Performance Analysis of M/M/1 Queuing Model

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Abstract–The poisson queuing model indicates the distribution of interval arrival time and services time is exponentially distributed based on the First-comeFirst-serve (FCFS) and the Laplace transforms analyze the various probability generating function. Finally, the steady state solutioninvestigated for the better performance and uses for queuing model on the basis of Kendall's notation.

Keywords –M/M/1 queuing model, Markovian process, Laplace transform, Exponential distribution, Poisson distribution and Probability generating function

1. INTRODUCTION

Queuing theory originated when a Danish mathematician A.K. Erlang published in 1909 his pioneering paper "The theory of probabilities and telephone conversations" on the study of congestion of telephone traffic. His studies are now classics in queuing theory. Until about 1940 the development of the new branch of applied probability was directed by the needs encountered in the design of automatic telephone exchanges[1]. Generally, a queuing system is characterized by the following factors.

(i) The input process
(ii) The queue discipline
(iii) The service mechanism

In queuing system, we will discuss two common concepts:

Utilization factor - Utilization plays the crucial role and is defined as the proportion of the system's resources which is used by the traffic which arrives at it. It should be strictly less than one for the system to function well. It is usually denoted by the symbol ρ .

Little's theorem

Little's theorem [2] describes the relationship between throughput rate (i.e. arrival and service rate), cycle time and work in process (*i.e.* number of customers/jobs in the system). The theorem states that the expected number of customers (N) for a system in steady state can be determined using the following equation:

$$L = \lambda T$$

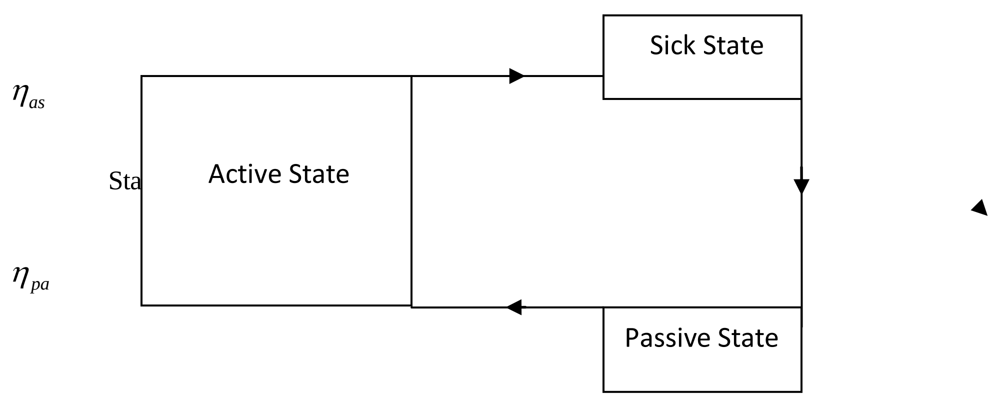
Here, λ is the average customer arrival rate and T is the average service time for a customer.

The steady-state solution, transient solution and busy period distribution for the first discipline and the steady-state solution for the second discipline are obtained.The steady-state probabilities of system size are obtained explicitly using iterative method and also discussed some useful measures of effectiveness. In the ensuing problem, we study a queuing system with three arrival rates. It is assumed that service rate is same for all three states of the input.

2. ANALYSIS OF QUEUING MODELOF THREE INPUTS

A stream of Poisson-type unit arrives at a single service station. The arrival pattern is multifarious, *i.e.*, there exists three different arrival rates λ_a (when input source is active), λ_s (when input source is sick) and zero (when input source is passive). The input source is operative in one state at a time. The service time of customers is exponentially distributed with Poissonian service rate μ corresponding to arrival rates λ_a, λ_s and 0 (zero). The state of the system, operating with arrival rate λ_a is designated as P, operating with arrival rate λ_s is designated as Q, and operating with arrival rate zero is designated as R. The system starts with input source in active state. The time duration for which it remains in active state is a random variable which is exponentially distributed with parameter η_{ap}. After the active state, the input source moves to the sick state, that is, the rate of arrival of units decreases considerably. The time period which is spent in sick state is also a random variable with parameter η_{sp}, which is different from active state. After the sick state, the input source shifts to passive state. In this state, units stop arriving at a service facility [14-15]. The input remains in the passive state for a random time with exponential rate η_{pa}, which is different from those of active and sick states.After the passive state, the input source again moves to active state and process continues in this way.

