Existence and uniqueness of mild solutions forimpulsive semilinear differential equations in a Banach space

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Abstract: This paper is devoted to study the uniqueness and existence of mild solutions for impulsive semilinear differential equations, we assume that the linear part generates a strongly continuous semigroup on a general Banach space in a Banachspace. The arguments are based upon Banach's, Mönch's and Darbo'sfixed point theorems and the technique of measure of noncompactness. Finally, an illustrative example is given to illustrate the results.

Keywords: Impulsive Differential Equations, semilinear, fixed point, existence and uniqueness, semigroup of linear operators, measure of noncompactness.

1. Introduction

Dietary Differential equations with impulses were considered for the first time by Milman and Myshkis [16] and then followed by a period of active research which culminated with the monograph by Halanay and Wexler[13]. Many phenomena and evolution processes in physics, chemical technology, population dynamics, and natural sciences may change state abruptly or be subject to short-term perturbations. These perturbations may be seen as impulses. Impulsive problems arise also in various applications in communications, mechanics (jump discontinuities in velocity), electrical engineering, medicine and biology. A comprehensive introduction to the basic theory is well developed in the monographs by Benchohra et al. [5], Graef et al. [11], Laskshmikantham et al. [3], and Samoilenko and Perestyuk [18].

In this paper, we consider the impulsive semilinear differential equations:

$$y'(t) = Ay(t) + f(t, y), t \in J := [0, T], t \neq t_k, k = 1, \dots, m \qquad (1)$$

$$y(t_k^+) - y(t_k^-) = I_k(y(t_k)), k = 1, \dots, m \qquad (2)$$

$$y(0) = y_0. \qquad (3)$$

Where $f: J \times E \to E$ is a given function verifying certain hypotheses which will be specified later, A is the infinitesimal generator of a C_0-semi group $\{T(t)\}_{t\geq 0}$, $I_k: E \to E, k = 1, \dots, m$ and $y_0 \in E, 0 = t_0 < t_1 < \cdots < t_m$, $\Delta y|t =_{t_k} = y(t_k^+) - y(t_k^-), y(t_k^+) = \lim_{h\to 0^+} y(t_k + h), y(t_k^-) = \lim_{h\to 0^+} y(t_k - h)$ represent the left and right limits of $y(t)$ at $t = t_k$ and E is a Banach space with the norm $\|.\|$.

Denote the solution of the problem (1)-(3) by y,

if

$$PC := \{y: J \to E: y \in AC((t_k, t_{k+1}), E), k = 1, \dots, m\}.$$

Evidently, PC(J, E) is a Banach space with norm

$$\|.\|_{PC} := \sup\{\|y(t)\|: t \in [0, T]\}.$$

This paper is organized as follows: In Section 2, we will recall briefly some basic definitions, some fixed point theorems and preliminary facts which will be needed in the following sections.

In Section 3, we give one of our main existence results for solutions of (1)-(3), with the proof based on Banach's fixed point theorem. In Section 4, we give two other existence results for solutions of (1)-(3). Their proofs involve the measure of noncompactness paired in one resultwith a Mönch fixed point theoremandpaired in the other result with a Darbo fixed point theorm. In Section 5, we present an illustrative and comparative example.

2. Preliminaries

In what follows we introduce definitions, notations, and preliminary facts which are used in the sequel. For more details, we refer to [20].

Definition 2.1[20]We say that $f: J \times E \to E$ is a Caratheodory function if

(a) $t \to f(t,u)$ is measurable for each $u \in E$,

(b) $u \to f(t,u)$ is continuous for almost by all $t \in J$.

Moreover, if

(c) For each $r > 0$, there exist a function $\Phi_r \in L^1(J, \mathbb{R}^+)$ such that

$$\|f(t,u)\| \leq \Phi_r(t)$$

for all $\|f(t,u)\| \leq r$; and for almost all $t \in J$.

Then the application f is said to be L^1-Caratheodory.

2.1. Semigroup of Linear Operator

Definition 2.2[17] A one-parameter family S(t) of bounded linear operators on aBanach space E is aC_0-semigroup (or strongly continuous) on E if

(i) $S(t)\circ S(s) = S(t + s)$; for t; $s \geq 0$; (semigroup property),

(ii) S(0) = I, (the identity on E),

(iii) the map $t \to S(t)x$ is strongly continuous for each $x \in E$, i.e:

$$\lim_{t\to 0} S(x) = x, \forall x \in E.$$

A semigroup of bounded linear operators S(t), is uniformly continuous if

$$\lim_{t\to 0}\|S(t) - I\| = 0.$$

Here I denotes the identity operators in E.

