GENERALIZED THEORY OF THERMOELASTICITY IN ISOTROPIC AND HOMOGENIOUS THERMOELASTIC SOLIDS

M.B.Bera [1], M.K.Mondal [2], B.S.Mahapatra [3], G.Roymahapatra[4], P.P.Acharjya[5]

[1](School of Applied Sciences & Humanities, Haldia Institute of Technology, India, mihirmath2005@gmail.com)
[2](School of Applied Sciences & Humanities, Haldia Institute of Technology, India, manojmondol@gmail.com)
[3](School of Applied Sciences & Humanities, Haldia Institute of Technology, India, biplab.22@gmail.com)
[4](School of Applied Sciences & Humanities, Haldia Institute of Technology, India, grm.chem@gmail.com)
[5](Department of Computer Science and Engineering, Haldia Institute of Technology, India, ppacharjya@gmail.com)

ABSTRACT
The present paper deals with the recent development of the theory of thermoelasticity. The change of basic equations of thermoelasticity, for different type of thermoelastic parameters, under different condition has been shown in this paper to achieve Lord and Shulman theory and Green and Naghdi theories of generalized thermoelasticity. Relevant literatures on thermoelasticity are also re-examined.
Keywords: Conservation of Internal Energy, Homogeneous Medium, Isotropic Medium, Rayleigh Waves, Thermoelasticity.

INTRODUCTION

The subject of study in this paper is thermoelasticity. Thermoelasticity deals with the interaction of temperature in an elastic solid.

All natural materials have elastic properties, in more or less measure. The distortion is sometimes so small that they require very sensitive instruments for their detections. The theory of elasticity is a branch of solid mechanics which deals with the methods of computation of stresses and strains in deformable solids produced by external forces and / or changes in temperature.

Due to the deformation of the body, the body temperature change is observed. Also if we impose some heat source from outside then we observed the change in body temperature and the deformation of the body. So the internal energy of the body becomes a function of deformation and temperature. In thermoelasticity we deal with these coupled processes, to find the expression of stress, stain and deformation in terms of heat source and the time. It has been seen that, this theory has great impact in several areas of engineering and technology like acoustic, aeronautics, chemical and nuclear engineering and in the analysis of stresses and displacements of structural or machine elements within the elastic limit and thereby to check the sufficiency of the strength, stiffness and stability.

The famous Hooke's [1] law of proportionality of stress and strain, forms the basis equation of the mathematical theory of elasticity, known as classical elasticity theory. Later on the general equations of equilibrium and vibration of elastic solids was proposed by Navier [2]. Depending on the Navier work, Cauchy [3] formulate linear theory of elasticity which remains virtually unchanged to the present day. Several researchers made significant contributions towards the development of this theory.

The classical theory which deals with the coupling between the strain and temperature fields was first studied by Duhamel [4] who derived the governing equations for the distribution of strain in an elastic medium subjected to temperature gradients. Neumann [5] and several others worked on Duhamel theory and solved a number of interesting problems. These authors postulated that the heat conduction equation should contain a term representing the time – rate of change of dilatation and that the theory should be based on coupled system of momentum and energy equations.

Biot [6] was the first to give a satisfactory derivation of the linear theory of coupled thermoelasticity. Later, Chadwick [7] presented both linear and non – linear versions of the theory and solved a few important linear dynamical problems. He showed that, in dynamical problems, the interactions between the thermal and strain fields are such that the two effects cannot be treated separately. A proof of uniqueness of solution of coupled equations of thermoelasticity was given by Weiner [8]. Analysis of wave propagation, including Rayleigh waves in thermo-elastic bodies have been given by Chadwick and Sneddon [9]

Extensions of coupled thermoelasticity have been made to cover more general type of solids like micro-elastic solids, electromagnetic solids, visco-elastic solids etc. These extensions are available in the work of Paria [10], Eringen [11], Perkus [12], Nowacki [13] and Dhaliwal and Singh [14].

In this paper we are illustrating the different type of mathematical models and their corresponding equations in the case of an isotropic and homogeneous thermo-elastic solid and then try to find out a generalized model to represents all types of models in a generalized form.