The transition rate from one state to another state is as shown in the following figure:



Further, service time is assumed to be exponentially distributed with parameter μ for all states of the input. The stochastic processes involved, viz., interarrival time of units and service time of customers are independent of each other. On the basis of simulation by Monte-Carlo software we get

(i) $L.T.'s$ of the probability generating function of the distribution of the number of units in the system for different states of the input.
(ii) $L.T.'s$ of the probabilities for different states of the input.
(iii) A particular case, when input does not move to sick state.
(iv) The explicit steady state results corresponding to (i).
(v) The explicit steady state probabilities corresponding to (ii).

3. ANALYSIS OFTHESTEADY STATE BEHAVIOR OF FINITE POPULATION QUEUING MODEL

The steady state solution can be obtained by the well-known property of the L, T., *viz.*[16]

$$\lim_{s\to 0} s\bar{F}(s) = \lim_{t\to\infty} F(t) \quad (1)$$

If the limit on the right exists.

Thus, if $\lim_{t\to\infty} P_n(t) = P_n$. We have, $\lim_{s\to 0} s\bar{P}_n(s) = P_n$, etc.

Using property (1) we have.

$$K_a(z)P(z) = \mu(z-1)P_0 + z\eta_{pa}R(z) \quad (2)$$

$$K_s(z)Q(z) = \mu(z-1)Q_0 + z\eta_{as}P(z) \quad (3)$$

$$K_p(z)R(z) = \mu(z-1)R_0 + z\eta_{sp}Q(z) \quad (4)$$

where $K_a(z) = \left[z\left\{\lambda_a(1-z)+\mu+\eta_{as}\right\}-\mu\right]$

$$K_s(z) = \left[z\left\{\lambda_s(1-z)+\mu+\eta_{sp}\right\}-\mu\right]$$

$$K_p(z) = \left[z(\mu+\eta_{sp})-\mu\right]$$

Solving equations (1-3)

$$P(z) = \frac{\mu(z-1)\left[K_s(z)K_p(z)P_0 + z\eta_{pa}K_s(z)R_0 + z^2\eta_{sp}\eta_{pa}Q_0\right]}{K_a(z)K_s(z)K_p(z) - z^3\eta_{as}\eta_{sp}\eta_{pa}} \quad (5)$$

$$Q(z) = \frac{\mu(z-1)\left[K_a(z)K_p(z)Q_0 + z\eta_{as}K_p(z)P_0 + z^2\eta_{as}\eta_{pa}R_0\right]}{K_a(z)K_s(z)K_p(z) - z^3\eta_{as}\eta_{sp}\eta_{pa}} \quad (6)$$

$$R(z) = \frac{\mu(z-1)\left[K_a(z)K_s(z)R_0 + z\eta_{sp}K_a(z)Q_0 + z^2\eta_{as}\eta_{sp}P_0\right]}{K_a(z)K_s(z)K_p(z) - z^3\eta_{as}\eta_{sp}\eta_{pa}} \quad (7)$$

$$S(z) = P(z) + Q(z) + R(z)$$

Hence, from equations (5-7)

$$S(z) = \frac{\mu(z-1)\sum_{s,p,a \text{ and P,Q,R}} [K_s(z)K_p(z)\mathrm{P}_0 + z\eta_{pa}K_s(z)\mathrm{R}_0(s) + z^2\eta_{sp}\eta_{pa}Q_0]}{K_a(z)K_s(z)K_p(z) - z^3\eta_{as}\eta_{sp}\eta_{pa}} \quad (8)$$

where, $\sum$ runs cyclically over a, s, p and P, Q, R.

$S(z)$ is known in terms of three unknowns, viz., P_0, Q_0 and R_0. We proceed to obtain these unknowns. Setting $z=1$ in equations of Rouche's theorem

$\eta_{as}P(1)=\eta_{pa}R(1)$ (9)

$\eta_{sp}Q(1)=\eta_{as}P(1)$ (10)

$\eta_{pa}R(1)=\eta_{sp}Q(1)$ (11)

Equations (9-11) lead to the following

$P(1)\equiv$ The Steady state probability for which input will remain in active state.

$$=\frac{\eta_{sp}\eta_{pa}}{\left(\eta_{sp}\eta_{pa}+\eta_{pa}\eta_{as}+\eta_{as}\eta_{sp}\right)} \quad (12)$$

