SOME APPLICATIONS OF SEPARATION AXIOMS ON ROUGH SETS

A. I. EL-MAGHRABI[a], SAEID JAFARI[b], RAJA MOHAMMAD LATIF[c], A. M. MUBARAKI[d]

[a] Department of Mathematics, Faculty of Science

Kafr El-Sheikh University Kafr El-Shaikh Egypt

E-mail: aelmaghrabi@yahoo.com

[b]Dr.rer.nat in Mathematics
(Graz University of Technology-Graz, Austria)
Professor of Mathematics
College of Vestsjaelland South, Herrestarede 11
Mathematical and Physical Science Foundation
4200 Slagelse Denmark
E-mails: jafaripersia@gmail.com&saeidjafari@topositus.com

[c]Department of Mathematics and Natural Sciences

Prince Mohammad Bin Fahd University

Al – Khobar Saudi Arabia

E-mails: rlatif@pmu.edu.sa&rajamlatif@gmail.com&dr.rajalatif@yahoo.com

[d]Department of Mathematics

Faculty of Science, Taibah University

Al-Madinah Al-Munawarah Saudi Arabia

E-mail: alimobaraki@hotmail.com

Article History: *Do not touch during review process(xxxx)*

Abstract: Rough set theory $[1,2]$ is one of the new methods that connect information systems and data processing to mathematics in general and especially to the theory of topological structures and spaces.This paper is aimed to improve some new types of separation axioms by using the concept of Z-open sets. So, we define new separation axioms called Z-T_k-Spaces, $k=0,1,2$. Also, some of its properties are investigated. Further, we defined Z-regular space and studied the relation between this space and some types of mappings. Furthermore, we introduce Z-normal spaces and several properties of this space are presented.

Keywords: Z-open set, Z-T_0-Space, Z-T_1-Space, Z-T_2-Space, Z-regular-Space, Z-normal-Space

2020 AMS Subject Classification:54D10; 54C05; 54C08

---

1. INTRODUCTION

N. El-Deeb, et.al $[3]$ introduced the notion p-regular spaces. The concept of *p-normal* $($*resp. pre-normal*$)$ spaces introduced by Paul, et.al $[4]$ (resp. T. M. J. Nour $[5]$). In this Paper, we define and study the notions of Z-T_k-Spaces, $k=0,1,2$. Also, some properties and characterizations of their notion are discussed. Further, we introduce and study the concept of Z-regular and Z-normal spaces. Also, some types of separation axioms had been applied in modifications of rough set approximations $[6,7]$ which is widely applied in many application fields.Furthermore, several properties of these spaces are presented and the relations between these spaces and some types of mappings are investigated.

2. PRELIMINARIES

A subset A of a topological space (X,τ) is called regular open (resp. regular closed) [8] if $A=Int[Cl(A)]$ (*resp.* $A=Cl[Int(A)]$). The δ-*interior* [9] of a subset A of X is the union of all regular open sets of X contained of A is denoted by δ-$Int(A)$. A subset A of a space X is called δ-*open* if it is the union of regular open sets. The complement of δ-*open* set is called δ-*closed*. Alternatively, a subset A of X is called δ-*closed* [9] if $A=\delta\text{-}Cl(A)$, where $\delta\text{-}Cl(A)=\{x\in X : A\cap Int[Cl(U)]\neq\phi, U\in\tau\ \&\ x\in U\}$. Throughout this paper (X,τ) and (Y,σ) (*or Simply* X *and* Y) represent non-empty topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X,τ), $Cl(A)$, $Int(A)$ and $X-A$ denoted the closure of A, the interior of A and the complement of A respectively. A subset A of a space X is called a-*open* [10] (*resp. preopen* [11], δ-*semiopen* [12], Z-*open* [13]), if $A\subseteq Int[Cl(\delta\text{-}Int(A))]$ (*resp.* $A\subseteq Int[Cl(A)]$, $A\subseteq Cl[\delta\text{-}Int(A)]$, $A\subseteq Cl[\delta\text{-}Int(A)]\cup Int[Cl(A)]$). The complement of an a-*open* (*resp. preopen*, δ-*semiopen*) set is called a-*closed* (*resp. pre-closed*, δ-*semi-closed*). The intersection of all Z-*closed* sets containing A is called the Z-*closure* of A and is denoted by $Z\text{-}Cl(A)$. The union of all Z-*open* sets contained in A is called the Z-*interior* of A and is denoted by $Z\text{-}Int(A)$. The family of all Z-*open* (*resp.* Z-*closed*) sets is denoted by $ZO(X)$ (*resp.* $ZC(X)$).

