}u\right)\right]. \tag{64}$$

Substituting the results from equations (15) to (17) in the equation (64) and applying the equation (19), we determine the components of the solution as follows

$$u_0(x,t)=u(x,0)=\left(2(r+1)\sec h^2(2rx)\right)^{\frac{1}{2r}}, \tag{65}$$

$$u_1(x,t)=S^{-1}\left[i\,\omega^{\alpha}S\left(\frac{\partial^2 u_0}{\partial x^2}+2|u_0|^{2r}u_0\right)\right]$$

$$=\frac{(4i)t^{\alpha}\left(2(r+1)\sec h^2(2rx)\right)^{\frac{1}{2r}}}{\Gamma(\alpha+1)}, \tag{66}$$

$$u_2(x,t)=S^{-1}\left[i\,\omega^{\alpha}S\left(\frac{\partial^2(u_0+u_1)}{\partial x^2}+2|(u_0+u_1)|^{2r}(u_0+u_1)\right)\right]$$

$$-S^{-1}\left[i\,\omega^{\alpha}S\left(\frac{\partial^2 u_0}{\partial x^2}-2|u_0|^{2r}u_0\right)\right]$$

$$= \frac{(4i)^2 t^{2\alpha} \left(2(r+1)\operatorname{sec}h^2(2rx)\right)^{\frac{1}{2r}}}{\Gamma(2\alpha+1)}, \tag{67}$$

$$u_3(x,t) = S^{-1}\left[i\,\omega^\alpha S\left(\frac{\partial^2 (u_0+u_1+u_2)}{\partial x^2} + 2\left|(u_0+u_1+u_2)\right|^{2r}(u_0+u_1+u_2)\right)\right]$$

$$-S^{-1}\left[i\,\omega^\alpha S\left(\frac{\partial^2 (u_0+u_1)}{\partial x^2} + 2\left|(u_0+u_1)\right|^{2r}(u_0+u_1)\right)\right]$$

$$= \frac{(4i)^3 t^{3\alpha} \left(2(r+1)\operatorname{sec}h^2(2rx)\right)^{\frac{1}{2r}}}{\Gamma(3\alpha+1)}, \tag{68}$$

and so on. The other components can be found accordingly.
Therefore, the approximate analytical solution in the series form can be obtained as

$$u(x,t) \cong \lim_{N\to\infty} \sum_{m=0}^{N} u_m(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) +, ...,$$

$$u(x,t) = \left(2(r+1)\operatorname{sec}h^2(2rx)\right)^{\frac{1}{2r}} \left[1 + \frac{(4it^\alpha)}{\Gamma(\alpha+1)} + \frac{(4it^\alpha)^2}{\Gamma(2\alpha+1)} +, ..., \right]$$

$$= \left(2(r+1)\operatorname{sec}h^2(2rx)\right)^{\frac{1}{2r}} \sum_{n=0}^{\infty} \frac{(4it^\alpha)^n}{\Gamma(n\alpha+1)},$$

thus, the exact solution can be given as

$$u(x,t) = \left(2(r+1)\operatorname{sec}h^2(2rx)\right)^{\frac{1}{2r}} E_\alpha\left[4it^\alpha\right]. \tag{69}$$

The same result was obtained by Sharma and Bairwa [25] using ILTM and Saba *et al.* [23] using HTAM.
If we put $\alpha = 1,\ r = 1,$ in equation (69) we have

$$u(x,t) = 2\operatorname{sec}h(2x)e^{4it}. \tag{70}$$

Which is the exactly the same solution obtained by earlier Mousa *et al.*[19] using HPM method.

5. Conclusion

In this article, we have derived analytical solutions to the time-fractional Schrödinger equations under initial conditions utilizing a newly developed analytical approach termed as the Sumudu transform iterative method. It is observed that the proposed technique is completely compatible with the versatility of these types of problems, and the obtained results are extremely accurate.

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