A Characterization of Strong and Weak Convergence in Fuzzy Metric Spaces

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Abstract: In this paper, we attempt to introduce the concept of fuzzy metric space and some of its properties, and we investigate the strong and weak convergence in fuzzy metric Spaces. Despite uncertainty in fuzzy random variables, crisp metrics have always been used. Here, we use the strong law of large numbers for fuzzy random variables in the fuzzy metric space for the bootstrap mean. Then the problem of constructing a satisfactory theory of fuzzy metric spaces has been investigated by several authors from different points of view. In particular, and by modifying a definition of fuzzy metric space given by Kramosil and Michalek, Georgeand Veeramani have introduced and studied the following interesting notion of a fuzzy metric space. A fuzzy metric space is $(P_F(X), d_F)$ such that $P_F(X)$ is a set and d_F is a function defined on $d_F : P_F(X) \times P_F(X) \rightarrow [0,1]$ satisfying certain axioms and d_F is called a fuzzy metric in $P_F(X)$. Finally, we developed ideas that many of known strong and weak convergence theorems can easily be derived from the fuzzy metric Spaces.

Keywords: Fuzzy metric space, Limit theorems, Cauchy sequence, fuzzy diameter, strong and weak convergence, Random set, Fuzzy random variable.

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1. Introduction

This paper aims to introduce as fuzzy metric spaces, fuzzy normed space weak and strong fuzzy metric spaces. In the convergence of sequences of fuzzy points and the completeness of induced fuzzy metric spaces are considered. In 1965, the concept of fuzzy Sets was introduced by L. A. Zadeh [1]. Since then many authors have expansively developed the theory of fuzzy Sets and applications–Especially, Deng [2], Erceg [3], Kavela and Seokkala [4],kramosil and Michalek[5] has introduced the concept of fuzzy metric spaces in different ways. How to define fuzzy metric is one of the fundamental problems in fuzzy mathematics which is wildly used in fuzzy optimization and pattern recognition. Here fuzzy numbers to define metric in ordinary spaces, firstly proposed by Kaleva, following which fuzzy normed spaces, fuzzy topology induced by fuzzy metric spaces, fixed point theorem and other properties of fuzzy metric spaces are studied by a few researchers, see for instance, Felbin (1992)[6], George (1994)[7], Gregori (2000)[8], Hadzic (2002)[9] etc more details [10]. In this work, we obtain sufficiently many of fuzzy metric spaces. Results of these researches have been applied to many practical problems in fuzzy environment. There are kinds of new fuzzy measure useful for solving problems in strong and weak convergence in fuzzy metric Spaces. In the sense of Xia and Guo (2003) [11] is the theory of fuzzy linear space. As the concept of Cauchy sequence is the classical case in the literature [12]. In particular, Gregori and Miñana [13] introduced and studied a concept of Cauchy sequence in the fuzzy fixed point theory are due to M. Grabiec [14] and George and Veeramani [15]. Here at present, we discussed of strong and weak convergence in fuzzy metric Spaces.

2. Preliminaries

In this section, we discuss the fuzzy metric and normed spaces. And also elaborate the elementary concepts of fuzzy set theory that will be used in the following sections.

Fuzzy metric space:

Let X be any non-empty set. Then the mapping $d_F : P_F(X) \times P_F(X) \to S_F^{+}(R)$ is said to be a fuzzy metric space if for any $\{(x, \lambda), (y, \gamma), (z, \rho)\} \subset P_F(X)$, which satisfies the following conditions:

$i.\quad d_F((x, \lambda), (y, \gamma)) \geq 0$

$ii.\quad d_F((x, \lambda), (y, \gamma)) = 0 \text{ if and only if } x = y \text{ and } \lambda = \gamma = \rho = 1$

$iii.\quad d_F((x, \lambda), (y, \gamma)) = d_F((y, \gamma), (x, \lambda)) \quad [Symmetric]$

$iv.\quad d_F((x, \lambda), (z, \rho)) \leq d_F((x, \lambda), (y, \lambda)) + d_F((y, \gamma), (z, \rho)) \quad [Triangle\ inequlity]$

Here, d_F is a fuzzy metric in $P_F(X)$.