$Q(1)\equiv$ The Steady state probability for which input will remain in sick state.

$$=\frac{\eta_{pa}\eta_{as}}{\left(\eta_{sp}\eta_{pa}+\eta_{pa}\eta_{as}+\eta_{as}\eta_{sp}\right)} \quad (13)$$

$R(1)\equiv$ The Steady state probability for which input will remain in passive state.

$$=\frac{\eta_{as}\eta_{sp}}{\left(\eta_{sp}\eta_{pa}+\eta_{pa}\eta_{as}+\eta_{as}\eta_{sp}\right)} \quad (14)$$

The denominator of $P(z)$, $Q(z)$ and $R(z)$, $[K_a(z)K_s(z)K_p(z)-z^3\eta_{as}\eta_{sp}\eta_{pa}]$ is of 5^{th} degree in z. So this must have five zeros. We now prove that it has three zeros inside and two zeros outside the unit circle. $K_a(z)=[z\{\lambda_a(1-z)+\mu+\eta_{as}\}-\mu]$ has two zeros, *viz.*, α_1 and α_2, whose values are given by

$$\alpha_1=\frac{1}{2\lambda_a}[(\lambda_a+\mu+\eta_{as})-\sqrt{\{(\lambda_a+\mu+\eta_{as})^2-4\lambda_a\mu\}}]$$

$$\alpha_2=\frac{1}{2\lambda_a}[(\lambda_a+\mu+\eta_{as})+\sqrt{\{(\lambda_a+\mu+\eta_{as})^2-4\lambda_a\mu\}}]$$

As proved earlier $K_a(z,s)$ has two real zeros, one inside and other outside unit circle $|z|=1$. Therefore, we say α_1 is inside and α_2 is outside of unit circle $|z|=1$. $K_s(z)=[z\{\lambda_s(1-z)+\mu+\eta_{sp}\}-\mu]$, has two real zeros, *viz.*, α_3 and α_4, whose values are given by

$$\alpha_3 = \frac{1}{2\lambda_s}[(\lambda_s + \mu + \eta_{sp}) - \sqrt{\{(\lambda_s + \mu + \eta_{sp})^2 - 4\lambda_s\mu\}}]$$

$$\alpha_4 = \frac{1}{2\lambda_s}[(\lambda_s + \mu + \eta_{sp}) + \sqrt{\{(\lambda_s + \mu + \eta_{sp})^2 - 4\lambda_s\mu\}}]$$

By the reasoning given earlier α_3 is inside and α_4 is outside of unit circle $|z| = 1$. $K_p(z) = \{z(\mu + \eta_{pa}) - \mu\}$ has one zero *viz.*, α_5

$\alpha_5 = \frac{\mu}{(\mu + \eta_{pa})}$, which is clearly inside of unit circle $|z| = 1$.

So, we conclude that factor $K_a(z,s)K_s(z)K_p(z)$ has three zeros, α_1, α_3 and α_5 inside and two zeros α_2 and α_4 are outside of unit circle $|z| = 1$.

Studying equation (5), a factor $(z-1)$ is common in numerator and denominator of $P(z)$. We cancel this factor. The denominator of $P(z)$ will now has four zeros, two inside and two outside of unit circle. We now proceed to prove that denominator of $P(z)$ has two real zeros outside the unit circle $|z| = 1$.

Let $f(z) = K_a(z)K_s(z)K_p(z) - z^3\eta_{as}\eta_{sp}\eta_{pa}$

$$\equiv (z-\alpha_1)(z-\alpha_2)(z-\alpha_3)(z-\alpha_4)(z-\alpha_5) - \frac{z^3\eta_{as}\eta_{sp}\eta_{pa}}{\lambda_a\lambda_s(\mu + \eta_{pa})} \quad (15)$$

Dividing $f(z)$ by $(z-1)$ and taking limit as z tends to infinity, we find that $\lim_{z\to\infty}\frac{f(z)}{z-1}$ >0. If we take limit as z tends to 1. Then, $\lim_{z\to 1}\frac{f(z)}{z-1} = \{\mu(\eta_{as}\eta_{sp} + \eta_{sp}\eta_{pa} + \eta_{pa}\eta_{as}) - \eta_{pa}(\lambda_s\eta_{as} + \lambda_a\eta_{sp})\}$. This is obtained by using L' Hospital's rule.