THE BASIC EQUATIONS OF GENERALIZED THERMOELASTICITY

The field equations of linear thermoelasticity for different models for isotropic and homogeneous thermo-elastic solid are as follows:

1.1 Classical Thermoelasticity (CTE) :

Constitutive relations :

$$\tau_{ij} = \lambda \Delta \ \delta_{ij} + 2\mu e_{ij} - \beta T \ \delta_{ij} \ ; \quad (i,j = 1,2,3) \tag{1}$$

where λ, μ are Lame constants, $\beta = (3\lambda + 2\mu)\alpha_t$, α_t is the coefficient of the linear thermal expansion of the material, τ_{ij} is the stress tensor, T is the increase in temperature above the reference temperature T_0 and $\Delta = u_{i,i}$ (dilatation).

Strain – Displacement relations :

$$e_{ij} = (u_{i,j} + u_{j,i})/2 \tag{2}$$

Classical Fourier law :

Equations (1) and (2) are to be supplemented by Classical Fourier law connecting the heat flux vector $\vec{q}$ with the temperature gradient $\vec{\nabla} T$ through the equation

$$q_i = -K\, T_{,i}\ , \quad i = 1, 2, 3 \tag{3}$$

i.e., the heat flux vector is the instantaneous result of a temperature gradient. Here $K > 0$is the thermal conductivity of the solid.

Law of conservation of internal energy :

$$-\, q_{i,i} + \rho\, Q = \rho\, c_e\ \dot{T}\ , \quad i = 1, 2, 3 \tag{4}$$

Where Q is the heat source and c_e is the specific heat of the solid at constant strain. A superposed dot denotes the partial derivative with respect to time.

Classical heat transport equation :

Equation (3) and (4) together give the parabolic type of heat transport equation as:

$$K\, \nabla^2 T + \rho\, Q = \rho\, c_e\ \dot{T} \tag{5}$$

Equations of motion :

Stress equations of motion :

$$\tau_{ij,j} + \rho\, f_i = \rho\, \ddot{u}_i\ , \quad (i, j = 1,2,3) \tag{6}$$

Where f_i are the body force components and τ_{ij} are given by equations (1).

Displacement equations of motion :

$$(\lambda + \mu) u_{j,ij} + \mu\, u_{i,jj} - \beta\, T_{,i} + \rho\, f_i = \rho\, \ddot{u}_i \tag{7}$$

Equations (1), (5) and (6) [or (7)] constitute the complete mathematical model of the classical theory of thermoelasticity (CTE).

1.2 Classical Coupled Thermoelasticity (CCTE) :

Law of conservation of internal energy :

$$-q_{i,i} + \rho Q = \rho c_e \dot{T} + \beta T_0 \dot{e}_{k,k} \tag{8}$$

where the term $(\beta T_0 \dot{e}_{k,k})$ brings about the coupling between strain and temperature, vide, Biot [6].

Classical heat transport equation :

Eliminating q_i between the equations (3) and (8), there results the classical heat transport equation as

$$K \nabla^2 T + \rho Q = \rho c_e \dot{T} + \beta T_0 \dot{u}_{k,k} \tag{9}$$

which is a parabolic type equation as in Biot [6].

The equations of motions (7) and the heat transport equation (9) along with the constitutive equation (1) constitute the complete mathematical model of the theory of Classical Coupled Thermoelasticity (CCTE).

Modification of the Coupled theory of Thermoelasticity

The classical uncoupled theory of thermoelsticity predicts two phenomena not compatible with physical observations. First, the equation of heat conduction of this theory does not contain any elastic term contrary to the fact that elastic changes produce heat effects. Second, the heat equation is of parabolic type predicting infinite speed of propagation for heat waves.

In order to overcome this paradox, efforts were made to modify coupled theory of thermoelasticity, on different grounds, to obtain a wave type heat conduction equation.

Kaliski [15] employed a heat conduction law on adhoc basis given by

$$\left(1 + \tau \frac{\partial}{\partial t}\right) q_i = -K T_{,i} \tag{10}$$

where τ is a non – negative constant. This law is a generalization of the classical Fourier law given by (3) and q_i represents the heat flux within the material to the temperature gradient.

1.3 Lord – Shulman (L – S) model

[Extended thermoelasticity (ETE)]

Using (10) in place of (3) we gets the following generalization of the heat conduction equation (9).

$$K \nabla^2 T = \left(1 + \tau \frac{\partial}{\partial t}\right) \left[\rho c_e \dot{T} + T_0 \beta \dot{u}_{k,k} - \rho Q\right] \tag{11}$$

Equation (11) is of hyperbolic type and so, thermoelasticity theory, for which (11) is the heat transport equation, is free from the paradox of infinite heat propagation speed. Equation (11) predicts the speed of $\sqrt{K/(\rho\, c_e\, \tau)}$ for thermal signals. This wave – type thermal disturbance is sometimes referred to as 'second sound', vide, Suhubi [16].