Theorem 2.3[17]If S(t) is a C_0-semigroup, then there exist $\omega \geq 0$ and $M \geq 1$ such that

$\|S(t)\|_{B(E)} \leq M\exp(\omega t)$ for $0 \leq t \leq \infty$.

Definition 2.4[17] Let S(t) be a semigroup of class C_0 defined on E. The infinitesimal generator A of S(t) is the linear operator defined by

$$A(x) = \lim_{t\to 0}\frac{S(h)x - x}{h}, \text{for } x \in D(A),$$

where$D(A) = \left\{x \in D(A)/\lim_{t\to 0}\frac{S(h)x-x}{h} \text{ exists in } E\right\}$.

Proposition 2.5[17]The infinitesimal generator A is a closed, linear and densely defined operator in E. If $x \in D(A)$, then S(t)(x) is a C^1-map and

$$\frac{d}{dt}S(t)x = A(S(t)x) = S(t)\big(A(x)\big) \text{ on } [0,\infty).$$

Now let us recall some fundamental facts of the notion of Kuratowski measure of noncompactness.

2.2. The measure of non-compactness in the sense of Kuratowski

Definition 2.6([2]) Let E be a Banach space and Ω_E the family of bounded subsets of E. The Kuratowski measure of noncompactness is the map $\gamma: \Omega_E \to \mathbb{R}^+$defined by

$$\gamma(B) = \inf\left\{\varepsilon > 0 : B \subseteq \bigcup_{i=1}^{n} B_i \text{ and diam } (B_i) \leq \varepsilon\right\}.$$

This measure of noncompactness satisfies some important properties ([2]):

(a) $\gamma(B) = 0 \Leftrightarrow \overline{B}$ is compact (B is relatively compact).

(b) $\gamma(B) = \gamma(\overline{B})$.

(c) $A \subseteq B \Rightarrow \gamma(A) \leq \gamma(B)$.

(d) $\gamma(A + B) \leq \gamma(A) + \gamma(B)$.

(e) $\gamma(cA) = |c|\gamma(A); c \in \mathbb{R}$.

(f) $\gamma(\text{conv}A) = \gamma(A)$.

Here $\overline{B}$ and convB denote the closure and the convex hull of the bounded set B, respectively. The details of γ and its properties can be found in **([2])**.

Then we have this following result:

Theorem 2.7[14] Let $C \subset L^1(J, E)$ be a countable set with $\|u(t)\| \leq h(t)$ for p.p. $t \in J$ and all $u \in C$; where $h \in L^1(J)$.then the function $\phi: t \to \gamma(C(t))$ belongs to$L^1(J)$ and satisfy.

$$\gamma\left(\left\{\int_0^T u(s)ds : u \in C\right\}\right) \leq 2\int_0^T \gamma\big(C(s)\big)ds.$$

Let us now recall Mönch's and Darbo's fixed point theorems.

Theorem 2.8 (Mönch's Fixed Point Theorem [1]) Let D be a bounded, closed and convex subset of a Banach space such that 0 ∈ D, and let N be a continuous mapping of D into itself. If the implication

$$V = \overline{\text{Conv}}\, N(V) \text{or } V = N(V) \cup \{0\} \Rightarrow \gamma(V) = 0$$

holds for every subset V of D, then N has a fixed point.

Theorem 2.9 (Darbo's Fixed Point Theorem [8]) Let X be a Banach space and C be a bounded, closed, convex and nonempty subset of X. Suppose a continuous mapping $N: C \to C$ is such that for all closed subsets D of C,

$$\gamma\big(N(D)\big) \leq k\gamma(D), \qquad (4)$$

where $0 \leq k < 1$; and γ is the Kuratowski measure of noncompactness. Then N has a fixed point in C.

Remark 2.10 Mappings satisfying the Darbo-condition (4) have subsequently been called k-set contractions.

3. Uniqueness of mild solutions

This section is devoted to the existence results for problem (1)-(3). First, we done a lemma and define what we express to be a mild solution of problem (1)-(3).

