ON A CLOSURE SPACE VIA SEMI-CLOSURE OPERATOR

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Abstract: In this paper, we show that a pointwise symmetric semi-isotonic semi-closure function is uniquely determined by the pairs of sets it separates. We then show that when the semi-closure function of the domain is semi-isotonic and the semi-closure function of the codomain is semi-isotonic and pointwise-semi-symmetric, functions that separate only those pairs of sets which are already separated, are semi-continuous.

Keywords: *semi-closure −separated, semi-closure function, semi-continuous function, semi-connected.*

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1. INTRODUCTION

The most important legacy of Norman Levine [3] was the introduction of semi-open sets which is one of thewell-known notions of generalized open sets. Throughout the present paper (X,τ) and (Y,σ) (or simply X and Y) denote topologicalspaces. Let A be a subset of X. We denote the interior and the closure of a set A by $Int(A)$ and $Cl(A)$, respectively. $A\subseteq X$ is called a semi-open set of X [3] if $A\subseteq Cl[Int(A)]$. The complement of a semi-open set is called semi-closed. The intersection of all semi-closedsets containingaset A iscalledthe semi-closure of A andisdenotedby $sCl(A)$.

Definition 1. (1) A generalized semi-closure space is a pair (X, sCl) consisting of a set X and a semi-closure function sCl, a function from the power set of X to itself.

(2) The semi-closure of a subset A of X, denoted by sCl, is the image of A under sCl.

(3) The semi-exterior of A is $sExt(A)=X-sCl(A)$, and the semi-interior of A is $sInt(A)=X-sCl(X-A)$.

(4) We say that A is semi-closed if $A=sCl(A)$, A is semi-open if $A=sInt(A)$ and N is a semi −neighborhood of x if $x\in sInt(N)$.

Definition 2. We say that a semi-closure function sCl defined on X is:

(1) semi-grounded if $sCl(\phi)=\phi.$

(2) semi-isotonic if $sCl(A)\subseteq sCl(B)$ whenever $A\subseteq B.$

(3) semi-enlarging if $A\subseteq sCl(A)$ for each subset A of X.

(4) semi-idempodent if $sCl(A)=sCl\left[sCl(A)\right]$ for each subset A of X.

(5) semi-sub-linear if $sCl(A\cup B)\subseteq sCl(A)\cup sCl(B)$ forall $A, B\subseteq X$.

Definition 3. (1) Two subsets A and B of X aresaidtobe *semi-closure-separated* inageneralize semi-closure space (X, sCl) $(or\ simply, sCl\text{-}separated)$ if $A\cap sCl(B)=\phi$ and $sCl(A)\cap B=\phi,$ or equivalently, if $A\subseteq sExt(B)=\phi$ and $B\subseteq sExt(A).$

(2) semi-exterior points are said to be semi-closure-separated in a generalized semi-closure space (X, sCl) if for each $A\subseteq X$ and each $x\in sExt(A)$, $\{x\}$ and A are *sCl-separated*.

Theorem 1.1. Let (X, sCl) be a generalized semi-closure space in which *semi-exterior* points are *sCl-separated* and let S be the pairs of *sCl-separated* sets in X. Then, for each subset A of X, the *semi-closure* of A is $sCl(A)=\{x\in X:\{\{x\},A\}\notin S\}$.

Proof. Inanygeneralized *semi-closure* space $sCl(A)\subseteq\{x\in X:\{\{x\},A\}\notin S\}$. Supposethat $y\notin\{x\in X:\{\{x\},A\}\notin S\}$, that is, $\{\{y\},A\}\in S.$ Then $\{y\}\cap sCl(A)=\phi,$ andso $y\notin sCl(A)$. Supposenowthat $y\notin sCl(A)$. Byhypothesis, $\{\{y\},A\}\in S$, andhence, $y\notin\{x\in X:\{\{x\},A\}\notin S\}$.

2. SOME FUNDAMENTAL PROPERTIES

Definition 4. A semi-closure function sCl defined on a set X is said to be pointwise semi-symmetric when for all $x,y\in X$, if $x\in sCl(\{y\})$, then $y\in sCl(\{x\})$.