For $\frac{f(z)}{(z-1)}$ to have even number of real zeros between 1 and ∞, $\lim_{z\to 1}\frac{f(z)}{z-1} > 0$, *i.e.*,

$$\{\mu(\eta_{as}\eta_{sp} + \eta_{sp}\eta_{pa} + \eta_{pa}\eta_{as}) - \eta_{pa}(\lambda_s\eta_{as} + \lambda_a\eta_{sp})\} > 0 \quad (16)$$

and this must be true, as this is the condition of ergodicity, which is proved as, effective arrival rate of units is $\{\lambda_a P(1) + \lambda_s Q(1)\}$, as it represents the total number of arrivals in one unit of time when the input is in working stage (active state and sick state).

Total number of units served by the system in one unit of time are $[\mu\{P(1) + Q(1) + R(1)\}]$. Condition of ergodicity demands that effective arrival rate be less them effective service rate. Therefore,

$\{\lambda_a P(1)+\lambda_s Q(1)\}<\mu\{P(1)+Q(1)+R(1)\}$ .Substituting the values of $P(1)$, $Q(1)$ and $R(1)$ from equations (12-14) respectively. We obtain,

$$\{\mu(\eta_{as}\eta_{sp}+\eta_{sp}\eta_{pa}+\eta_{pa}\eta_{as})-\eta_{pa}(\lambda_s\eta_{as}+\lambda_a\eta_{sp})\}>0 \quad (17)$$

We find that (16) and (17) are identical and this gives the condition of ergodicity. This concludes that $\lim_{z\to\infty}\frac{f(z)}{z-1}$ and $\lim_{z\to 1}\frac{f(z)}{z-1}$ have like signs, so an even number of zeros of $f(z)$ lie in between 1 and ∞ .We proceed to prove that $\frac{f(z)}{z-1}$ has two zeros say z_1 *and* z_2, which lie outside $|z|=1$. Considering $\alpha_4>\alpha_2$, we have from equation (15),

$$\lim_{z\to\alpha_2}\frac{f(z)}{(z-1)}=-\frac{\alpha_2^3\eta_{as}\eta_{sp}\eta_{pa}}{(\alpha_2-1)\lambda_a\lambda_s(\mu+\eta_{pa})}<0\,.$$

Sign changes between 1 and α_2. So there is a real zero, say z_1, in between 1 and α_2.

$$\lim_{z\to\alpha_4}\frac{f(z)}{(z-1)}=-\frac{\alpha_4^3\eta_{as}\eta_{sp}\eta_{pa}}{(\alpha_4-1)\lambda_a\lambda_s(\mu+\eta_{pa})}<0$$

Like sign between α_2 and α_4. But there is a change of sign in between α_4 and ∞. So there is a real zero, say z_2, is between α_4 and ∞. This conclude that two real zeros of $\frac{f(z)}{z-1}$, z_1 and z_2 lie in the interval $[1,\alpha_2)$ and $[\alpha_4,\infty)$ respectively.

The two zeros of the denominator in (5) which are inside $|z|=1$ must vanish its numerator, because $P(z)$ is a well defined functions inside the unit circle. Thus, cancelling two factors in the numerator and in the denominator corresponding to these zeros, then equations (5) reduces to the following form:

$$P(z)=\frac{A}{(z-z_1)}+\frac{B}{(z-z_2)} \quad (18)$$

where A and B are to determined.Setting $z=1$, $P(1)=\frac{A}{1-z_1}+\frac{B}{1-z_2}$

Using (12), $A=-\left[\frac{B(z_1-1)}{(z_2-1)}+\frac{(z_1-1)\eta_{sp}\eta_{pa}}{(\eta_{as}\eta_{sp}+\eta_{sp}\eta_{pa}+\eta_{pa}\eta_{as})}\right]$. Therefore, $\mathrm{P}(z)$ in term of B is

$$P(z)=\frac{B(z_2-z_1)(z-1)}{(z_2-1)(z-z_2)(z-z_1)}-\frac{(z_1-1)\eta_{sp}\eta_{pa}}{(z-z_1)(\eta_{as}\eta_{sp}+\eta_{sp}\eta_{pa}+\eta_{pa}\eta_{as})} \quad (19)$$

$$P_n = B\left[\frac{(z_1-1)}{(z_2-1)z_1^{n+1}} - \frac{1}{z_2^{n+1}}\right] + \frac{(z_1-1)\eta_{sp}\eta_{pa}}{z_1^{n+1}(\eta_{as}\eta_{sp}+\eta_{sp}\eta_{pa}+\eta_{pa}\eta_{as})}, \quad n \ge 0 \,(20)$$

$$P_0 = \frac{B(z_1-z_2)}{z_1 z_2 (z_2-1)} + \frac{(z_1-1)\eta_{sp}\eta_{pa}}{z_1(\eta_{as}\eta_{sp}+\eta_{sp}\eta_{pa}+\eta_{pa}\eta_{as})} \,(21)$$

