COMMON FIXED POINT THEOREMS FOR EIGHT OCCASIONALLY WEAKLY COMPATIBLE MAPPINGS IN FUZZY METRIC SPACE

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Abstract: In this paper, we prove common fixed point theorems for eight occasionally weakly Compatible (owc) mappings.

1. Introduction

In 1965 L.A.Zadeh [7] introduced the concept of fuzzy sets. Later with the concept of fuzzy sets, O.Kramosil and J.Michalek [8] introduced fuzzy metric spaces afterwards the notion of fuzzy metric spaces was modified with the help of continuous t-norm by A.George and P.Veeramani [1]. S. Sessa [14] improved commutativity condition in fixed point theorem by introducing the notion of weakly commuting maps in metric space.R.Vasuki [11] proved fixed point theorems for R-weakly commuting mapping . The concept of compatible maps by [4] and weakly compatible maps by [5] in fuzzy metric space is generalized by A.Al.Thagafi and NaseerShahzad [2] by introducing the concept of occasionally weakly compatible mappings.In this paper we prove some fixed point theorems for Eight occasionally weakly compatible owcmappings which generalises the results of [10]

2. Preliminaries

Definition 2.1 [3]- A binary operation $*$: [0,1] x [0,1]→ [0,1] is a continuous t-norms if it satisfies following assertions:
a) $*$ is commutative and associative;
b) $*$ is continuous;
c) a*1 = afor all aϵ[0,1];
d) a *b ≤ c *d whenever a ≤ c and b ≤d , and a,b, c, d ∈[0,1] .

Definition 2.2[3]- A 3-tuple (X,M,$*$) is said to be a fuzzy metric space if X is an arbitrary set, $*$is a continuous t -norm and M is a fuzzy set on X^2x(0,∞) satisfying the following conditions,∀x, y, z ϵ X, s, t >0,
(f1) M(x, y, t) >0; for all t >0
(f2) M(x, y, t) = 1 if and only if x = y.
(f3) M(x, y, t) =M(y, x, t);
(f4)M(x, y, t) $*$M(y, z, s) ≤ M(x, z, t + s)
(f5)M(x, y,.) :(0,∞)→[0,1] is continuous.
Then M is called a fuzzy metric on X . Then M(x, y,t) denotes the degree of nearness between x and y with respect to t .

Example 2.3 (Induced fuzzy metric [6]) **Let** (X,d) be a metric space. Denote a $*$b = ab for all a, b∈ [0,1]and let M_d be fuzzy sets on X^2x[0,∞] defined as follows:
M(x, y, t) = $\frac{t}{t+d(x,y)}$Then (X,M_d ,$*$) is a fuzzy metric space.

Definition 2.4 [6]: Two self mappings f and g of a fuzzy metric space (X,M,*) are called compatible if $\lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1$whenever $\{x_n\}$is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x$ for all x ∈ X.

Lemma 2.5: Let (X,M,*) be fuzzy metric space. If there exists q ∈ [0,1]such that M(x, y,qt) ≥M(x, y,t) for allx, yϵXand t >0, then x =y .

Definition 2.6: Let X be a set, f and g are self-mapping of X. A point xϵX is called a coincidence point of f and g ifffx=gx. We shall call w =fx=gxa point of coincidence of f and g.

Definition 2.7 [8]**:** A pair of maps S and T is called weakly compatible pair if they commute at coincidence points.
The concept of occasionally weakly compatible is introduced by A. Al-Thagafi and NaseerShahzad [2].It is stated as follows:

Definition 2.8: Two self maps f and g of a set X are called occasionally weakly compatible (owc) iff there is a point x ∈ Xwhich is a coincidence point of f and g at which f and g commute.

Example 2.9 [2]**:** Let R be the usual metric space. Define S,T : R→Rby Sx= 2x and Tx= x^2 for all xϵR. Then Sx= Txfor x ϵ 0, 2, but ST0 = TS0, and ST2 ≠ TS2. S and T are occasionally weakly compatible self maps but not weakly compatible.

Lemma 2.10 [5]:Let X be a set, f and g owcself maps of X . If f and g have a unique point of coincidence, w= fx= gx, thenw is the unique common fixed point of f and g.