Fuzzy linear normed space:

Suppose that L is a fuzzy linear space. The mapping $\|.\|: L \to S_F^{+}(\mathbb{R})$ is called fuzzy linear normed space for any $(x, \lambda), (y, \gamma) \in L$ if satisfies the following condition:

i) $\|(x, \lambda\| \geq 0$

ii) $\|(x, \lambda\| = 0 \text{ if and only if } x = 0 \text{ and } \lambda = 1$

iii) $\|(x, \lambda\| = |k|.\|(x, \lambda\| \ \forall\ k \in \mathbb{R}$

iv) $\|(x, \lambda) + (y, \gamma)\| \leq \|(x, \lambda\| + \|(x, \gamma\|$ [triangle inequality]

The fuzzy normed space is denoted by $(L, \|.\|)$

Convergent in a fuzzy metric space:

Let $(P_F(X), d_F)$ be a fuzzy metric space. A sequence $\{x_n\}$ in a metric space $(P_F(X), d_F)$ is said to be convergent if there exists $(x, \lambda) \in P_F(X)$ s. t. $\lim_{n\to\infty} d_F((x_n, \lambda_n), (x, \lambda)) = 0 \quad \forall\ n \in N$

Diameter in a fuzzy metric:

Let $(P_F(X), d_F)$ be a fuzzy metric space and $S \subset P_F(X)$. The diameter of S is denoted by diam S, defined as

$$diam\ S = \sup\{d_F((x, \lambda), (y, \gamma)) : (x, \lambda), (y, \gamma) \in S\}$$

Cauchy sequence in a fuzzy metric space:

Let $(P_F(X), d_F)$ be a fuzzy metric space and $\{x_n\}$ be a sequence in it. Then the sequence $\{x_n\}$ is said to be a Cauchy sequence if for all $\varepsilon \in (0, 1]$, there exists a positive integer N s. t.

$$d_F((x_n, \lambda_n), (x_m, \lambda_m)) > 1 - \varepsilon \quad \forall\ n, m \in N$$

Complete in a fuzzy metric space:

Let $(P_F(X), d_F)$ be a fuzzy metric space. An induced fuzzy metric space $(P_F(X), d_F)$ is said to be complete if any Cauchy sequence in it has a unique limit in the space.

3. Characterization of Strong and Weak Convergence in Fuzzy Metric Spaces

We begin this section natural define with the following theorem related characterization of strong and weak convergence of metric spaces.

Strong convergence in a fuzzy metric space:

Let $(P_F(X), d_F)$ be a fuzzy metric space and $\{x_n\}$ be a sequence in it. Then a sequence $\{x_n\}$ is said to be strongly convergent if for all $\varepsilon \in (0, 1)$, there exists $(x, \lambda) \in P_F(X)$ and depending on ε s. t.

$$d_F((x_n, \lambda_n), (x, \lambda)) > 1 - \varepsilon \quad \forall\ n \in N$$

Weak convergence in a fuzzy metric space:
Let $(P_F(X), d_F)$ be a fuzzy metric space and $\{x_n\}$ be a sequence in it. The maps $d_F : P_F(X)\times P_F(X)\rightarrow S_F{}^{+}(R)$ and $f : P_F(X)\rightarrow P_F(X)$. Then a sequence $\{x_n\}$ is said to be strongly convergent if for all $\varepsilon\in(0, 1)$, there exists $(x, \lambda)\in P_F(X)$ and depending on ε s. t. $d_F(f(x_n, \lambda_n), f(x, \lambda))>1-\varepsilon \quad \forall\ n\in N$

3. 1. Theorem (Weak convergence): Suppose that $\{x_n\}$ is a weak convergent sequence in a fuzzy metric space $P_F(X)$.Then

(a) The weak limit of $\{x_n\}$ is unique.

(b) Every subsequence of $\{x_n\}$converges weakly to x.

Proof: (a) Suppose there are two limits of $\{x_n\}$which are x and y. Then we have

$(x_n, \lambda_n)\rightarrow(x, \lambda) \quad \therefore f(x_n, \lambda_n)\rightarrow f(x, \lambda)$

$(y_n, \gamma_n)\rightarrow(y, \gamma) \quad \therefore f(x_n, \lambda_n)\rightarrow f(y, \gamma)$

Since $f(x_n, \lambda_n)$is a sequence of numbers, its limit is unique.