A generalized *semi-closure* space (X, sCl) is said to be semi-R_0 when for all $x,y\in X$, if x is in each *semi-neighborhood* of y, then y is in each *semi-neighborhood* of x.

Corollary 2.1. Let (X, sCl) a generalized semi-closure space in which *semi-exterior* points are *sCl-separated*. Then sCl is pointwise semi-symmetric and (X, sCl) is semi-R_0.

Proof. Suppose that *semi-exterior* points are *sCl-separated* in (X, sCl). If $x\in sCl(\{y\})$, then $\{x\}$ and $\{y\}$ are not *sCl-separated* and hence, $y\in sCl(\{x\})$. Hence, sCl is pointwise semi-symmetric. Suppose that x belongs to every semi-neighborhood of y, that is, $x\in M$ whenever $y\in sInt(M)$. Letting $A=X-M$ and rewriting contrapositive, $y\in sCl(A)$ whenever $x\in A$. Suppose $x\in sInt(N)$. $x\notin sCl(X-N)$, so x is *sCl-separated* from $X-N$. Hence $sCl\{x\}\subseteq N$, $x\in\{x\}$, so $y\in sCl(\{x\})\subseteq N$. Hence (X, sCl) is semi-R_0.

While these three axioms are not equivalent in general, they are equivalent when the semi-closure function is *semi-isotonic:*

Theorem 2.2. Let (X, sCl) beageneralized semi-closure spacewith sCl semi-isotonic. Then the following statements areequivalent:

(1) sExterior points are *sCl-separtaed*.

(2) sCl is pointwise semi-symmetric.

(3) (X, sCl) is semi-R_0.

Proof. Suppose that (2) is true. Let $A \subseteq X$, and suppose $x \in sExt(A)$. Then, as sCl is semi-isotonic, foreach $y \in A$, $x \notin sCl(\{y\})$, andhence, $y \notin sCl(\{x\})$. Hence $A \cap sCl(\{x\}) = \phi$. Hence (2) implies (1), and by the previous corollary, (1) implies (2). Suppose now that (2) is true and let $x, y \in X$ such that x is in every semi-neighborhood of y, that is, $x \in N$, whenever $y \in sInt(N)$. Then $y \in sCl(A)$ whenever $x \in A$, and in particular, since $x \in \{x\}$, $y \in sCl(\{x\})$. Hence $x \in sCl(\{y\})$. Thus if $y \in B$, then $x \in sCl(\{y\}) \subseteq sCl(B)$, as sCl s is semi-isotonicHenceif $x \in sInt(C)$, then $y \in C$, thatis, y isinevery semi-neighborhood of x. Hence, (2) implies (3).

Finally, suppose that (X, sCl) is semi-R_0 and suppose that $x \in sCl(\{y\})$. Since sCl is semi-isotonic, $x \in sCl(B)$ whenever $y \in B$, or, equivalently, y is in every semi-neighborhood of x. Since (X, sCl) is semi-R_0, $x \in N$ whenever $y \in sInt(N)$. Hence, $y \in sCl(A)$ whenever $x \in A$, and in particular, since $x \in \{x\}$, $y \in sCl(\{x\})$. Hence (3) implies (2).

Theorem 2.3. Let S be a set of unordered pairs of subsets of a set X such that, for all $A, B, C \subseteq X$.

(1) If $A \subseteq B$ and $\{B, C\} \in S$, then $\{A, C\} \in S$ and

(2) If $\{\{x\}, B\} \in S$ foreach $x \in A$ and $\{\{y\}, A\} \in S$ foreach $y \in B$, then $\{A, B\} \in S$.

Then there exists a unique pointwise semi-symmetric *semi-isotonic* semi-closure function sCl on X which semi-closure-separates theelementsof S.