Lemma 2.1. [13] If $A\in aO(X,\tau)$ and $B\in ZO(X,\tau)$, then $A\cap B\in ZO(A,\tau_A)$.

Definition 2.1. A space X is said to be *p-regular* [3] (*resp.* δs-*regular*) if for each closed set F and each point $x\in X-F$, there exist disjoint *pre-open* (*resp.* δ-*semiopen*) sets U and V such that $F\subseteq U$ and $x\in V$.

Definition 2.2. A space X is said to be pre-normal [5] or *p-normal* [4] (*resp.* δs-*normal*) if for any pair of disjoint closed sets A and B, there exist disjoint preopen (*resp.* δ-*semiopen*) sets U and V such that $A\subseteq U$ and $B\subseteq V$.

Definition 2.3. [13] A mapping $f:(X,\tau)\to(Y,\sigma)$ is called Z-*continuous* if, for each open set U of (Y,σ), $f^{-1}(U)$ is Z-*open* in (X,τ).

Definition 2.4. [14] A mapping $f:(X,\tau)\to(Y,\sigma)$ is called:

(1) Z-*open* if $f(V)\in ZO(Y)$, for each open set V in X.

(2) Z-*closed* if $f(V)\in ZC(Y)$, for each closed set V in X.

(3) *pre-*Z-*open* if $f(V)\in ZO(Y)$, for each $V\in ZO(X)$.

(4) *pre-*Z-*closed* if $f(V)\in ZC(Y)$, for each $V\in ZC(X)$.

(5) Z-*irresolute* if $f^{-1}(V)\in ZO(X)$, for each $V\in ZO(Y)$.

Lemma 2.2. [14] Let $f:(X,\tau)\to(Y,\sigma)$ be a *pre-Z-closed* function and $B,C\subseteq Y$. If U is a *Z-open* set containing $f^{-1}(B)$, then there exists a *Z-open* set V containing B such that $f^{-1}(B)\subseteq f^{-1}(V)\subseteq U$.

3. SEPARATION AXIOMS VIA Z – OPEN SETS

Definition 3.1. A space (X,τ) is said to be:

(i) *Z-T_0-Space* if every two distinct points $x,y\in X$, there exists a *Z-open* set $U\subseteq X$ such that either $x\in U$, $y\notin U$ or $y\in U$, $x\notin U$.

(ii) *Z-T_1-Space* [14] if, for any pair of distinct points x, y of X, there exist two *Z-open* sets U,V of X such that $x\in U$, $y\notin U$ and $y\in V$, $x\notin V$.

(iii) *Z-T_2-Space* [14] if, for every two distinct points $x,y\in X$, there exist two disjoint *Z-open* sets U,V such that $x\in U$, $y\in V$.

Remark 3.1. (i) The following diagram is shown that the relation between spaces *Z-T_i-Space*, where $i=0,1,2$.

$$\begin{array}{ccccc}
T_2\text{-Space} & \longrightarrow & T_1\text{-Space} & \longrightarrow & T_0\text{-Space} \\
\downarrow & & \downarrow & & \downarrow \\
Z\text{-}T_2\text{-Space} & \longrightarrow & Z\text{-}T_1\text{-Space} & \longrightarrow & Z\text{-}T_0\text{-Space}
\end{array}$$

(ii) None of the above implications is reversible as is shown by the following examples.

Remark 3.2. The following example is an application of some types of separation axioms in the concept of the rough set approximations.

Example 3.1. If we have the following information system, where the objects $U=\{x_1,x_2,x_3,x_4\}$ represent, the attribute of Mathematics and the values of the numbers scored by the students in the examination given by the following table.

$Object(U)$	x_1	x_2	x_3	x_4
$MA(V)$	86	91	88	92

Also, the relation R on the set of objects is defined by xRy if and only if $|V(x)-V(y)|\le 3$. Hence, we get the following classifications corresponding to every subclass of the attributes for Mathematics: $x_1R=x_3R=\{x_1,x_3\}$, $x_2R=x_3R=\{x_2,x_3\}$, $x_2R=x_4R=\{x_2,x_4\}$. Then $S_V=\{\{x_1,x_3\},\{x_2,x_3\},\{x_2,x_4\}\}$ and hence the topology generated by the previous classes is given by $\tau_{V1}=\{U,\phi,\{x_2\},\{x_3\},\{x_2,x_3\},\{x_2,x_4\},\{x_2,x_3,x_4\}\}$. So, this space is a *Z-T_0-Space* but it is not *Z-T_1-Space.*

Example 3.2. Let $X=\{a,b,c,d\}$ and let $\tau=\{\phi,\{a\},\{c\},\{a,b\},\{a,c\},\{a,b,c\},\{a,c,d\},X\}$ be a topology on X. Then (X,τ) is a *Z-T_1-Space* but it is not *Z-T_2-Space.*

Example 3.3. Let $X=\{a,b,c\}$ with topology $\tau=\{\phi,\{a\},X\}$. Then, (X,τ) is *Z-T_0-Space* but it is not *T_0-Space*. Since for $b,c\in X$ with $b\ne c$, there does not exist an open set containing one of them but not containing the other.