The constant τ that appears in the equation (10) represents the time required to establish the steady state of heat conduction and is known as 'thermal relaxation parameter'. A table of values of τ for various metals has been given in Francis [17], Engelbracht [18] and relevant references therein.

The theory based on equations (1), (7) and (11) contains only one such relaxation time parameter τ and is often referred to as extended thermoelasticity (ETE) or Lord and Shulman (L – S) theory of generalized thermoelasticity for homogeneous and isotropic solids.

1.4 Green – Lindsay Model (G – L model).

[Temperature Rate dependent Thermoelasticity (TRDTE)]

The generalized thermoelasticity theory, formulated by Green and Lindsay [19] and independently by Suhubi [16]. This theory is referred to as 'temperature rate dependent thermoelasticity' or briefly TRDTE. It has contained special features that contrast with L – S theory having one relaxation time parameter. In G – L model, Fourier law of heat conduction is unchanged whereas the classical energy equation and the stress – strain – temperature relations are modified. Two constitutive constants α and α_0, having the dimension of time, appear in the governing equations in place of one relaxation time τ in L – S model. The equations as proposed in G – L model are as follows :

Modified Energy Equation :

$$-q_{i,i} + \rho Q = \rho\, c_e\left(\dot{T} + \alpha_0 \ddot{T}\right) + \beta\, T_0\, \dot{u}_{i,i} \tag{12}$$

Modified Constitutive Equation with Temperature Rate term :

$$\tau_{i,j} = \lambda u_{i,i}\delta_{ij} + 2\mu e_{ij} - \beta\left(T + \alpha\dot{T}\right)\delta_{ij}\,, \quad i,j = 1,2,3 \tag{13}$$

Fourier Law :

$$q_i = -K\, T_{,i} \tag{14}$$

Coupled Heat Transport Equation :

$$K\nabla^2 T + \rho Q = \rho\, c_e\left(\dot{T} + \alpha_0 \ddot{T}\right) + \beta\, T_0\, \dot{u}_{i,i} \tag{15}$$

The constitutive constants $\alpha,\ \alpha_0$ satisfy the relation $\alpha \geq \alpha_0 > 0$.
Equations (12) – (15) constitute the (G – L) model of the generalized thermoelasticity.

1.5 Green and Naghdi (G – N) model :

Another generalization of the theory is presented by Green and Naghdi [20]. They developed three models for generalized thermoelasticity of homogeneous isotropic materials which are labelled as model I, II and III. The nature of these theories are such that when the respective theories are linearized, model I reduces to the classical heat conduction equation theory based on Fourier law. The linearized versions of model II and III permit propagation of thermal waves at finite speed.

1.5.1 Green – Naghdi (G – N) model II without energy dissipation (TEWOED) :

The basic equations are as follows :

Modified Energy Equation :

$$-q_{i,i} + \rho Q = \rho\, c_e\, \dot{T} + \beta\, T_0\, \dot{u}_{i,i} \tag{16}$$

Heat Conduction Law :

$$q_i = -K^* v_{,i}\,, \quad \dot{v} = T \tag{17}$$

Equations (16) and (17) are combined together to give a hyperbolic equation as

$$K^*\nabla^2 T + \rho\dot{Q} = \rho\, c_e\, \ddot{T} + \beta\, T_0\, \ddot{u}_{i,i} \tag{18}$$

Here K^* (> 0)is a material constant. Finite wave speed is clearly $\sqrt{K^*/(\rho\, c_e)}$. Model II, in particular, exhibits a feature that is not presented in the other established thermo-elastic models. It reveals that no damping term appears in the system of equations and therefore the G – N theory type II is known as the thermoelasticity without energy dissipation (TEWOED) [21]. It also provided the proof of the uniqueness of solution for the corresponding initial – boundary value problem.