Proof. Define sCl by $sCl(A) = \{x \in X : \{\{x\}, A\} \notin S\}$ forevery $A \subseteq X$. If $A \subseteq B \subseteq X$ and $x \in sCl(A)$, then $\{\{x\}, A\} \notin S$. Hence $\{\{x\}, B\} \notin S$, thatis, $x \in sCl(B)$. Thus sCl is semi-isotonic. Also, $x \in sCl(\{y\})$ ifandonlyif $\{\{x\}, \{y\}\} \notin S$ ifandonlyif $y \in sCl(\{x\})$, and thus sCl is pointwise semi-symmetric. Supposethat $\{A, B\} \in S$. Then $A \cap sCl(B) = A \cap \{x \in X : \{\{x\}, B\} \notin S\} = \{x \in A : \{\{x\}, A\} \notin S\} = \phi$. Similarly, $sCl(A) \cap B = \phi$. Hence,if $\{A, B\} \in S$, then A and B are *sCl-seprated*.

Nowsupposethat A and B are *sCl-seprated*. Then $\{x \in A : \{\{x\}, B\} \notin S\} = A \cap sCl(B) = \phi$ and $\{x \in B : \{\{x\}, A\} \notin S\} = sCl(A) \cap B = \phi$. Hence, $\{\{x\}, B\} \in S$ foreach $x \in A$ and $\{\{y\}, A\} \in S$ for each $y \in B$, and thus, $\{A, B\} \in S$.

Furthermore,manypropertiesofsemi-closurefunctionscanbeexpressedintermsofthe sets theyseparate:

Theorem 2.4. Let S be the pairs of *sCl-separted* sets of a generalized semi-closure space (X, sCl) in which *sExterior* points are semi-closure-separates. Then sCl is

(1) semi-grouped if and only if for all $x \in X$, $\{\{x\}, \phi\} \in S$.

(2) semi-enlarging if and only if for all $\{A, B\} \in S$, A and B are disjoint.

(3) semi-sub-linear if and only if $\{A, B \cup C\} \in S$ whenever $\{A, B\} \in S$ and $\{A, C\} \in S$.

Moreover, if sCl is semi-enlarging and for all A, $B \subseteq X$, $\{\{x\}, A\} \notin S$ whenever $\{\{x\}, B\} \notin S$ and $\{\{y\}, A\} \notin S$ for each $y \in B$, then sCl is semi-idempodent. Also, if sCl is semi-isotonic and semi-idempodent, then $\{\{x\}, A\} \notin S$ whenever $\{\{x\}, B\} \notin S$ and $\{\{y\}, A\} \notin S$ for each $y \in B$.

Proof. Recall that by Theorem 1.1, $sCl(A) = \{x \in X : \{\{x\}, A\} \notin S\}$ for every $A \subseteq X$. Suppose that for all $x \in X$, $\{\{x\}, \phi\} \in S$. Then $sCl(\phi) = \{x \in X : \{\{x\}, \phi\} \notin S\} = \phi$. Hence sCl is semi-grounded. Conversely, if $\phi = sCl(\phi) = \{x \in X : \{\{x\}, \phi\} \notin S\}$, then $\{\{x\}, \phi\} \in S$, for all $x \in X$. Suppose that for all $\{A, B\} \in S$, A and B are disjoint. Since $\{\{a\}, A\} \notin S$ if $a \in A$, $A \subseteq sCl(A)$ for each $A \subseteq X$. Hence, sCl is semi-enlarging. Conversely, suppose that sCl is semi-enlarging and $\{A, B\} \in S$. Then $A \cap B \subseteq sCl(A) \cap B = \phi$. Suppose that $\{A, B \cup C\} \in S$ whenever $\{A, B\} \in S$. and $\{A, C\} \in S$. Let $x \in X$ and $B, C \subseteq X$ such that $\{\{x\}, B \cup C\} \notin S$. Then $\{\{x\}, B\} \notin S$ or $\{\{x\}, C\} \notin S$. Hence $sCl(B \cup C) \subseteq sCl(B) \cup sCl(C)$, and therefore, sCl is semi-sub-linear. Conversely, suppose that sCl is semi-sub-linear and let $\{A, B\}$, $\{A, C\} \in S$. Then we obtain $sCl(B \cup C) \cap A \subseteq (sCl(B) \cup sCl(C)) \cap A = ((sCl(B) \cap A) \cup (sCl(C)) \cap A) = \phi$ and $(B \cup C) \cap sCl(A) = (B \cap sCl(A)) \cup (C \cap sCl(A)) = \phi$. Suppose that sCl is semi-enlarging and suppose that $\{\{x\}, A\} \notin S$ whenever $\{\{x\}, B\} \notin S$ and $\{\{y\}, A\} \notin S$ for every $y \in B$. Then $sCl(sCl(A)) \subseteq sCl(A)$. If $x \in sCl(sCl(A))$, then $\{\{x\}, sCl(A)\} \notin S$. $\{\{y\}, A\} \notin S$, for each $y \in sCl(A)$; hence $\{\{x\}, A\} \notin S$. And since sCl is semi-enlarging, $sCl(A) \subseteq sCl(sCl(A))$. Thus $sCl(sCl(A)) = sCl(A)$, for each $A \subseteq X$. Finally, suppose that sCl is semi-isotonic and semi-idempodent. Let $x \in X$ and A, $B \subseteq X$ such that $\{\{x\}, B\} \notin S$ and, for each $y \in B$, $\{\{y\}, A\} \notin S$. Then $x \in sCl(B)$ and for each $y \in B$, $y \in sCl(A)$, that is, $B \subseteq sCl(A)$. Hence, $x \in sCl(B) \subseteq sCl(sCl(A)) = sCl(A)$.