Example 3.4. Let $X=\{a,b,c,d\}$ with topology $\tau=\{\phi,\{a\},\{b\},\{a,b\},X\}$. Then (X,τ) is Z-T_2-*Space* $(resp.\ Z\text{-}T_1\text{-}Space)$, but it is not T_2-*Space* $(resp.\ T_1\text{-}Space)$, Since for $c,d\in X$ with $c\neq d$, there do not exist two disjoint open sets containing b, c.

Theorem 3.1. Let (X,τ) be a topological space. Then, the following statements are equivalent:

(i) X is Z-T_0-*Space.*

(ii) for every two distinct points $x,y\in X$, then $Z\text{-}Cl(\{x\})\neq Z\text{-}Cl(\{y\})$.

Proof. (i)$\Rightarrow$(ii): Let x, y be any two distinct points of X. We show that $Z\text{-}Cl(\{x\})\neq Z\text{-}Cl(\{y\})$. Since (X,τ) is Z-T_0-*Space,* then there exists a Z-*open* set G of X such that $x\in G$ and $y\notin G$. Hence $y\in X-G$ and $X-G$ is a Z-*closed* set. So, $y\in Z\text{-}Cl(\{y\})$ as $x\notin X-G$. Therefore $Z\text{-}Cl(\{x\})\neq Z\text{-}Cl(\{y\})$.

(ii)$\Rightarrow$(i): Suppose that for every $x,y\in X$, $x\neq y$ and $Z\text{-}Cl(\{x\})\neq Z\text{-}Cl(\{y\})$. Let $z\in X$ such that $z\in Z\text{-}Cl(\{x\})$, hence $z\notin Z\text{-}Cl(\{y\})$. If $x\in Z\text{-}Cl(\{y\})$, then $\{x\}\subseteq Z\text{-}Cl(\{y\})$ which implies that $Z\text{-}Cl(\{x\})\subseteq Z\text{-}Cl(\{y\})$ and hence $z\in Z\text{-}Cl(\{y\})$ which is a contradiction, thus $x\notin Z\text{-}Cl(\{y\})$ which implies that $x\in[Z\text{-}Cl(\{y\})]^c$ and hence $[Z\text{-}Cl(\{y\})]^c$ is a Z-*open* set containing x but not y. Therefore, X is Z-T_0-*Space.*

Theorem 3.2. Let (X,τ) be a topological space. Then the following statements are equivalent:

(i) X is Z-T_1-*Space.*

(ii) $A=\bigcap\{N : N \text{ is a } Z\text{-neighborhood of } x\}$, for any subset A of X.

(iiii) $\{x\}=\bigcap\{N : N \text{ is a } Z\text{-neighborhood of } x\}$, for every $x\in X$.

(iv) If $x\in X$, then $\{x\}$ is Z-*closed*.

(v) $Z\text{-}d(\{x\})=\phi$, for every point $x\in X$.

Proof. (i)$\Rightarrow$(ii): Let $A\subseteq X$ and W be the intersection of a Z-*neighborhood* of A such that $A\subseteq W$. Also, suppose that $x\in A$, $y\in X-A$ and $y\in W$, then there exists a Z-*open* neighborhood N_x of X and not containing y. Assume that $U=\bigcup\{N_x : x\in A\}$. Then U is a Z-*open* neighborhood of A not containing y. But, $W\subseteq U$, hence $y\notin W$ which is a contradiction with assumption. Therefore, $A=W$.

(ii)$\Rightarrow$(iii): Obvious,

(iii)$\Rightarrow$(iv): Let x, y be two distinct points of X. Then, there exists a Z-*neighborhood* U_y of y which does not contain x. Hence, $y\notin\{x\}$, thus $y\in U_y\subseteq X-\{x\}$. So, $X-\{x\}=\bigcup\{U_y : U_y \text{ is } Z\text{-open}, y\in X-\{x\}\}$ whichis the union of Z-*open* sets. Therefore, $\{x\}$ is a Z-*closed* subset of X, for each $x\in X$.