1.5.2 Green – Naghdi (G – N) model III with energy dissipation (TEWED) :

In this model the basic equations are as follows :

Modified Energy Equation :

$$-q_{i,i} + \rho Q = \rho\, c_e\, \dot{T} + \beta\, T_0\, \dot{u}_{i,i} \tag{19}$$

Heat Conduction Law :

$$q_i = -\left(K\, T_{,i} + K^* v_{,i}\right), \quad \dot{v} = T \tag{20}$$

Equations (19) and (20) are combined together to give a hyperbolic equation as

$$K\nabla^2\dot{T} + K^*\nabla^2 T + \rho\dot{Q} = \rho\, c_e\, \ddot{T} + \beta\, T_0\, \ddot{u}_{i,i} \tag{21}$$

Equation (21) admits propagation of damped thermoelastic waves, where the damping being due to the term $\dot{T}$ in the equation.

1.6 Unified system of equations :

It has been seen that the basic equations of generalized thermoelasticity for an isotropic elastic body can be written in the form :

$$\tau_{ij} = \lambda u_{k,k} \delta_{ij} + 2\mu e_{ij} - \beta (T + \alpha \dot{T})\delta_{ij} \quad (22)$$

$$K\nabla^2 T = \rho c_e(\dot{T} + \alpha_0 \ddot{T}) + \left(1 + \tau \frac{\partial}{\partial t}\right)(\xi\beta T_0 \dot{u}_{k,k} - \rho Q) \quad (23)$$

$$\mu \nabla^2 u_i + (\lambda + \mu) u_{k,ki} - \beta (T + \alpha\dot{T})_{,i} + \rho f_i = \rho \ddot{u}_i \quad (24)$$

It is observed that when

(i) $\alpha = 0, \alpha_0 = \tau, \xi = 1$; the above equations reduce to the basic equations as in L – S model

(ii) $\tau = 0, \xi = 1$; the above equations reduce to the basic equations as in G – L model

(iii) $\alpha = \alpha_0 = \tau = \xi = 0$; the equations reduce to the classical equations of thermoelasticity.

Thus, the CTE, the ETE (L – S model) and TRDTE (G – L model) can be studied in a unified way on the basic of equations (22), (23), and (24).

A unified version of the equation of generalized thermoelasticity in the case of Classical theory of thermoelasticity (CTE), Classical coupled theory of thermoelasticity (CCTE), Temperature rate dependent thermoelasticity (TRDTE), Thermoelasticity with energy dissipation (TEWED) may be presented as follows:

The constitutive relations in the generalized theory of thermoelasticity are

$$\tau_{ij} = \lambda u_{i,i} \delta_{ij} + 2\mu e_{ij} - \beta (T + \delta_{1k} \alpha \dot{T})\delta_{ij} \quad (25)$$

The generalized heat conduction equation is

$$\left(\delta_{1k} + \delta_{2k}\frac{\partial}{\partial t}\right) K\nabla^2 T + \delta_{2k} K^* \nabla^2 T + \left(\delta_{1k} + \delta_{2k}\frac{\partial}{\partial t}\right) \rho Q =$$

$$\rho c_e[\delta_{1k}\dot{T} + (\delta_{1k}\alpha_0 + \delta_{2k})\ddot{T}] + \beta T_0 \left(\xi\delta_{1k} + \delta_{2k}\frac{\partial}{\partial t}\right) \dot{u}_{i,i} \quad (26)$$

The equations of motion are

$$\mu \nabla^2 u_i + (\lambda + \mu) u_{j,ji} - \beta (T + \alpha\dot{T})_{,i} + \rho f_i = \rho \ddot{u}_i \quad (27)$$

where τ_{ij} is the stress tensor, λ, μ are Lame constants, $\beta = (3\lambda + 2\mu)\alpha_t$, α_t being the coefficient of linear thermal expansion of the material, ρ is the density, K is the thermal conductivity, K^* is the additional material constant, c_e is the specific heat of the solid at constant strain, T is the temperature increase over the absolute reference temperature T_0, α and α_0 are the relaxation times, f_i are the components of the body forces, Q is the heat source and δ_{ij} is the Kronecker delta.

In the expression (25) – (27), we see that

(i) If $\alpha = 0, \alpha_0 = 0, k = 1$ and $\xi = 0$, then they reduce to the equations of Classical theory of thermoelasticity (CTE).

(ii) If $\alpha = 0$, $\alpha_0 = 0$, $k = 1$ and $\xi = 1$, then they reduce to the equations of Classical coupled theory of thermoelasticity (CCTE).