3. Main Results

3.1 Theorem: Let (X, M, *) be fuzzy metric space and let P, Q, S, T, A, B, C and D be self mappings of X. Let the pairs (CP, ST) and (DQ, AB) be occasionally weakly compatibleif there exist q ∈ (0, 1)such that

(3.3.1) M (CPx, DQy, qt) ≥ min {M (STx, ABy, t), M (STx, CPx, t),

$$\frac{aM(STx, CPx, t) + bM(DQy, ABy, t) + cM(CPx, ABy, t)}{a + b + cM(DQy, STx, t)},$$

$$\frac{aM(CPx, DQy, t) + bM(DQy, STx, t) + cM(CPx, ABy, t)}{a + b + c}\}$$

for all x, y ∈X,t >0 and a, b, c, d ≥ 0 with a and b (c and d) cannot be simultaneously zero, then there exist a unique common fixed point of CP, DQ, ST and AB in X. If (C, P), (D, Q), (A, B),
(S, T), (DQ, B) and (T, CP) are commuting pairs and Px= P^2x , Qx= Q^2x for all x ∈ X then A, B, C, D, S, T, P and Q have a unique common fixed point in X.

Proof: Let the pairs (CP, ST)and (DQ, AB) be occasionally weakly compatible, so there are points x, y ∈ Xsuch that CPx= STxand DQy= ABy. We say that
CPx= DQy. If not, by inequality (3.3.1)
M (CPx, DQy, qt) ≥ min {M (STx, ABy, t), M (STx, CPx, t),

$$\frac{aM(STx, CPx, t) + bM(DQy, ABy, t) + cM(CPx, ABy, t)}{a + b + cM(DQy, STx, t)},$$

$$\frac{aM(CPx, DQy, t) + bM(DQy, STx, t) + cM(CPx, ABy, t)}{a + b + c}\}$$

≥ min { M (CPx, DQy, t), M (CPx, CPx, t),

$$\frac{aM(CPx, PCx, t) + bM(DQy, DQy, t) + cM(CPx, DQy, t)}{a + b + cM(DQy, CPx, t)},$$

$$\frac{aM(CPx, DQy, t) + bM(DQy, CPx, t) + cM(CPx, DQy, t)}{a + b + c}\}$$

$$\geq \min\{M(CPx, DQy, t), 1, \frac{a + b + cM(CPx, DQy, t)}{a + b + cM(CPy, DQx, t)},$$

$$\frac{(a + b + c)M(CPx, DQy, t)}{a + b + c}\}$$

≥ min {M(CPx,DQy,t),1,1,M(CPx,DQy,T)}
= M (CPx, DQy, t).

Therefore CPx =DQyi.e., CPx= STx= DQy= ABy. Suppose that there is another point z such that CPz= STzthen by inequality (3.3.1) we have
CPz= STz= DQy= AByso CPx= CPzand w=CPx= STxis the unique point of coincidence of CP and ST. By Lemma 2.10 wis the only common fixed point of CP and ST. Similarly there is a unique point z ∈X such that z = DQz= ABz.
Assume that w ≠ z. We have, by inequality (3.3.1)
M (w, z, qt) =M (CPw, DQz, qt)
M (CPw, DQz, qt) ≥ min {M (STw, ABz, t), M (STw, CPw, t),

$$\frac{aM(STw, CPw, t) + bM(DQz, ABz, t) + cM(CPw, ABz, t)}{a + b + cM(DQz, STw, t)},$$

$$\frac{aM(CPw, DQz, t) + bM(DQz, STw, t) + cM(CPw, ABz, t)}{a + b + c}\}$$

$$\geq \min\{M(w, z, t), M(w, w, t), \frac{aM(w, w, t) + bM(z, z, t) + cM(w, z, t)}{a + b + cM(z, w, t)},$$

$$\frac{aM(w, z, t) + bM(z, w, t) + cM(w, z, t)}{a + b + c}\}$$

$\geq$ min {M (w, z, t), 1, $\frac{a+b+cM(w,z,t)}{a+b+cM(z,w,t)}$, $\frac{(a+b+c)M(w,z,t)}{a+b+c}$ }

= min {M (w, z, t), 1, 1, M (w, z, t)}.