(iv)$\Rightarrow$(i): Let $x, y\in X$ with $x\neq y$. Then $\{x\}^c$ is a Z-*open* set containing y but not x. Similarly. $\{y\}^c$ is a Z-*open* set containing x but not y. Therefore X is a Z-T_1-*Space.*

$(\text{iv}) \Leftrightarrow (\text{v})$: Obvious.

Theorem 3.3. Let X be a topological space. Then the following statements are equivalent:

(i) X is Z-T_2-*Space*.

(ii) for each $x \in X$ with $y \neq x$, there is a Z-*open* set U containing x such that $y \notin Z\text{-}Cl(U)$.

(iii) For each $x \in X$, $\{x\} = \bigcap\{Z\text{-}Cl(U) : U \text{ is a } Z\text{-open set containing } x\}$.

Proof. $(\text{i}) \Rightarrow (\text{ii})$: For each $x \in X$ with $x \neq y$, then by hypothesis, there are Z-*open* sets U, V such that $x \in U$, $y \in V$ and $U \cap V = \phi$, hence, $x \in U \subseteq X - V$. Put $F = X - V$, then F is Z-*closed*, $U \subseteq F$ and $y \notin F$. This implies that $y \notin \bigcap\{F : F \text{ is } Z\text{-closed and } U \subseteq F\} = Z\text{-}Cl(U)$

$(\text{ii}) \Rightarrow (\text{i})$: Let $x, y \in X$ and $x \neq y$. Then by (ii), there is U a Z-*open* set U containing x such that $y \notin Z\text{-}Cl(U)$. Thus $y \in X - [Z\text{-}Cl(U)]$, $X - [Z\text{-}Cl(U)]$ is Z-*open* and $U \cap [X - (Z\text{-}Cl(U))] = \phi$. Therefore, (X, τ) is Z-T_2-*Space*.

$(\text{iii}) \Leftrightarrow (\text{i})$: Obvious.

Theorem 3.4. Every a-*open* subspace of a Z-T_i-*Space* is a Z-T_i-*Space*, where $\text{i} = 0, 1, 2$.

Proof. We prove that the theorem for Z-T_2-*Space*. Let Y be an a-*open* subspace of a Z-T_2-*Space* and x, y be two distinct points of Y. Then, x, y are two distinct points of X. But X is Z-T_2-*Space*, then there exist two disjoint Z-open sets U, V containing x, y respectively. Suppose that $U_1 = Y \cap U$ and $V_1 = Y \cap V$. Then by Lemma 2.1, U_1, V_1 are Z-*open* sets of Y containing x, y respectively and $U_1 \cap V_1 = Y \cap (U \cap V) = \phi$. Therefore, (Y, τ_Y) is a Z-T_2-*Space*.

Remark 3.3. The following example shows that the property of being a Z-T_2-*Space* is not hereditary.

Example 3.5. In Example 3.4, Let $A = \{b,c,d\} \notin a\text{-}O(X, \tau)$ with a topology on A to be $\tau_A = \{\phi, \{b\}, \{b,c\}, \{b,d\}, A\}$. Then (A, τ_A) is not Z-T_2-*Space*. Since $b, c \in X$ with $b \neq c$, there do not exist two disjoint Z-*open* sets of A such that one containing b and the other containing c.

Theorem 3.5. Let $f : (X, \tau) \to (Y, \sigma)$ be an injective Z-*irresolute* mapping and Y is a Z-T_i-*Space*. Then X is Z-T_i, where $\text{i} = 0, 1, 2$. $\text{i} = 0, 1, 2$.

Proof. We prove that the theorem for a Z-T_0-*Space*. Let $x, y \in X$ with $x \neq y$ and Y be a Z-T_0-*Space*. Then there exists a Z-*open* set G of Y such that either $f(x) \in G$ and $f(y) \in G$ with $f(x) \neq f(y)$. Then by using injective Z-*irresoluteness* of f, we conclude that $f^{-1}(G)$ is a Z-*open* set of X such that either $x \in f^{-1}(G)$, or $y \in f^{-1}(G)$. Therefore, X is a Z-T_0-*Space*.

Definition 3.2. A mapping $f : (X, \tau) \to (Y, \sigma)$ is called strongly Z-*irresolute* if for each Z-*open* set U of (Y, σ), $f^{-1}(U)$ is open in (X, τ).

Proposition 3.1. Let $f:(X,\tau)\to(Y,\sigma)$ be an injective strongly Z-*irresolute* mapping and Y be a Z-T_i-*Space*. Then, (X,τ) is T_i, where $\mathrm{i}=0,1,2$.