Definition 3.1A function y ∈PC(J; E) is said to be a mild solution of the problem (1)-(3) if y is the solution of the impulsive integral equation

$$y(t) = T(t)y_0 + \int_0^t T(t-s)\, f(s,y)ds + \sum_{0<t_k<t} T(t-t_k)\, I_k\big(y(t_k)\big), t\ \in\ J.$$

A is the generator of a strongly continuous semigroup$S(t), t \in J$ in the Banach space E. There exists a constant M > 0 such that

$$\|S(t)\| \le M \text{ for all } t \in J.$$

Let us introduce the following hypotheses:

(A1) There exists a constant $d \ge 0$ such that

$$\|f(t,y) - f(t,\bar{y})\| \le d\ \|y - \bar{y}\|, \text{for each } t \in J, \forall y, \bar{y}\ \in E.$$

(A2) There exist constants $c_k \ge 0$ such that

$$\|I_k(y) - I_k(\bar{y})\| \le c_k\|y - \bar{y}\|, \text{for each } k = 1, \dots, m, \forall y, \bar{y}\ \in E.$$

Theorem 3.2 Assume that assumptions (A_1) and (A_2) hold, if $\theta < 1$; where

$\theta = MdT + M\sum_{k=1}^{m} c_k$,

then, the problem (1)-(3) has a unique solution on J.

Proof: Transform the problem (1)-(3) into a fixed point problem. Consider the operator $N: PC(J,E) \to PC(J,E)$ defined by

$$N(y)(t) = T(t)y_0 + \int_0^t T(t-s)\, f(s,y)ds + \sum_{0<t_k<t} T(t-t_k)\, I_k\big(y(t_k)\big).$$

Note that a fixed point of *N* is a mild solution of the problem (1)-(3). We will show that N is a contraction. Indeed, considery, $\bar{y} \in PC(J,E)$. Thus, for $t \in J$, we have:

$\|N(y)(t) - N(\bar{y})(t)\| \le \int_0^t \|T(t-s))\|\ \big\|f\big(t,y(s)\big) - f(t,\bar{y}(s))\big\|ds$

$+\sum_{0<t_k<t} \|T(t-t_k)\|\|I_k y(t_k) - I_k(\bar{y}(t_k))\|$

$\le Md\int_0^t \|T(t-s))\|\ \|y(s) - \bar{y}(s)\|ds + M\sum_{0<t_k<t} c_k\ \|y(t_k) - \bar{y}(t_k)\|$

$\le MdT\|y - \bar{y}\|_{PC} + M\sum_{k=1}^{m} c_k\ \|y - \bar{y}\|_{PC}$

$$\le \left(MdT + M\sum_{k=1}^{m} c_k\right)\|y - \bar{y}\|_{PC}.$$

Consequently,

$$\|N(y) - N(\bar{y})\|_{PC} \le (MdT + M\sum_{k=1}^{m} c_k)\ \|y - \bar{y}\|_{PC}$$

$$\le \theta\|y - \bar{y}\|_{PC}.$$

Since $\theta < 1$, N is a contraction. By the Banach fixed point theorem [9] we concludethat N has a unique fixed point in $PC(J,E)$ and the problem (1)-(3) has a uniquesolution on$[0,T]$.

4. Existence of mild solutions

In this section we apply a technique based on noncompactness measure assumption.In the following,we prove existence results, for the problem (1)-(3) by using a Mönch'sand Darbo's fixed point theorems.

(H1) A is the generator of a strongly continuous semigroup $T(t), t \in J$ which is compact for t > 0 in the Banach space E. Let M > 0 be such that

$\|S(t)\| \leq M \, for \, all \, t \in J.$

(H2) $f: J \times E \to E$ satisfies the Caratheodory conditions.

(H3) There exists $p \in C(J, \mathbb{R}^+)$, such that

$$\|f(t,y)\| < p(t)\|y)\|, for \, all \, t \in J \text{and each } y \in E.$$

(H4) There exists c > 0, such that

$$\|I_k(y)\| \leq c\|y)\|, for \text{ each } y \in E.$$

(H5) For each bounded set $B \subset E$, we have

$$\gamma\big(I_k(y)\big) < c\,\gamma(B), k = 1, \dots, m.$$

(H6) For each $t \in J$ and each bounded set $B \subset E$, we have

$$\gamma(f(t,B)) \leq p(t)\gamma(B).$$

Theorem 4.1 Assume that conditions $(H1) - (H6)$ hold. Let $p^* = \sup_{t \in J} p(t)$. If

$$Mp^* + mMc < 1, \qquad (5)$$

then the problem (1)-(3) has at least one mild solution on J.