So, M (w, z, qt) $\geq$ M (w, z, t)

Therefore by lemma 2.5 we have z = w.

z is a common fixed point of CP, DQ, AB and ST.

To prove uniqueness let u be another common fixed point of CP, DQ, ST, and AB. then

M (z, u, qt) =M (CPz, DQu, qt)

M (CPz, DQu, qt) $\geq$ min { M (STz, ABu, t), M (STz, CPz, t),

$\frac{aM(STz,CPz,t)+bM(DQu,\ ABu,t)+cM(CPz,ABu,t)}{a+b+cM(DQu,\ STz,t)}$,

$\frac{aM(CPz,DQu,\ t)+bM(DQu,\ STz,t)+cM(CPz,ABu,t)}{a+b+c}$ }

$\geq$min {M (z, u, t), M (z, z, t),

$\frac{aM(z,z,t)+bM(u,u,t)+cM(z,u,t)}{a+b+cM(u,z,t)}$,

$\frac{aM(z,u,t)+bM(u,\ z,t)+cM(z,u,t)}{a+b+c}$ }

$\geq$ min {M (z, u, t), 1, $\frac{a+b+cM(z,u,t)}{a+b+cM(u,z,t)}$, $\frac{(a+b+c)M(z,u,t)}{a+b+c}$ }

= min {M (z, u, t), 1, 1, M (z, u, t)}.

So, M (z, u, qt) $\geq$ M (z, u, t).

Therefore by lemma 2.5 z = u.

Therefore z is a unique common fixed point of CP, DQ, AB and ST in X. By using the commutativity of the pairs (C, P), (D, Q), (A, B), (S, T) (DQ, B) and (T, CP) and Px= P^2x , Qx= Q^2x for all x $\in$ X then we can easily prove that z is a unique common fixed point of A, B,C, D, S, T, P and Q in X.

We get the result for seven self maps by taking $Q = I_X$ ($orP = I_X$) identity map in X , for six self maps by taking$Q = P = I_X$ for five self maps by taking $Q = P = I_X = T$ ($orB = I_X = P = Q$) in the theorem 3.1.

If $B = T = C = D = I_X$ is taken as an identity mapping of X in theorem 3.1 then we get the result for four self maps. The results for three self maps is obtained by taking $P = Q,\ T = B = C = D = I_X$ and also by taking S = A, T=B = I_X. Similarly for two self maps is obtained by taking $P = Q,\ A = S,\ T = B = I_X$ in theorem 3.1.

3.4 Corollary: Let (X, M, *) be complete fuzzy metric space and let P, Q, S, T, A, B, C and D be self-mappings of X .Let the pairs (CP, ST) and (DQ, AB) be occasionally weakly compatible mappings if there exist q $\in$ (0, 1) such that

(3.4.1) M (CPx, DQy, qt) $\geq$ φ [min {M (STx, ABy, t), M (STx, CPx, t),

$\frac{aM(STx,CPx,t)+bM(DQy,\ ABy,t)+cM(CPx,ABy,t)}{a+b+cM(DQy,\ STx,t)}$,

$\frac{aM(CPx,DQy,\ t)+bM(DQy,\ STx,t)+cM(CPx,ABy,t)}{a+b+c}$ }

for all x, y $\in$X , t >0 and a, b, c, d $\geq$ 0 with a and b (c and d) cannot be simultaneously zero , φ: [0,1] $\rightarrow$ [0,1]such that φ (t) >t for all 0 <t <1, then there is a unique common fixed point of CP, DQ, ST and AB in X. If (C, P), (D, Q), (A, B), (S, T) (DQ, B) and (T, CP) are commuting pairs and Px= P^2x , Qx= Q^2x for all x $\in$ X then A, B,C, D, S, T, P and Q have a unique common fixed point in X.

Proof: The Proof follows from theorem 3.1.