Theorem 3.6. Let $f:(X,\tau)\to(Y,\sigma)$ be bijective Z-*open* mapping and (X,τ) is T_i-*Space*. Then (Y,σ) is Z-T_i, where $\mathrm{i}=0,1,2$.

Proof. We prove that the theorem for a Z-T_0-*Space*. Let y_1, y_2 be distinct points of Y. Then there exist x_1, x_2 in X such that $f(x_1)=y_1$ and $f(x_2)=y_2$. But X is T_0-*Space*, then there exists an open set U of X such that either $x_1\in U$ or $x_2\in U$. Since f isbijective Z-*open*, then $f(U)$ is Z-*open* set of Y such that either $y_1\in f(U)$ or $y_2\in f(U)$. Therefore, Y is a Z-T_0-*Space*.

Definition 3.3. A mapping $f:(X,\tau)\to(Y,\sigma)$ is called quasi Z-*open* if, for each Z-*open*, set U of (X,τ), $f(U)$ is open in (Y,σ).

Theorem 3.7. Let $f:(X,\tau)\to(Y,\sigma)$ be a bijective quasi Z-*open* mapping and (X,τ) is a Z-T_i-*Space*. Then, (Y,σ) is T_i, where $\mathrm{i}=0,1,2$.

Proof. Obvious.

Theorem 3.8. Let $f:(X,\tau)\to(Y,\sigma)$ be a bijective *pre*-Z-*open* mapping and (X,τ) is a Z-T_i-*Space*. Then, (Y,σ) is Z-T_i, where $\mathrm{i}=0,1,2$.

Proof. Obvious.

4. Z-REGULAR SPACES

Definition 4.1. A space (X,τ) is said to be Z-*regular* if, for each closed set F and each point $x\in X-F$, then there exist two disjoint Z-*open* sets U and V such that $F\subseteq U$ and $x\in V$.

Remark 4.1. Every δs-*regular* ($resp.$ p-*regular*) space is Z-*regular*. But the converse is not true in general as is shown by the following example.

Example 4.1. Let $X=\{a,b,c,d\}$ with topology $\tau=\{X,\phi,\{a\},\{b,c\},\{a,b,c\}\}$. Then space (X,τ) is Z-*regular* but it is not δs-*regular* and it is not p-*regular*.

Theorem 4.1. Let (X,τ) be a topological space. Then the following statements are equivalent:

(i) X is Z-*regular* space.

(ii) For each closed set $F\subseteq X$ and $p\in X-F$, there exists a Z-*open* set U such that $p\in U\subseteq Z\text{-}Cl(U)\subseteq X-F$.

Proof. (i)$\Rightarrow$(ii): Let X be a Z-*regular* space, $F\subseteq X$ and $p\in X-F$. Then there exist Z-*open* sets U, V such that $p\in U$, $F\subseteq V$ and $U\cap V=\phi$, hence $F\subseteq V=X-[Z\text{-}Cl(U)]$ which implies that $p\in U\subseteq Z\text{-}Cl(U)\subseteq X-F$.

$(\text{ii}) \Rightarrow (\text{i})$: Let $p \in X$ and $F \subseteq X - \{p\}$ be a closed set. Then by hypothesis, there exists a Z-*open* set U of X such that $p \in U \subseteq Z\text{-}Cl(U) \subseteq X - F$ and hence $F \subseteq X - [Z\text{-}Cl(U)]$ is a Z-*open* set of X disjoint with U. Therefore, X is a Z-*regular* space.

Theorem 4.2. In a Z-*regular* space (X, τ), for any two points x, y of X, then either $Cl(\{x\}) = Cl(\{y\})$ or $Cl(\{x\}) \cap Cl(\{y\}) = \phi$.

Proof. Assume that $Cl(\{x\}) \neq Cl(\{y\})$. Then either $x \notin Cl(\{y\})$ or $y \notin Cl(\{x\})$. Suppose $y \notin Cl(\{x\})$. Since X is Z-*regular*, then there exists a Z-*open* set G of X such that $Cl(\{x\}) \subseteq G$ and $y \in X - G$. But $X - G$ is Z-*closed* and $Cl(\{y\}) \subseteq X - G$. Hence, $Cl(\{x\}) \cap Cl(\{y\}) \subseteq G \cap (X - G) = \phi$.

Theorem 4.3. Every a-*open* subspace of a Z-*regular* space is Z-*regular*.