Hence $f(x, \lambda)=f(y, \lambda)$ (1)

By the linearity and continuity of f, then we have

$f[(x, \lambda)-(y, \gamma)]= f(x, \lambda)- f(y, \gamma)$

$\Rightarrow f[(x, \lambda)-(y, \gamma)]= f(x, \lambda)- f(x, \gamma)$

$\Rightarrow (x, \lambda)-(y, \gamma)=0$

$\Rightarrow (x, \lambda)=(y, \gamma)$

Hence the weak limit of $\{x_n\}$ is unique.

(b) This follows from the fact that $\{f(x_n)\}$is a convergent sequence of number, converges and has the same limit as the sequence.

3. 2. Theorem: Let $(P_F(X), d_F)$be a fuzzy metric space. Then

a) A convergent sequence in $P_F(X)$is bounded and its limit is unique.

b) If$(x_n, \lambda_n)\rightarrow(x, \lambda)$ *and* $(y_n, \gamma_n)\rightarrow(y, \gamma)$ *in* $P_F(X)$,

then $d_F((x_n, \lambda_n), (y_n, \gamma_n))\rightarrow d_F((x, \lambda), (y, \gamma))$

Proof: Let$\{x_n\}$ be a convergence sequence in $P_F(X)$which converges to x. Then

$\lim_{n\to\infty} d_F((x_n, \lambda_n), (x, \lambda))=0$

a) Let $\{x_n\}$be a convergent sequence in $P_F(X)$which converges to x. Then

$\lim_{n\to\infty} d_F((x_n, \lambda_n), (x, \lambda))=0$

$\Rightarrow d_F((x_n, \lambda_n), (x, \lambda))>1-\varepsilon, \quad where\ \varepsilon\in(0, 1)$

Taking $\varepsilon=\frac{1}{2}$, there exists an integer N s. t.

$d_F((x_n, \lambda_n), (x, \lambda))>1-\frac{1}{2}=\frac{1}{2},\ n>N$

For all n, then we have

$d_F((x_n, \lambda_n), (x, \lambda))>1+a$

Where $a=\{d_F((x_1, \lambda_1), (x, \lambda)), \ . \ . \ . \ , d_F((x_n, \lambda_n), (x, \lambda))\}$

This shows that $\{x_n\}$is bounded.

For the uniqueness of the limit:

Let us assume that $(x_n, \lambda_n) \to (x, \lambda)$ *and* $(x_n, \lambda_n) \to (y, \gamma)$ *as* $n \to \infty$ (2)

By triangle inequality we have,

$$d_F((x, \lambda), (y, \gamma)) \le d_F((x, \lambda), (x_n, \lambda_n)) + d_F((x_n, \lambda_n), (y, \gamma))$$
$$\Rightarrow d_F((x, \lambda), (y, \gamma)) \to 0 + 0$$
$$\Rightarrow (x, \lambda) = (y, \gamma)$$

Hence the limit is unique.

b) We have,

$$d_F((x_n, \lambda_n), (y_n, \gamma_n)) \le d_F((x_n, \lambda_n), (x, \lambda),) + d_F((x, \lambda), (y, \gamma)) + d_F((y, \gamma), (y_n, \gamma_n))$$
$$\Rightarrow d_F((x_n, \lambda_n), (y_n, \gamma_n)) - d_F((x, \lambda), (y, \gamma)) \le d_F((x_n, \lambda_n), (x, \lambda)) + d_F((y, \gamma), (y_n, \gamma_n)) \tag{3}$$

Interchanging x_n *and* x, y_n *and* y *as well as* λ_n *and* λ, γ_n *and* γ

$$d_F((x, \lambda), (y, \gamma)) - d_F((x_n, \lambda_n), (y_n, \gamma_n)) \le d_F((x, \lambda), (x_n, \lambda_n)) + d_F((y_n, \gamma_n), (y, \gamma))$$
$$\Rightarrow -[d_F((x_n, \lambda_n), (y_n, \gamma_n)) - d_F((x, \lambda), (y, \gamma))] \le d_F((x_n, \lambda_n), (x, \lambda)) + d_F((y_n, \gamma_n), (y, \gamma)) \tag{4}$$