Proof: Let

$$r \geq \frac{M\|y_0)\|}{1 - Mp^* - mMc} \qquad (6)$$

andconsider

$D_r = \{y \in PC(J,E): \|y)\|_{PC} \leq r\}.$

Clearly, the subset D_r is closed, bounded and convex. We shall show that N defined in Theorem 3.2 satisfies the assumptions of Mönch's fixed point theorem. The proof will be given in three steps.

Step l: N is continuous.

Let $\{y_n\}$ a sequence such that $y_n \to y$ in PC(J,E). Then for each $t \in J$

$\|N(y_n)(t) - N(y)(t)\| \leq \int_0^t T(t-s) \left\|f\big(s, y_n(s)\big) - f(s, y(s))\right\| ds$

$$+\sum_{0<t_k<t} T(t-t_k) \left\|I_k\big(y_n(t_k)\big) - I_k(y(t_k))\right\|.$$

Since I_k is continuous and f is of the Caratheodory type, then by Lebesgue's dominated convergence theorem, we have

$$\|N(y_n) - N(y)\|_{PC} \to 0 \text{ as } n \to \infty.$$

Step 2: N is an application of D_r into itself.

For each y in Dr by (H3) and (6), for each t in J we have:

$$\|N(y)(t)\| \leq \|y_0\|\|T(t)\| + \int_0^t \|T(t-s)\| \, \|f(s,y(s))\| ds$$

$$+\sum_{0<t_k<t} \|T(t-t_k)\| \, \|I_k(y(t_k))\|$$

$\leq M\|y_0\| + M\int_0^t p(t) \, \|y\| ds + \sum_{0<t_k<t} Mc \, \|y\|$

$\leq M\|y_0\| + Mp^*\|y\| + mMc\|y\|$

$\leq r.$

Step 3: $N(D_r)$ is bounded and equicontinuous.

By step 2, it is obvious that $N(D_r) \subset PC(J;E)$ is bounded.

For the equicontinuity of $N(D_r)$. Either τ_1, τ_2 in $J, \tau_1 < \tau_2$ and either y in D_r

$$\|Ny(\tau_2) - Ny(\tau_1)\| \leq \|T(\tau_2)y_0 - T(\tau_1)y_0\|$$

$$+\left\|\int_0^t T(\tau_2 - s)f(s,y)ds - \int_0^t T(\tau_1 - s)f(s,y)ds\right\|$$

$$+\left\|\sum_{0<t_k<t} T(\tau_2 - t_k)I_k\big(y(t_k)\big) - \sum_{0<t_k<t} T(\tau_2 - t_k)I_k\big(y(t_k)\big)\right\|$$

$$\leq \| [T(\tau_2) - T(\tau_1)y_0] \| + \| \int_0^t [T(\tau_2 - s) - T(\tau_1 - s)]f(s,y)ds \|$$

$$+\| \int_{\tau_1}^{\tau_2} T(\tau_2 - s)f(s,y)ds \|$$

$$+\| \sum_{0<t_k<\tau_1} [T(\tau_2 - t_k) - T(\tau_1 - t_k)]\, I_k(y(t_k) \|$$

$$+\| \sum_{\tau_1<t_k<\tau_2} T(\tau_2 - t_k)\, I_k\big(y(t_k)\big) \| \qquad (*)$$

If $\tau_1 = 0$, then the right side of $(*)$ can be made small when τ_2 is small.

If $\tau_1 > 0$ then we can find a small $\varepsilon > 0$with $\tau_1 - \varepsilon > 0$, then it follows

$$\| [Ny(\tau_2) - Ny(\tau_1) \| \leq \| [T(\tau_2) - T(\tau_1)]y_0 \|$$

$$+\| \int_0^{\tau_1-\varepsilon} [T(\tau_2 - s) - T(\tau_1 - s)]\, f(s,y)ds \|$$