Proof. Let Y be an a-*open* subspace of a Z-*regular* space and F be a closed set of Y with $x \notin F$. Then there exist two disjoint Z-*open* sets U, V such that $F \subseteq U$ and $x \in V$. Let $U_1 = Y \cap U$ and $V_1 = Y \cap V$. Hence by Lemma 2.1, U_1, V_1 are $Z-open$ subsets of Y and $F \subseteq U_1$, $x \in V_1$ and $U_1 \cap V_1 = Y \cap (U \cap V) = \phi$. Therefore, (Y, τ_Y) is Z-*regular*.

Remark 4.2. The following example is shown that the property of being a Z-*regular* space is not hereditary.

Example 4.2. Consider Example 4.1, and let $A = \{a, c\} \notin$ a-$O(X, \tau)$ with topology on A given by $\tau_A = \{\phi, \{c\}, A\}$. (A, τ_A) is not Z-*regular* space. Since for a closed set $\{a\}$ of X and $c \notin \{a\}$, then there do not exist disjoint Z-*open* sets U and V of A such that $c \in U$ and $\{a\} \subseteq V$.

Theorem 4.4. If $f : (X, \tau) \to (Y, \sigma)$ is an injective Z-*irresolute* (*resp. strongly* Z-*irresolute*) and closed mapping, then X is Z-*regular* (*resp. regular*), if Y is a Z-*regular* space.

Proof. We prove the theorem for a Z-*regular* space. Let V be any closed set of X and $x \in X - V$. Since f is closed, then $f(V)$ is closed in Y and $f(x) \in Y - f(V)$. But Y is Z-*regular*, then there exist disjoint Z-*open* sets G, H such that $f(x) \in G$ and $f(V) \subseteq H$. Hence, by Z-*irresolute*, we obtain $x \in f^{-1}(G)$, $V \subseteq f^{-1}(H)$ and $f^{-1}(G) \cap f^{-1}(H) = \phi$. Therefore, X is Z-*regular*.

Theorem 4.5. If $f : (X, \tau) \to (Y, \sigma)$ is a bijective continuous and *pre*-Z-*open* mapping, then Y is Z-*regular*, if X is Z-*regular*,

Proof. Let F be any closed set of Y and $y \in Y - F$. Since, f is a bijective continuous map, then $f^{-1}(F)$ is closed in X and $f(x) = y$. Hence, $x \in X - f^{-1}(F)$. But X is Z-*regular*, then there exist disjoint Z-*open* sets G, H such that $x \in G$ and $f^{-1}(F) \subseteq H$. Since f is a bijective *pre*-Z-*open* map, then $y \in f(G)$, $F \subseteq f(H)$ and $f(G) \cap f(H) = \phi$. Therefore, Y is Z-*regular*.

Corollary 4.1. The property of being a Z-*regular* space is a topological property.

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a Z-*homeomorphism*. Then f is a *pre*-Z-*open* continuous mapping. Assume that $A \subseteq Y$ is a closed set and $y \in Y - A$. Then $f^{-1}(A)$ is a closed subset of X and $x \in X - f^{-1}(A)$.

But X is Z-*regular*, then there exist disjoint Z-*open* sets G, H such that $x \in G$ and $f^{-1}(A) \subseteq H$. By using *pre-Z-open*, we obtain $y \in f(G)$, $A \subseteq f(H)$ and $f(G) \cap f(H) = \phi$. Therefore, Y is a Z-*regular* space.

5. Z-NORMAL SPACES

Definition 5.1. A space X is said to be Z-*normal* if for any two pairs of disjoint closed sets A and B, there exist two disjoint Z-*open* sets U and V such that $A \subseteq U$ and $B \subseteq V$.

Remark 5.1. (i) Every δs-*normal* (*resp. p-normal*) space is Z-*normal*.

(ii) Every Z-*regular* space is Z-*normal*.

The following examples illustrate that the converse of the above remark is not true.

Example 5.1. Let $X = \{a,b,c,d\}$ with topology $\tau = \{X, \phi, \{b,d\}, \{a,b,d\}, \{b,c,d\}\}$. Then space (X, τ) is Z-*normal* but it is not δs-*normal*.

Example 5.2. Let $X = \{a,b,c,d\}$ with topology $\tau = \{X, \phi, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}, \{a,b,d\}\}$. Then space (X, τ) is Z-*normal* but it is not *p-normal*.

Example 5.3. Let $X = \{a,b,c\}$ with topology $\tau = \{X, \phi, \{a\}, \{b\}, \{a,b\}. \{a,c\}\}$. Then space (X, τ) is Z-*normal* but it is not Z-*regular*.

Theorem 5.1. Let (X, τ) be a topological space. Then the following statements are equivalent:

(i) X is Z-*normal* space.