(iii) If $k = 1$ and $\xi = 1$, then they reduce to the equations of Temperature rate dependent thermoelasticity (TRDTE).

(iv) If $k = 2$ and $\alpha = 0$, then they reduce to the equations of Thermoelasticity with energy dissipation (TEWED).

The thermal relaxation times satisfy the inequality $\alpha \geq \alpha_0 > 0$.

Conclution:

Generalized estimating equations are a convenient and general approach to the analysis of several kinds of wide-ranging correlated data. Starting from Hook's Law, scientists have proposed different model equation to mathematically explain the diverse thermoelastic behaviour of different materials. Here we have derived a unified system to represents all types of models in a generalized form, for isotropic and homogeneous thermoelastic solids. By this way, from a optimal model, anybody can find out their required solution. It will minimize the time and cost of calculation in every aspect, and above all it will be learners friendly.

References:

[1] R. Hooke (1678): De Potentia restitutiva, London.

[2] C. L. M. H. Navier (1827): Memoire sur les lois de l'equilibre et du movement des corps solides elastiques,
Mem. Acad. Sci., Paris, tom VII , pp. 375 -- 393.

[3] A. L. Cauchy (1823): Bulletin des Sciences a la Societe philomathique.

[4] J. M. C. Duhamel (1837): Second Memoire Sur Les Phenomes Thermomechaniques, *J. de l'Ecole Polytech., tome 15,*
cahier 25, pp. 1 -- 57.

[5] F. E. Neumann (1835): Vorlesung uber die Theorie des Elasticitat der festen Korper und des Lichtathers, *Teubner,*
Leipzig.

[6] M. A. Biot (1956): Thermoelasticity and irreversible thermodynamics, *J. Appl. Phys.*, vol. 27(3), pp. 240 -- 253.

[7] P. Chadwick and I. N. Sneddon, (1958): Plane waves in an elastic solid conducting heat, *J. Mech. Phys. Solids.*,
vol. 6, pp. 223 -- 230.

[8] J. H. Weiner (1957): A uniqueness problem for coupled thermoelastic problems, *Quart. Appl. Math.*, vol. 15(1),
pp. 102 -- 105.

[9] P. Chadwick and I. N. Sneddon (1958): Plane waves in an elastic solid conducting heat, *J. Mech.*

Phys. Solids., vol. 6,
pp. 223 -- 230.
[10] G. Paria (1959): Coupling of elastic and thermal deformations, *Appl. Sci. Res.: Sec. A*, vol. 7, pp. 463 -- 475.
[11] A. C. Eringen (1970): Foundations of micropolar thermoelasticity, *Springer - Verlag, Wien*
[12] H. Parkus (1954): Stress in a centrally heated disc, *Proc. 2nd U. S. Nat. Congr. Appl. Mech.*, pp. 307.
[13] W. Nowacki (1959): Some Dynamic Problems of Thermoelasticity, *Archiwum Mechaniki Stosowanej*, vol. 11(6),
pp. 737 -- 750.
[14] R. S. Dhaliwal and A. Singh (1980): Dynamic Coupled Thermoelasticity, *Hindustan Pub., Delhi*.
[15] S. Kaliski (1965): Wave equations of thermoelasticity, *Bull. Acad. Pol. Sci. Ser. Sci. Tech.*, vol. 13(5), pp. 253 -- 260.
[16] E. S. Suhubi (1982): A generalized theory of simple thermomechanical materials, *Int. J. Engrg. Sci.*, vol. 20,
pp. 367 -- 371.
[17] P. H. Francis (1972): Thermo - mechanical effects in elastic wave propagation - a survey, *J. Sound. Vibe.*, vol. 21(2),
pp. 181 -- 192.
[18] J. Engelbrecht (1983): Nonlinear wave processes of deformation of solids, *Pitman Advanced Publishing Program,*
London.
[19] A. E. Green and K. A. Lindsay (1972): Thermoelasticity, *J. Elasticity*, vol. 2(1), pp. 1 -- 7.
[20] A. E. Green and P. M. Naghdi (1992): On undamped heat waves in an elastic solid, *J. Thermal Stresses*, vol. 15,
pp. 252 -- 264.
[21] A. E. Green and P. M. Naghdi (1993): Thermoelasticity without energy dissipation, *J. Elasticity*, vol. 31,
pp. 189 -- 208.