Substituting the values of $P(z)$ and P_0 from (19) and (21) in equation (2).

$$R(z) = \frac{B(z_2-z_1)(z-1)\{z(\mu - z_1 z_2 \lambda_a) + z_1 z_2(\lambda_a+\mu+\eta_{as}) - \mu(z_1+z_2)\}}{\eta_{pa}(z_2-1) z_1 z_2 (z-z_1)(z-z_2)}$$

$$-\frac{(z_1-1)\eta_{sp}\{z(\mu - z_1\lambda_a) + z_1(\lambda_a+\eta_{as}) - \mu\}}{(z-z_1) z_1(\eta_{as}\eta_{sp}+\eta_{sp}\eta_{pa}+\eta_{pa}\eta_{as})} \,(22)$$

$$R_n = \frac{B(\mu - z_1 z_2 \lambda_a)}{z_1 z_2 \eta_{pa}}\left[\frac{(z_1-1)}{(z_2-1)z_1^{n}} - \frac{1}{z_2^{n}}\right] + \frac{B}{z_1 z_2 \eta_{pa}}\left[\frac{(z_1-1)}{(z_2-1)z_1^{n+1}} - \frac{1}{z_2^{n+1}}\right]\{z_1 z_2(\lambda_a+\mu+\eta_{as}) - \mu(z_1+z_2)\}$$
$$+\frac{(z_1-1)\eta_{sp}[z_1\{\lambda_a((1-z_1)+\mu+\eta_{as}\} - \mu]}{z_1^{n+1}(\eta_{as}\eta_{sp}+\eta_{sp}\eta_{pa}+\eta_{pa}\eta_{as})}, \qquad n \ge 0 \,(23)$$

$$R_0 = \frac{(z_1-1)\eta_{sp}\{z_1(\lambda_a+\eta_{as}) - \mu\}}{z_1^2(\eta_{as}\eta_{sp}+\eta_{sp}\eta_{pa}+\eta_{pa}\eta_{as})} - \frac{B(z_2-z_1)\{z_1 z_2(\lambda_a+\mu+\eta_{as}) - \mu(z_1+z_2)\}}{z_1^2 z_2^2 \eta_{pa}(z_2-1)} \qquad (24)$$

Substituting the values of $R(z)$ and R_0 from (22) and (24) in equation (4)

$$Q(z) = \frac{B(z_2-z_1)(z-1)}{\eta_{pa}\eta_{sp} z_1^2 z_2^2 (z_2-1)(z-z_1)(z-z_2)}[z_1 z_2\{z(\mu+\eta_{pa}) - \mu\}\{\mu - z_1 z_2 \lambda_a\}$$
$$+\{z_1 z_2(\lambda_a+\mu+\eta_{as}) - \mu(z_1+z_2)\}\{z_1 z_2(\mu+\eta_{pa}) + z\mu - \mu(z_1+z_2)\}]$$

$$-\frac{(z_1-1)}{z_1^2(z-z_1)(\eta_{as}\eta_{sp}+\eta_{sp}\eta_{pa}+\eta_{pa}\eta_{as})}[\{z(\mu+\eta_{pa}) - \mu\}(\mu - z_1\lambda_a)z_1$$
$$+(z\mu + z_1\eta_{pa} - \mu)\{z_1(\lambda_a+\eta_{as}) - \mu\}] \qquad (25)$$