Definition 5. Let $(X, (sCl)_X)$ and $(Y, (sCl)_Y)$ be generalized semi-closure spaces. Then a function $f : X \to Y$ is said to be

(1) semi-closure-preserving if $f[(sCl)_X(A)] \subseteq (sCl)_Y(f(A))$ for each $A \subseteq X$.

(2) semi-continuous if $(sCl)_X(f^{-1}(B)) \subseteq f^{-1}[(sCl)_Y(B)]$ for each $B \subseteq Y$.

In general, neither condition implies the other. However, we easily obtain the following result:

Theorem 2.5. Let $(X, (sCl)_X)$ and $(Y, (sCl)_Y)$ be generalized semi-closure spaces and let $f : X \to Y$.

(1) If f is semi-closure-preserving and $(sCl)_Y$ is semi-isotonic, then f is semi-continuous

(2) If f is semi-continuous and $(sCl)_X$ is semi-isotonic, then f is semi-closure-preserving

Proof. Suppose that f is semi-closure-preserving and $(sCl)_Y$ is semi-isotonic. Let $B \subseteq Y$. Then $f[(sCl)_X(f^{-1}(B))] \subseteq (sCl)_Y[f(f^{-1}(B))] \subseteq (sCl)_Y(B)$ and therefore we obtain $(sCl)_X[f^{-1}(B)] \subseteq f^{-1}[f(sCl)_X(f^{-1}(B))] \subseteq f^{-1}[(sCl)_Y(B)]$. Suppose that f is semi-continuous and $(sCl)_X$ is semi-isotonic. Let $A \subseteq X$. $(sCl)_X(A) \subseteq (sCl)_X[f^{-1}(f(A))] \subseteq f^{-1}[(sCl)_Y(f(A))]$ and hence $f[(sCl)_X(A)] \subseteq f[f^{-1}((sCl)_Y(f(A)))] \subseteq (sCl)_Y[f(A)]$.

Definition 6. Let $(X,(sCl)_X)$ and $(Y,(sCl)_Y)$ be generalized semi-closure spaces and let $f: X \to Y$ be a function. If for all A, $B \subseteq X$, $f(A)$ and $f(B)$ are not $(sCl)_Y$-*separated* whenever A and B are not $(sCl)_X$-*separated*, then we say that f is non-semi-separating.

Note that f is non-semi-separating if and only if A and B are $(sCl)_X$-*separted* whenever $f(A)$ and $f(B)$ are $(sCl)_Y$-*separated*.

Theorem 2.6. Let $(X,(sCl)_X)$ and $(Y,(sCl)_Y)$ be generalized semi-closure spaces and let $f: X \to Y$ be a function.

(1) If $(sCl)_Y$ is semi-isotonic and f is non-semi-continuous, then $f^{-1}(C)$ and $f^{-1}(D)$ are $(sCl)_X$-*separated* whenever C and D are $(sCl)_Y$-*separated*.

(2) If $(sCl)_X$ is semi-isotonic and $f^{-1}(C)$ and $f^{-1}(D)$ are $(sCl)_X$-*separated* whenever C and D are $(sCl)_Y$-*separated*, then f is non-semi-separating.