3.5 Theorem: Let (X, M, *) be complete fuzzy metric space and let P, Q, S, T, A, B, C and Dbe self-mappings of X .Let the pairs (CP, ST) and (DQ, AB) be occasionally weakly compatible mappings if there exist q$\in$ (0, 1) such that

(3.5.1)M (CPx, DQy, qt) $\geq$ φ {M (STx, ABy, t), M (STx, CPx, t), M (DQy, STx, t),

$\frac{aM(STx,CPx,t)+bM(DQy,\ ABy,t)+cM(CPx,ABy,t)}{a+b+cM(DQy,\ STx,t)}$,

$$\frac{aM(CPx, DQy, t) + bM(DQy, STx, t) + cM(CPx, ABy, t)}{a+b+c}\}$$

for all x, y ∈X , t >0 and a, b, c, d ≥ 0 with a and b cannot be simultaneously zero,
φ :$[0,1]^5$→ [0,1] such that φ(t, 1, t, 1, t) >tfor all 0 <t <1, then there is a unique common fixed point of CP, DQ, ST and AB. Moreover If (C, P), (D, Q), (A, B), (S, T) (DQ, B) and (T, CP) are commuting pairs and Px= P^2x , Qx= Q^2x for all x ∈ X then A, B,C, D, S, T, P and Q have a unique common fixed point in X.

Proof: Let the pairs (CP, ST) and (DQ, AB) be occasionally weakly compatible mappings, so there are points $x, y \in X$such that CPx= STxand DQy=ABy. We state that CPx= DQy. If not, by inequality (3.5.1)

M (CPx, DQy, qt) ≥ φ {M (CPx, DQy, t), M (CPx, CPx, t), M (DQy, CPx, t),

$$\frac{aM(CPx, CPx, t) + bM(DQy, DQy, t) + cM(CPx, DQy, t)}{a+b+cM(DQy, CPx, t)},$$

$$\frac{aM(CPx, DQy, t) + bM(DQy, CPx, t) + cM(CPx, DQy, t)}{a+b+c}\}$$

= φ {M (CPx,DQy, t), 1,M (CPx, DQy, t),

$$\frac{a+b+cM(CPx, DQy, t)}{a+b+cM(CPx, DQy, t)}, \frac{(a+b+c)M(CPx, DQy, t)}{a+b+c}\}$$

= φ {M (CPx, DQy, t), 1, M (CPx, DQy, t), 1, M (CPx, DQy, t)}

So, M (CPx, DQy, qt) ≥ M (CPx, DQy, qt)

a contradiction, therefore CPx= DQy, i.e. CPx= STx= DQy=ABy. Suppose that there is another point z such that CPz= STzthen by (3.5.1),CPz= STz= DQy=ABy, so CPx= CPzand w =CPx= ABxis the unique point of coincidence of CP and AB. By lemma 2.10w is a unique common fixed point of CP and ST .Similarly there is a unique point $z \in X$such that

z = DQz= ABz.

Assume that w ≠ z. We have, by inequality (3.3.1)

M (w, z, qt) = M (CPw, DQz, qt)

M (CPw, DQz, qt) ≥ φ {M (STw, ABz, t), M (STw, CPw, t), M (DQz, STw, t),

$$\frac{aM(STw, CPw, t) + bM(DQz, ABz, t) + cM(CPw, ABz, t)}{a+b+cM(Qz, STw, t)},$$

$$\frac{aM(CPw, DQz, t) + bM(DQz, STw, t) + cM(CPw, ABz, t)}{a+b+c}\}$$

= φ {M (w, z, t), M (w, w, t), M (z, w, t),

$$\frac{aM(w, w, t) + bM(z, z, t) + cM(w, z, t)}{a+b+cM(z, w, t)},$$

$$\frac{aM(w, z, t) + bM(z, w, t) + cM(w, z, t)}{a+b+c}\}$$

= φ {M (w, z, t), 1, M (w, z, t),

$$\frac{a+b+cM(w, z, t)}{a+b+cM(z, w, t)}, \frac{(a+b+c)M(w, z, t)}{a+b+c}\}$$

= min {M (w, z, t), 1, M (w, z, t), 1, M (w, z, t)}

M (w, z, qt) ≥ M (w, z, t)

Therefore we have z = w, by Lemma 6.5,

z is a common fixed point of CP, DQ, ST and AB.