$$Q_n = \frac{B}{\eta_{sp}\eta_{pa}} \left[\begin{array}{l} \frac{(z_1-1)(\mu - z_1 z_2 \lambda_a)\{z_1(\mu+\lambda_a)-\mu\}}{z_1^{n+2} z_2 (z_2-1)} - \frac{(\mu - z_1 z_2 \lambda_a)\{z_2(\mu+\eta_{pa})-\mu\}}{z_1 z_2^{n+2}} \\ + \frac{(z_1-1)\{z_1 z_2(\lambda_a+\mu+\eta_{as})-\mu(z_1+z_2)\}\{z_1\mu + z_1 z_2(\mu+\eta_{pa})-\mu(z_1+z_2)\}}{z_1^{n+3} z_2^2 (z_2-1)} \\ - \frac{\{z_1 z_2(\lambda_a+\mu+\eta_{as})-\mu(z_1+z_2)\}\{z_2\mu + z_1 z_2(\mu+\eta_{pa})-\mu(z_1+z_2)\}}{z_1^2 z_2^{n+3}} \end{array} \right]$$

$$+ \frac{(z_1-1)\{z_1(\mu+\eta_{pa})-\mu\}}{z_1^{n+3}(\eta_{as}\eta_{sp}+\eta_{sp}\eta_{pa}+\eta_{pa}\eta_{as})}[\{z_1(\lambda_a+\eta_{as})-\mu\}+z_1(\mu - z_1\lambda_a)], \quad n \geq 0 \qquad (26)$$

$$Q_0 = \frac{B(z_2-z_1)}{z_1^3 z_2^3 (z_2-1)\eta_{sp}\eta_{pa}} \left[\begin{array}{l} z_1 z_2 \mu(\mu - z_1 z_2 \lambda_a) - \{z_1 z_2(\lambda_a+\mu+\eta_{as}) \\ -\mu(z_1+z_2)\}\{z_1 z_2(\mu+\eta_{pa})-\mu(z_1+z_2)\} \end{array} \right]$$

$$+ \frac{(z_1-1)}{z_1^3(\eta_{as}\eta_{sp}+\eta_{sp}\eta_{pa}+\eta_{pa}\eta_{as})}[(z_1\eta_{pa}-\mu)\{z_1(\lambda_a+\eta_{as})-\mu\} - z_1\mu(\mu - z_1\lambda_a)] \qquad (27)$$

Equations (1-8) give the values of $P(z)$, P_n, P_0; $R(z)$, R_n, R_0 and $Q(z)$, Q_n Q_0 respectively in terms of B. If B is known, these are all obtained explicitly.Setting $z = \alpha_5$ in equation (4), we get

$Q(\alpha_5) = \frac{\eta_{pa}}{\eta_{sp}} R_0$. Substituting the value of R_0 from (24),

$$Q(\alpha_5) = \frac{B(z_1-z_2)\{z_1 z_2(\lambda_a+\mu+\eta_{as})-\mu(z_1+z_2)\}}{z_1^2 z_2^2 (z_2-1)\eta_{sp}} + \frac{\eta_{pa}(z_1-1)\{z_1(\lambda_a+\eta_{as})-\mu\}}{z_1^2(\eta_{as}\eta_{sp}+\eta_{sp}\eta_{pa}+\eta_{pa}\eta_{as})}$$

Substituting $z = \alpha_5$ in $Q(z)$ gives by (25) and equating two values of $Q(\alpha_5)$ thus obtained, we get

$$\frac{B(z_1-z_2)}{\eta_{sp} z_2^2 (z_2-1)\{\mu - z_2(\mu+\eta_{pa})\}} \left[\begin{array}{l} \{z_1 z_2(\lambda_a+\mu+\eta_{as})-\mu(z_1+z_2)\}[z_1 z_2(\mu+\eta_{pa})^2+\mu^2 \\ -\mu(z_1+z_2)(\mu+\eta_{pa})\} - \{\mu - z_1(\mu+\eta_{pa})\}\{\mu - z_2(\mu+\eta_{pa})\}] \end{array} \right]$$

$$= \frac{(z_1-1)\{z_1(\lambda_a+\eta_{as})-\mu\}}{(\eta_{as}\eta_{sp}+\eta_{sp}\eta_{pa}+\eta_{pa}\eta_{as})}[\{\mu^2+z_1\eta_{pa}-\mu)(\mu+\eta_{pa})\}+\eta_{pa}\{\mu - z_1(\mu+\eta_{pa})\}] \quad (28)$$

Equation (28) gives value of B in term of known quantities.

4. CONCLUSION

By this analysisof the given model queuing model with three different type inputs we get when the system size becomes large then probability become smaller and the derived steady state probabilities formula is useful for the next incoming research work like telecommunication model of short time frame.

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