Proof. Let C and D be $(sCl)_Y$-*separated* subsets, where $(sCl)_Y$ is semi-isotonic. Let $A = f^{-1}(C)$ and let $B = f^{-1}(D)$. $f(A) \subseteq C$ and $f(B) \subseteq D$ and since $(sCl)_Y$ is semi-isotonic, $f(A)$ and $f(B)$ are also $(sCl)_Y$-*separated*. Hence, A and B $(sCl)_X$-*separated* in X. Suppose that $(sCl)_X$ is semi-isotonic and let A, $B \subseteq X$ such that $C = f(A)$ and $D = f(B)$ are $(sCl)_X$-*separated*. Then $f^{-1}(C)$ and $f^{-1}(D)$ are $(sCl)_X$-*separated* and since $(sCl)_X$ is semi-isotonic, $A \subseteq f^{-1}[f(A)] = f^{-1}(C)$ and $B \subseteq f^{-1}[f(B)] = f^{-1}(D)$ are $(sCl)_X$-*separated*. as well.

Theorem 2.7. Let $(X,(sCl)_X)$ and $(Y,(sCl)_Y)$ be generalized semi-closure spaces and let $f: X \to Y$ be a function. If f is semi-closure-preserving, then f is non-semi-separating.

Proof. Suppose that f is semi-closure-preserving and A, $B \subseteq X$ are not $(sCl)_X$-*separated*. Suppose that $(sCl)_X(A) \cap B \neq \phi$. Then $\phi \neq f\left[(sCl)_X(A) \cap B\right] \subseteq f\left[(sCl)_X(A)\right] \cap f(B) \subseteq (sCl)_Y(f(A)) \cap f(B)$. Similarly, if $A \cap (sCl)_X(B) \neq \phi$, then $f(A) \cap (sCl)_Y(f(B)) \neq \phi$. Hence $f(A)$ and $f(B)$ are not $(sCl)_Y$-*separated*.

Corollary 2.8. Let $(X,(sCl)_X)$ and $(Y,(sCl)_Y)$ be generalized *semi-closure* spaces with $(sCl)_X$ semi-isotonic and let $f: X \to Y$ be a function. If f is *semi-continuous*, then f is non-semi-separating.

Proof. If f is semi-continuous and $(sCl)_X$ is semi-isotonic, then by Theorem 2.5 (2) f is semi-closure-preserving. Hence by Theorem 2.7, f is non-semi-separating.

Theorem 2.9. Let $(X,(sCl)_X)$ and $(Y,(sCl)_Y)$ be generalized semi-closure spaces with semi-exterior points are $(sCl)_Y$-*separated* in Y and let $f: X \to Y$ be a function. Then f is semi-closure-preserving if and only if f is non-semi-separating.

Proof. By Theorem 2.7, if f is semi-closure-preserving, then f is non-semi-separating. Suppose that f is non-semi-separating and let $A \subseteq X$. If $(sCl)_X = \phi$, then $f((sCl)_X(A)) = \phi \subseteq (sCl)_Y(f(A))$. Suppose $(sCl)_X(A) \neq \phi$. Let S_X and S_Y denote the pairs of $(sCl)_X$-*separated* subsets of X and the pairs of $(sCl)_Y$-*separated* subsets of Y, respectively. Let

$y \in f\left[(sCl)_X(A)\right]$ and let $x \in (sCl)_X(A) \cap f^{-1}(\{y\})$. Since $x \in (sCl)_X(A)$, $\{\{x\}, A\} \notin S_X$ andsince f is non-semi-separating, $\{\{y\}, f(A)\} \notin S_Y$. Since semi-exterior pointsare $(sCl)_Y$*-separated*, $y \in (sCl)_Y\left[f(A)\right]$. Thus $f\left[(sCl)_X(A)\right] \subseteq (sCl)_Y\left[f(A)\right]$ for each $A \subseteq X$.

Corollary 2.10. Let $(X, (sCl)_X)$ and $(Y, (sCl)_Y)$ be generalized semi-closure spaces with semi-isotonic closure functions and with $(sCl)_Y$*-pointwise-semi-symmetric* and let $f: X \to Y$ be a function. Then f is semi-continuous if and only if f is non-semi-separating.