From (2) and (3) together imply

$$|d_F((x_n, \lambda_n), (y_n, \gamma_n)) - d_F((x, \lambda), (y, \gamma))| \le d_F((x_n, \lambda_n), (x, \lambda)) + d_F((y_n, \gamma_n), (y, \gamma)) \tag{5}$$

Given that

$$(x_n, \lambda_n) \to (x, \lambda) \text{ as } n \to \infty \quad \therefore d_F((x_n, \lambda_n), (x, \lambda)) \to 0$$
$$(y_n, \gamma_n) \to (y, \gamma) \text{ as } n \to \infty \quad \therefore d_F((y_n, \gamma_n), (y, \gamma)) \to 0 \tag{6}$$

From (4) and (5), then we get,

$$|d_F((x_n, \lambda_n), (y_n, \gamma_n)) - d_F((x, \lambda), (y, \gamma))| \le d_F((x_n, \lambda_n), (x, \lambda)) + d_F((y_n, \gamma_n), (y, \gamma)) \to 0 + 0$$
$$\Rightarrow d_F((x_n, \lambda_n), (y_n, \gamma_n)) - d_F((x, \lambda), (y, \gamma)) \to 0$$
$$\Rightarrow d_F((x_n, \lambda_n), (y_n, \gamma_n)) \to d_F((x, \lambda), (y, \gamma))$$

4. Result and Discussion

Here, we discuss the extended result of strong and weak convergence in fuzzy metric spaces related cases.

4. 1. Problem: Suppose L is a fuzzy linear space in R^n. The two arbitrary fuzzy points (x, λ) *and* (y, γ) on L is defined by $d_{FE}((x, \lambda), (y, \gamma)) = (d_E(x, y), \min\{\lambda, \rho\})$

Where d_E is the Euclidean space, Then show that (L, d_{FE}) is a strong fuzzy metric space where L denotes the set of fuzzy points on the fuzzy set L

Solution: Suppose L is a fuzzy linear space in R^n. The distance between arbitrary two fuzzy points $(x, \lambda), (y, \gamma) \in L$ s. t. $d_{FE}((x, \lambda), (y, \gamma)) = (d_E(x, y), \min\{\lambda, \gamma\})$

Now we will prove that (L, d_{FE}) is a strong fuzzy metric space.

i. We have $d_E(x, y) \ge 0 \quad \forall\ x, y \in L$

So that $d_{FE}((x, \lambda), (y, \gamma)) = (d_E(x, y), \min(\lambda, \gamma)) \ge 0 \Rightarrow d_{FE}((x, \lambda), (y, \gamma)) \ge 0$

ii. Let $d_{FE}((x, \lambda), (y, \gamma)) = 0$

Now we will prove that $x = y$ *and* $\lambda = \gamma = 1$.Then we have,

$$(d_E(x, y), \min\{\lambda, \gamma\}) = 0$$
$$\Rightarrow (d_E(x, y), \min\{\lambda, \gamma\}) = (0, 1)$$
$$\Rightarrow d_E(x, y) = 0 \quad and \ \min\{\lambda, \gamma\} = 1$$
$$\Rightarrow x = y \qquad \Rightarrow \lambda = \gamma = 1$$