$$+\| \int_{\tau_1}^{\tau_2} T(\tau_2 - s)f(s,y)ds \|$$

$$+\| \sum_{0<t_k<\tau_1} [T(\tau_2 - t_k) - T(\tau_1 - t_k)]I_k(y(t_k) \|$$

$$+\| \sum_{\tau_1<t_k<\tau_2} T(\tau_2 - t_k) I_k I_k(y(t_k) \|$$

$$\leq \| [T(\tau_2) - T(\tau_1)]y_0 \|$$

$$+ \int_0^{\tau_1-\varepsilon} \| T(\tau_2 - s) - T(\tau_1 - s) \|\| f(s,y) \| ds$$

$$+\int_{\tau_1-\varepsilon}^{\tau_1} \| T(\tau_2 - s) - T(\tau_1 - s \|\| f(s,y) \| ds$$

$$+\int_{\tau_1}^{\tau_2} \| T(\tau_2 - s) \|\| f(s,y) \| ds$$

$$+\sum_{0<t_k<\tau_1} \| [T(\tau_2 - t_k) - T(\tau_1 - t_k)] \|\| I_k(y(t_k)) \|$$

$$+M \sum_{\tau_1<t_k<\tau_2} \| I_k(y(t_k)) \|.$$

So when for $\tau_1 \to \tau_2$ the right side of the above inequality tends to zero, that is

to say

$$\| Ny(\tau_2) - Ny(\tau_1) \| \to 0$$

Now, let V be a subset of D_r such that $V \in \overline{conv}(N(V) \cup (0))$.

V is bounded and equicontinuous and therefore the function $v \to v(t) = \gamma(V(t))$is continuous over J. From $(H_5), (H_6)$, and the properties of the measure we have for each $t \in J$:

$$v(t) \leq \gamma(N(V) \cup \{0\})$$

$$\leq (N(V))$$

$$\leq \int_0^t Mp(s)\, \gamma(V(s))ds + \sum_{0<t_k<t} M\, \gamma(V(s))$$

$$\leq \int_0^t Mp(s)\gamma\big(V(s)\big)ds + \sum_{0<t_k<t} Mcv(s)$$

$\leq \| v \|_{PC} \ (Mp^* + mMc)$.

This means that

$$\| v \|_{PC} \ (1 - [Mp^* + mMc]) \leq 0$$

By (5), it follows that $\| v \|_{PC} = 0$, that is v(t) = 0 for each t ∈ J, and then V (t) is relatively compact in PC(J,E). In view of the Ascoli-Arzela theorem, V is relatively compact in D_r. Applying now Theorem 2.8, we conclude that N has a fixed point which is a solution of the problem (1) - (3).

For the next theorem we replace the condition (5) by

$$M(1 + mc) < 1. \qquad (7)$$

Now, consider the Kuratowski measure of noncompactnessγ_C defined on the family of bounded subsets of the space PC(J,E) by

$$\gamma_C(H) = \sup_{s\in J} e^{-\tau L(s)} \gamma\big(H(s)\big),$$

Where $L(t) = \int_0^t \bar{l}(s)ds,\ \ \bar{l}(t) = Mp(t),\ \ \tau > \frac{1}{1-M(1+mc)}$

Our next result is based on the Darbo fixed point theorem 2.9.

Theorem 4.2 Assume that conditions $(H_1) - (H_6)$ and (7) are satisfied. Then the problem (1)-(3) has at least one mild solution on J.

Proof: As in Theorem 4.1, we can prove that the operator $N: B_q \rightarrow B_q$ defined in that theorem is continuous and $N(B_q)$ is bounded.

Now we show that the operator $N: B_q \rightarrow B_q$ is a strict set contraction, i.e., there is a constant $0 \leq \lambda < 1$ such that $\gamma(N(H)) \leq \lambda\gamma(H)$ for any $H \subset B_q$. In particular, we need to prove that there exists a constant $0 \leq \lambda < 1$ such that $\gamma_C(N(H)) \leq \lambda\gamma_C(H)$ for any $H \subset B_q$. For each t ∈ J we have

$$\gamma(N(H)(t)) \leq M\gamma(H(0)) + M\int_0^t p(s)\,\gamma(H(s))ds$$

$$+\textstyle\sum_{0<t_k<t} Mc\gamma(H(t_k))$$

$$\leq M\gamma\big(H(0)\big) + M\int_0^t p(s)e^{\tau L(s)}\, e^{-\tau L(s)}\gamma(H(s))ds$$

$$+\textstyle\sum_{0<t_k<t} Mc\gamma(H(t_k))$$

$$\leq\ M\gamma\big(H(0)\big) + M \sup_{s\in J} e^{-\tau L(s)}\,\gamma(H(s))\ \int_0^t p(s)\, e^{\tau L(s)} ds$$