(ii) For every open sets U, V of X whose union is X, there exist Z-*closed* sets A and B such that $A \subseteq U$, $B \subseteq V$ and $A \cup B = X$.

(iii) For every closed set H and every open set K containing H, there exists a Z-*open* set U of X such that $H \subseteq U \subseteq Z\text{-}Cl(U) \subseteq K$.

Proof. (i) $\Rightarrow$ (ii): Let U, V be two open sets of X such that $X = U \cup V$. Then $X - U$, $X - V$ are disjoint closed sets. But, X is Z-*normal*, then there exist disjoint Z-*open* sets U_1, V_1 such that $X - U \subseteq U_1$ and $X - V \subseteq V_1$. Assume that $A = X - U_1$, $B = X - V_1$. Then A, B are Z-*closed* sets such that $A \subseteq U$, $B \subseteq V$ and $A \cup B = X$.

(ii) $\Rightarrow$ (iii): Let H be a closed set of X and K be an open set containing H. Then $X - H$ and K are open sets whose union is X. Then by (ii), there exist Z-*closed* sets M_1, M_2 such that $M_1 \subseteq X - H$, $M_2 \subseteq K$ and $M_1 \cup M_2 = X$. Hence $H \subseteq X - M_1$, $X - K \subseteq X - M_2$ and $(X - M_1) \cap (X - M_2) = \phi$. Put $U = X - M_1$ and $V = X - M_2$. Then U and V are disjoint Z-*open* sets of X such that $H \subseteq U \subseteq X - V \subseteq K$. Since $X - V$ is a Z-*closed* set of X, then $Z\text{-}Cl(U) \subseteq X - V$ and hence $H \subseteq U \subseteq [Z\text{-}Cl(U)] \subseteq K$.

(iii) $\Rightarrow$ (i): Let H_1, H_2 be two disjoint closed sets of X. If we put, $K = X - H_2$, then $H_2 \cap K = \phi$. But $H_1 \subseteq K$, where K is an open set of X. Then by (iii), there exists a Z-*open* set U of X such that

$H_1 \subseteq U \subseteq Z\text{-}Cl(U) \subseteq K$. It follows that $H_2 \subseteq X - [Z\text{-}Cl(U)] = V$. Hence V is a Z-*open* set of X and $U \cap V = \phi$. Hence H_1, H_2 are separated by Z-*open* sets U and V. Therefore, X is Z-*normal*.

Theorem 5.2. A Z-*normal* and a Z-T_1-*Space* is Z-*regular*.

Proof. Suppose that A is a closed set of X and $x \notin A$. Since X is a Z-T_1-*Space*, then $\{x\}$ is Z-*closed* in X. But X is Z-*normal*, then there exist Z-*open* sets U, V of X such that $x \in \{x\} \subseteq U$, $A \subseteq V$ and $U \cap V = \phi$, Therefore, X is Z-*regular*.

Theorem 5.3. Every a-*open* subspace of Z-*normal* space is Z-*normal*.

Proof. Obvious.

Remark 5.2. The following example shows that the property of being Z-*normal* space is not hereditary.

Example 5.4. In Example 5.1, let $A = \{a,b,c\} \notin$ a-$O(X,\tau)$ and $\tau_A = \{\phi,\{b\},\{a,b\},\{b,c\},A\}$. Then (A,τ_A) is not Z-*normal* space. Since the pair of closed sets $\{a\}$, $\{c\}$ of X have no disjoint Z-*open* sets of A.

Theorem 5.4. If $f:(X,\tau)\to(Y,\sigma)$ is a bijective continuous and Z-*open* mapping from a normal space X onto space Y, then Y is Z-*normal*.

Proof. Let A, B be two disjoint closed sets of Y and f be continuous mapping. Then, $f^{-1}(A)$, $f^{-1}(B)$ are disjoint closed sets of X, hence by the normality of X, there exist two disjoint open sets U, V such that $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V$. But, f is a bijective and Z-*open* map, then $A \subseteq f(U)$, $B \subseteq f(V)$ and $f(U) \cap f(V) = \phi$. Hence, Y is Z-*normal*.

Corollary 5.1. The property of being Z-*normal* space is a topological property.

Theorem 5.5. If $f:(X,\tau)\to(Y,\sigma)$ is a *pre-Z-closed* and continuous mapping from Z-*normal* space X onto space Y, then Y is Z-*normal*.