Conversely, Let $x = y$ *and* $\lambda = \gamma = 1$

Now we will prove that $d_{FE}((x,\lambda),(y,\gamma))=0$
Then we have
$d_{FE}((x,\lambda),(y,\gamma))=(d_E(x,y),\min\{\lambda,\gamma\})$
$\Rightarrow(d_E(x,y),\min\{\lambda,\gamma\})=(0,1)=0$
$\Rightarrow d_{FE}((x,\lambda),(y,\gamma))=0$
iii. Since (L,d_E) is a metric space.
$\therefore \quad d_E(x,y)=d_E(y,x)$
Then we have,
$d_{FE}((x,\lambda),(y,\gamma))=(d_E(x,y),\min\{\lambda,\gamma\})=(d_E(y,x),\min\{\lambda,\gamma\})$
$\Rightarrow d_{FE}((x,\lambda),(y,\gamma))=d_{FE}((y,\gamma),(x,\lambda))$
iv. Let $(x,\lambda),\ (y,\gamma),\ (z,\rho)\in L$ be arbitrary three fuzzy points.
Since (R^n,d_E) is a metric space, s. t. $d_E(x,z)\le d_E(x,y)+d_E(y,z)$ (7)
$If\ \lambda\in F, then\ \ y=(1-\lambda)x+\lambda z$
Let $\alpha=\min\{\lambda,\rho\}$. Then we have $\{x,z\}\subset L_\alpha$
Since L is a fuzzy linear space, then $y\in L_\alpha\ s.t.\ \ \gamma=L(y)\ge\alpha=\min\{\lambda,\rho\}$
This implies that $\min\{\lambda,\gamma,\rho\}=\min\{\lambda,\rho\}$ (8)
$\therefore\ d_{FE}((x,\lambda),(z,\rho))=(d_E(x,z),\min\{\lambda,\rho\})$
$\Rightarrow d_{FE}((x,\lambda),(z,\rho))\le(d_E(x,y)+d_E(y,z),\min\{\lambda,\gamma,\rho\})$
$\Rightarrow d_{FE}((x,\lambda),(z,\rho))\le(d_E(x,y),\min\{\lambda,\gamma,\rho\})+(d_E(y,z),\min\{\lambda,\gamma,\rho\})$
$\Rightarrow d_{FE}((x,\lambda),(z,\rho))\le(d_E(x,y),\min\{\lambda,\gamma\})+(d_E(y,z),\min\{\gamma,\rho\})$
$\Rightarrow d_{FE}((x,\lambda),(z,\rho))\le d_{FE}((x,\lambda),(y,\lambda))+d_{FE}((y,\gamma),(z,\rho))$
Therefore (L,d_{FE}) is a strong fuzzy metric space.
4. 2. Problem: Let (X,d) be a fuzzy metric space and $r\in R^+$, then show that (X,d_1) is a metric space, Where d_1 is define by $d_1(x,y)=\sum_{k=1}^{\infty}\frac{d(x_k,\ y_k)}{1+rd(x_k,\ y_k)}$
Solution:
i) We have $d(x_k,y_k)\ge 0\quad\forall\ k=1,2,3,\dots\dots\dots$
$\Rightarrow\ \frac{d(x_k,y_{k\_}}{1+rd(x_k,y_k)}\ge 0$
$\Rightarrow d_1(x,y)\ge 0$
ii) Let $d_1(x,y)=0$.
Now we will prove that $x=y$
Then we have $\sum_{k=1}^{\infty}\frac{d(x_k,y_k)}{1+rd(x_k,y_k)}=0$
$\Rightarrow\frac{d(x_k,y_k)}{1+rd(x_k,y_k)}=0$
$\Rightarrow\ d(x_k,y_k)=0$
$\Rightarrow|x_k-y_k|=0$
$\Rightarrow\ x_k-y_k=0$
$\Rightarrow\ x_k=y_k$
$\Rightarrow x=y$
Conversely let $x=y$.
Now we will prove that $d_1(x,y)=0$
$\Rightarrow\ x_k=y_k$
$\Rightarrow\ x_k=y_k$
$\Rightarrow|x_k-y_k|=0$
$\Rightarrow\ d(x_k,y_k)=0$
$\Rightarrow\frac{d(x_k,y_k)}{1+rd(x_k,y_k)}=0$