To prove uniqueness let u be another common fixed point of CP, DQ, ST and AB. Then

M (z, u, qt) = M (CPz, DQu, qt)

M (CPz, DQu, qt) ≥ min {M (STz, ABu, t), M (STz, CPz, t), M (DQu, CPz, t),

$$\frac{aM(STz, CPz, t) + bM(DQu, ABu, t) + cM(CPz, ABu, t)}{a+b+cM(DQu, STz, t)},$$

$$\frac{aM(CPz, DQu, t) + bM(DQu, STz, t) + cM(CPz, ABu, t)}{a+b+c}\}$$

$= \min \{M (z, u, t), M (z, z, t), M (u, z, t),$

$$\frac{aM(z,z,t) + bM(u,u,t) + cM(z,u,t)}{a+b+cM(u,z,t)},$$

$$\frac{aM(z,u,t) + bM(u,z,t) + cM(z,u,t)}{a+b+c}\}$$

$= \min \{M (z, u, t), 1, M (z, u, t), \frac{a+b+cM(z,u,t)}{a+b+cM(u,z,t)},$

$$\frac{(a+b+c)M(z,u,t)}{a+b+c}\}$$

$= \min \{M (z,u,t), 1,M(z,u,t),1,M(z,u,t)\}$

So $M (z, u, qt) \geq M (z, u, t)$.

Therefore by lemma 2.5 we have z = u.

Therefore z is a unique common fixed point of CP, DQ, ST and AB in X. By using the commutativity of the pairs (C, P), (D, Q), (A, B), (S, T) (DQ, B) and (T, CP) and $Px= P^2x$, $Qx= Q^2x$ for all $x \in X$. We can easily prove that z is a unique common fixed point of A, B, C, D, P, Q, S and T in X.

We get results for seven self mappings by taking $Q = I_X$ $(or P = I_X)$, for six self maps by taking $P = Q = I_X$, for five self maps by taking $P = Q = I_X = T (Or B = I_X = P = Q)$, for four self maps by taking $T = B = C = D = I_X$ identity map in X, for three self maps by taking $P = Q$, $T = B = C = D = I_X$ and also by taking $S = A$, $T = B = C = D = I_X$, for two self maps by taking $P = Q, A = S$, $B = T = C = D = I_X$ in theorem 3.5.

3.6 Example : Let $(X, M, *)$ be a fuzzy metric space, where $X = [3,14)$, with $t-norm$ $*$is defined by $a * b = ab$ for all$a, b \in [0,1]$ and $M(x,y,t) = \begin{cases} \frac{t}{t+|x-y|} & if t > 0 \\ 0 & if t = 0 \end{cases}$ for all $x, y \in X$.

Define the self mappings $A, B, C, D, P, Q, S and T$ by

$$Cx = \begin{cases} 3, & if x \in \{3\} \cup (8,14) \\ 11, & if x \in (3,8] \end{cases};$$

$$Dx = \begin{cases} 3, & if x \in \{3\} \cup (8,14) \\ 5 & if x \in (3,8] \end{cases};$$

$$Ax = \begin{cases} 3 & if x = 3 \\ 13 & if x \in (3,8] \\ \frac{x+1}{3} & if x \in (8,14) \end{cases};$$

$$Sx = \begin{cases} 3 & \text{ifx} = 3 \\ 10 & \text{ifx} \in (3,8] \\ \frac{4x-23}{3} & \text{ifx} \in (8,14) \end{cases};$$

$Px = Qx = Bx = Tx = x$ forall$x \in [3,14)$,

Also $CP(X) = \{3,11\} \not\subseteq [3,11) = ST(X)$ and

$$DQ(X) = \{3,5\} \not\subseteq [3,5) \cup \{13\} = AB(X)$$

And (CP, ST) and (DQ, AB) are occasionally weakly compatible, all the conditions of theorem 3.1 are satisfied and 3 is the common fixed point of the pairs (CP, ST) and (DQ, AB) which also remains a point coincidence and 3 is the unique common fixed point of A, B, C, D, P, Q, S and T.

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