$$+\textstyle\sum_{0<t_k<t} Mc\gamma(H(t_k))$$

$$\leq M\gamma\big(H(0)\big) + M\gamma_C(H)\int_0^t p(s)e^{\tau L(s)}\, ds$$

$$+\textstyle\sum_{0<t_k<t} Mc\gamma(H(t_k))$$

$$\leq M\gamma\big(H(0)\big) + \gamma_C(H)\int_0^t \bar{l}(s)e^{\tau L(s)}\, ds$$

$$+\textstyle\sum_{0<t_k<t} Mc\gamma(H(t_k))$$

$$\leq M\gamma\big(H(0)\big) + \gamma_C(H)\int_0^t \left(\frac{e^{\tau L(s)}}{\tau}\right)' ds$$

$$+ \sum_{0<t_k<t} Mc\gamma(H(t_k))$$

$$\leq M\gamma\big(H(0)\big) + \gamma_C(H)\frac{1}{\tau}e^{\tau L(t)} + \sum_{0<t_k<t} Mc\gamma\big(H(t_k)\big).$$

Then

$$e^{-\tau L(t)}\gamma(N(H)(t)) \leq Me^{-\tau L(t)}\gamma(H)(0) + \gamma_C(H)\tfrac{1}{\tau} + \textstyle\sum_{0<t_k<t} Mc\, e^{-\tau L(t)}\,\gamma(H(t_k))$$

$$\leq M\sup_{s\in J} e^{-\tau L(t)}\,\gamma(H)(0) + \gamma_C(H)\frac{1}{\tau} + \sum_{0<t_k<t} M\, c\sup_{s\in J} e^{-\tau L(t)}\ \gamma\big(H(t_k)\big).$$

Consequently,

$$\gamma_C(NH) \leq \left[M + \tfrac{1}{\tau} + \textstyle\sum_{k=1}^{m} Mc\right]\gamma_C(H)$$

$$\leq \left[\tfrac{1}{\tau} + M(1+mc)\right]\gamma_C(H).$$

So, the operator N is a set contraction. By the Darbo fixed point Theorem 2.9 we deduce that N has a fixed point which is a mild solution of problem (1)-(2).

5. Example

Let E be a Banach space, $J \coloneqq [0,2\pi], \lambda_i$, i=0,1 and c_k, k=1,...,m are the constants such that $\lambda_0 > 0, 0 < \lambda_1 < \frac{1}{4\pi}$ and $0 < \sum_{k=1}^{m} c_k < \frac{1}{2}$.

Consider the following impulsive differential equation:

$$y'(t) = \lambda_0^2 y(t) + \lambda_1 y(t), t \in J, t \neq t_k, k = 1, \dots, m \quad (8)$$

$$y(t_k^+) - y(t_k^-) = c_k\big(y(t_k)\big), k = 1, \dots, m \quad (9)$$

$$y(0) = y_0. \quad (10)$$

Here

$f(t, y(t))=\lambda_1 y(t)$,

$$I_k\big(y(t_k)\big) = c_k y(t_k), k = 1, \dots, m$$

and

$$A = \lambda_0^2.$$

Firstly, we show that f and I_k are Lipschitz functions. Indeed, let $x, y \in E$, then

$$\| f(t,x) - f(t,y) \| = \| \lambda_1 x - \lambda_1 y \|$$
$$\leq \lambda_1 \| x - y \|$$

and

$$\| I_k(x) - I_k(y) \| = \| c_k x - c_k y \|$$
$$\leq c_k \| x - y \|.$$

We take $T(t) = e^{-\lambda_0^2 t}$, $\| T(t) = e^{-\lambda_0^2 t} \| \leq \| e^0 \|$=M for all $t \in J$.
We have

$$\theta = 2\,\pi M\lambda_1 + M\sum_{k=1}^{m} c_k < \frac{2\pi}{4\pi} + \frac{1}{2} = 1.$$

It is clear that $\theta < 1$, therefore, all the conditions of Theorem 3.2 are satisfied. Hence the problem (8)-(10) has a unique mild solution.

Conclusion

In this paper, we have considered an impulsivesemilinear differential equations. The existence of a solutions have been investigated using the measures of non-compactness, semigroup, Banach's, Mönch'sand Darbo's theorems.

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