$\Rightarrow \sum_{k=1}^{\infty} \frac{d(x_k, y_{k\_}}{1 + rd(x_k, y_k)} = 0$
$\Rightarrow d_1(x, y) = 0$
So $d_1(x, y) = 0 \Leftrightarrow x = y$
iii) Since (X, d) is a matrix space, so $d(x_k, y_k) = d(x_k, y_k)$
Then we have $d_1(x, y) = \sum_{k=1}^{\infty} \frac{d(x_k, y_k)}{1+rd(x_k, y_k)} = \sum_{k=1}^{\infty} \frac{d(y_k,\ x_k)}{1+rd(y_k,\ x_k)}$
$d_1(x, y) = d_1(y, x)$
iv) Define $f: R^+ \longrightarrow R^+$ by $f(t) = \frac{t}{1+rt} (t \epsilon R^+)$
$f^{'}(t) = \frac{1 + rt - rt}{(1 + rt)^2} = \frac{1}{(1 + rt)^2}$
So, f is strictly increasing on R^+, Also, we have $d_1(x, y) = d(x_k, z_k) + d(z_k, y_k)$
This means $d_1(x, y) = \sum_{k=1}^{\infty} \frac{d(x_k, y_k)}{1+rd(x_k, y_k)} \leq \sum_{k=1}^{\infty} [\frac{d(x_k, z_k)+d(z_k, y_k)}{1+r(d(x_k, z_k)+d(z_k, y_k))}]$
$\leq \sum_{k=1}^{\infty} \frac{d(x_k,\ z_k)}{1 + rd(x_k,\ z_k)} + \sum_{k=1}^{\infty} \frac{d(z_k,\ y_k)}{1 + rd(z_k,\ y_k)}$
$\leq d_1(x, z) + d_1(z, y)$
$\Rightarrow d_1(x, y) \leq d_1(x, z) + d_1(z, y)$
Hence, $d_1(x, y)$is a matrix space with the given meric.

4. 3. Problem: Let $(L, \|.\|)$ be a fuzzy linear normed space. Prove that the fuzzy norm in an inner products space L which holds the following conditions:

i) $\|(x, \lambda)\| \geq 0$
ii) $\|(x, \lambda)\| = 0\ if\ and\ only\ if\ x = 0\ and\ \lambda = 1$
iii) $\|(x, \lambda)\| = |k|. \|(x, \lambda)\|\ \forall\ k \in \mathbb{R}$
iv) $\|(x, \lambda) + (y, \gamma)\| \leq \|(x,\ \lambda)\| + \|(x, \gamma)\|$

Where $(x, \lambda), (y, \gamma) \in L$

Solution:

i) By the definition of linear product, we get
$\langle (x, \lambda), (x, \lambda) \rangle \geq 0$
$\Rightarrow \|(x, \lambda)\|^2 \geq 0$
$\Rightarrow \|(x, \lambda)\| \geq 0$
ii) Let $\|(x, \lambda)\| = 0$

Then we have
$\langle (x, \lambda), (x, \lambda) \rangle = 0$
$\Leftrightarrow \langle (x, \lambda) - 1, (x, \lambda) \rangle = 0$
$\Leftrightarrow (x, \lambda) - 1 = 0\ and\ (x, \lambda) = 0 \Rightarrow\ x = 0$
$\Leftrightarrow (x, \lambda) = 1$
$\lambda = 1$
iii) We have $\|k\ (x, \lambda)\|^2 = \langle k(x, \lambda), k(x, \lambda) \rangle$

$= k\ \langle (x, \lambda), k(x, \lambda) \rangle$
$= k\ \overline{\langle k(x, \lambda), (x, \lambda) \rangle}$
$= k\ \langle \overline{k(x, \lambda)}, \overline{(x, \lambda)} \rangle$
$= k\bar{k} \langle \overline{(x, \lambda)}, \overline{(x, \lambda)} \rangle$
$= |k|^2 \overline{\langle (x, \lambda), (x, \lambda) \rangle}$
$= |k|^2 \langle (x, \lambda), (x, \lambda) \rangle$
$= |k|^2 \|(x, \lambda)\|^2$
$\Rightarrow \|k\ (x, \lambda)\| = |k| \|x, \lambda\|$
iv) We have $\|(x, \lambda) + (y, \lambda)\|^2 = \langle (x, \lambda) + (y, \gamma), (x,\ \lambda) + (y, \gamma) \rangle$