Proof. Let M_1, M_2 be two disjoint closed sets of Y and f be a continuous map, then $f^{-1}(M_1)$, $f^{-1}(M_2)$ are disjoint closed sets in X. But X is Z-*normal*, then there exist two disjoint Z-*open* sets U, V such that $f^{-1}(M_1) \subseteq U$ and $f^{-1}(M_2) \subseteq V$. Hence by Lemma 2.2, there exist two Z-*open* sets A, B such that $M_1 \subseteq A$, $M_2 \subseteq B$ and $A \cap B = \phi$. Therefore, Y is Z-*normal*.

Corollary 5.2. Let $f:(X,\tau)\to(Y,\sigma)$ be a surjective Z-*closed* and continuous mapping. Then Y is Z-*normal*, if X is normal.

Theorem 5.6. Let $f:(X,\tau)\to(Y,\sigma)$ be an injective Z-*irresolute* and closed mapping. Then X is Z-*normal*, if Y is a Z-*normal* space.

Proof. Let F_1, F_2 be two disjoint closed sets of X and f be a closed mapping. Then $f(F_1)$ and $f(F_2)$ are disjoint closed sets of Y. Hence by Z-*normality* of Y, then there exist two disjoint Z-*open* sets U, V such that $f(F_1) \subseteq U$ and $f(F_2) \subseteq V$. But f is an injective and Z-*irresolute* map, hence $F_1 \subseteq f^{-1}(U)$, $F_2 \subseteq f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \phi$. Therefore, X is Z-*normal*.

Proposition 5.1. Let $f:(X,\tau)\to(Y,\sigma)$ be an injective strongly Z-*irresolute* and closed mapping. Then X is normal, if Y is Z-*normal* space.

Proof. Similar to Theorem 5.6.

6. CONCLUSION

Rough set theory is one of the new methods that connect information systems and data processing to mathematics in general and especially to the theory of topological structures and spaces.This paper aims to improve some new types of separation axioms. Consequently, we introduce a modification of some new spaces by the concept ofsets. So, this manuscript is considered a starting point of much of the work in real-life applications.

ACKNOWLEDGMENT

This paper is dedicated to Professor Maximilian Ganster on the occasion of his retirement. The third author namely Dr. Raja Mohammad Latif is highly and gratefully indebted to Prince Mohammad Bin Fahd University, Saudi Arabia, for providing research facilities during the preparation of this research paper.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interest.

References

[1] Z. Pawlak, Rough sets, Int. J. Inform. Comp. Sci., 11(1982), pp.341-356.

[2] Z. Pawlak, Rough sets, Rough relations and Rough functions, Fundamental Informaticae., 27(1996), pp. 103-108.

[3] N. El-Deeb, I. A. Hasanein, A. S. Mashhour, and T. Noiri, On p-regular spaces, Bull. Math. Soc. Sci. Math. R.S. Romaine, 27(75) (1983), pp. 311-315.

[[4] Paul and Bhattacharyya, On p-normal spaces, Soochow J. Math., 21 (3) (1995), pp. 273-289.T. M. J. Nour, Contributions to the theory of bitopological spaces, Ph.D. Thesis, Delhi University, India, (1989).

[5] T. M. J. Nour, Contributions to the theory of bitopological spaces, Ph.D. Thesis. Delhi University, India, (1968).

[6] A. Galton: A generalized topological view of motion in discrete space, Theoretical Computer Science , 305 (2003), pp. 111 – 134.

[7] Z. Pawlak :Some remarks on conflict analysis, European Journal of Operational Research, 166 (2005), pp. 649-654.

[8] M. H. Stone, Application of the theory of Boolean rings to general topology, TAMS, 41 (1937), pp. 375 - 381.

[9] N. V. Velicko, H-closed topological spaces, Amer. Math. Soc. Transl., 78(1968), pp.103-118.

[10] E. Ekici, On a-open sets, A^*-sets and decompositions of continuity and super continuity, Annales Univ. Sci. Budapest, 51(2008), pp.39 -51.

[11] A. S. Mashhour; M. E. Abd El-Monsef and S. N. El-Deeb, On pre-continuous and weak pre-continuous mappings, Proc. Math. Phys. Soc. Egypt, 53 (1982), pp. 47-53.

[12] J. H. Park; B .Y. Lee and M. J. Son, On δ-semi-open sets in topological spaces, J. Indian Acad. Math., 19 (1) (1997), pp. 59 - 67.

[13] A. I. El-Maghrabi and A. M. Mubarki, Z-open sets and Z-continuity in topological spaces, International Journal of Mathematical Archive, 2(10), (2011), pp.1819-1827.

[14] A. I. El-Maghrabi and A. M. Mubarki, Characterizations of mappings via Z-open sets, Studies in Mathematical Sciences, 5 (2) (2012), pp. 55-65.