$= \langle (x, \lambda), (x, \lambda) + (y, \gamma) \rangle + \langle (y, \gamma), (x, \gamma) + (y, \gamma) \rangle$
$= \overline{\langle (x, \lambda) + (y, \gamma), (x, \lambda) \rangle} + \overline{\langle (x, \gamma) + (y, \gamma), (y, \gamma) \rangle}$
$= \langle \overline{(x, \lambda) + (y, \gamma)}, \overline{(x, \lambda)} \rangle + \langle \overline{(x, \gamma) + (y, \gamma)}, \overline{(y, \gamma)} \rangle$
$= \langle \overline{(x, \lambda)} + \overline{(y, \gamma)}, \overline{(x, \lambda)} \rangle + \langle \overline{(x, \gamma)} + \overline{(y, \gamma)}, \overline{(y, \gamma)} \rangle$
$= \langle \overline{(x, \lambda)}, \overline{(x, \lambda)} \rangle + \langle \overline{(y, \gamma)}, \overline{(x, \lambda)} \rangle + \langle \overline{(x, \lambda)}, \overline{(y, \gamma)} \rangle + \langle \overline{(y, \gamma)}, \overline{(y, \gamma)} \rangle$
$= \overline{\langle (x, \lambda), (x, \lambda) \rangle} + \overline{\langle (y, \gamma), (x, \lambda) \rangle} + \overline{\langle (x, \lambda), (y, \gamma) \rangle} + \overline{\langle (y, \gamma), (y, \gamma) \rangle}$
$= \langle (x, \lambda), (x, \lambda) \rangle + \langle (y, \gamma), (x, \lambda) \rangle + \overline{\langle (x, \lambda), (y, \gamma) \rangle} + \langle (y, \gamma), (y, \gamma) \rangle$

$= \|(x,\lambda)\|^2 + 2\, Re\, \langle (x,\lambda),(y,\gamma)\rangle + \|(y,\gamma)\|^2$

$\Rightarrow\ \|(x,\lambda)+(y,\lambda)\|^2 = \|(x,\lambda)\|^2 + 2\ |\langle (x,\lambda),(y,\gamma)\rangle| + \|(y,\gamma)\|^2$ (9)

For Cauchy- Schwartz inequality (10)

We have $|\langle (x,y)\rangle| \le \|x\|\|y\|$

Form (9), then we gets, $\|(x,\lambda)+(y,\lambda)\|^2 \le \|(x,\lambda)\|^2 + 2\ \|(x,\lambda)\|\|(y,\gamma)\| + \|(y,\gamma)\|^2$

$\Rightarrow\ \|(x,\lambda)+(y,\lambda)\|^2 \le (\|(x,\lambda)\| + \|(y,\gamma)\|)^2$

$\Rightarrow \|(x,\lambda)+(y,\lambda)\| \le \|(x,\lambda)\| + \|(y,\gamma)\|$

Proposition: Suppose $\left(P, \|.\|_{XY}\right)$ is a fuzzy linear normed space in R^n. For any $(x,\lambda) \in P$

Such that $\langle (x,\lambda),(x,\lambda)\rangle = \|(x,\lambda)\|^2{}_{XY}$, i.e. $\langle (x,\ \lambda),\ (y,\ \gamma)\rangle = \left(\langle x,\ y\rangle,\ \min\{\lambda,\ \gamma\}\right)$

Proof: By definition of inner product space,

$$\langle (x,\ \lambda),\ (x,\ \lambda)\rangle = \left(\langle x,\ x\rangle,\ \lambda\right)$$

$$\Rightarrow \langle (x,\ \lambda),\ (x,\ \lambda)\rangle = \left(\|x\|^2{}_E,\ \lambda\right)$$

$$\Rightarrow \langle (x,\ \lambda),\ (x,\ \lambda)\rangle = \left(\|x\|_E,\ \lambda\right)\left(\|x\|_E,\ \lambda\right)$$

$$\Rightarrow \langle (x,\ \lambda),\ (x,\ \lambda)\rangle = \|(x,\ \lambda)\|^2{}_{FE}$$

Conclusion

In this paper, we introduce and study a concept of strong and weak convergence in fuzzy metric spaces $\left(P_F(X), d_F\right)$ such that $P_F(X)$ is a set. Here this paper is to investigate for which of these concepts of a characterization of strong and weak convergence in fuzzy metric space. We hope that this work will be useful for fuzzy metric space related to normed spaces. All expected results in this paper are a better solution of different fuzzy metric space related theorem. In future, we will discuss of strong and weak convergence in fuzzy metric space properties related to physical problems.

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Author contributions

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