Proof. Since $(sCl)_Y$ is semi-isotonic and pointwise-semi-symmetric, semi-Exterior points are semi-isotonic separated in $(Y, (sCl)_Y)$ (Theorem 2.2 (1)). Since both semi-closure functions are semi-isotonic, f is semi-closure-preserving (Theorem 2.5) if and only if f is semi-continuous. Hence, we can apply Theorem 2.9.

3. SEMI-CONNECTED GENERALIZED SEMI-CLOSURE SPACES

Definition 7. Let (X, sCl) beageneralized semi-closure space. Then X issaidtobe semi-connected if X isnotaunionofdisjointnontrivialsemi-closure-separatedpairofsets.

Theorem 3.1. Let (X, sCl) be a generalized semi-closure space with semi-grounded semi-isotonic semi-enlarging sCl. Then, the following statements are equivalent:

(1) (X, sCl) is semi-connected.

(2) X cannot be a union of non-empty disjoint semi-open sets.

Proof. $(1) \Rightarrow (2)$: Let X beaunionofnon-emptydisjoint semi-open sets A and B. Then, $X = A \cup B$ andthisimpliesthat $B = X - A$ and A isa semi-open set.Thus, B is semi-closed andhence $A \cap sCl(B) = A \cap B = \phi$. Byusing a similarway,weobtain $sCl(A) \cap B = \phi$. Hence, A and B are semi-closure-separated and hence X is not semi-connected. This is a contradiction.

$(2) \Rightarrow (1)$: Suppose that X is not semi-connected. Then $X = A \cup B$, where A, B are disjoint semi-closure-separated sets,i.e. $A \cap sCl(B) = sCl(A) \cap B = \phi$. Wehave $sCl(B) \subseteq X - A \subseteq B$. Since sCl is semi-enlarging, weobtain $sCl(B) = B$ andhence, B is semi-closed. By using $sCl(A) \cap B = \phi$ and similar way, it is obvious that A is semi-closed. This is a contradiction.

Definition 8. Let (X, sCl) be a generalized semi-closure space with semi-grounded semi-isotonic sCl. Then, (X, sCl) is called T_1-semi-grounded semi-isotonic space if $sCl\{x\} \subseteq \{x\}$ for all $x \in X$.

Theorem 3.2. Let (X, sCl) be a generalized semi-closure space with λ-grounded semi-isotonic sCl. Then, the following statements are equivalent:

(1) (X, sCl) is semi-connected.

(2) Any semi-continuous function $f: X \to Y$ is constant for all T_1-semi-grounded semi-isotonic spaces $Y = \{0,1\}$.

Proof. $(1) \Rightarrow (2)$: Let X be semi-connected Supposethat $f: X \to Y$ is semi-continuous andit is not constant.Then there exists aset $U \subseteq X$ such that $U = f^{-1}(\{0\})$ and $X - U = f^{-1}(\{1\})$. Since f is semi-continuous and Y is T_1-λ-grounded semi-isotonic space,thenwe have

$Cl_\lambda(U) = sCl\left[f^{-1}(\{0\})\right] \subseteq f^{-1}\left[sCl(\{0\})\right] \subseteq f^{-1}(\{0\}) = U$ andhence $sCl(U) \cap (X-U) = \phi$. Byusing a similarwaywehave $U \cap sCl(X-U) = \phi$. Thisisacontradiction.Thus, f isconstant.

$(2) \Rightarrow (1)$: Suppose that X is not semi-connected. Then there exist semi-closure-separated sets U and V such that $U \cup V = X$. We have $sCl(U) \subseteq U$ and $sCl(V) \subseteq V$ and $X - U \subseteq V$. Since sCl is *semi-isotonic* and U, V are semi-closure-separated, then $sCl(X-U) \subseteq sCl(V) \subseteq X - U$. If we consider the space (Y, sCl) by $Y = \{0,1\}$, $sCl(\phi) = \phi$, $sCl(\{0\}) = \{0\}$ $sCl(\{1\}) = \{1\}$ and $sCl(Y) = Y$, then space (Y, sCl) is a T_1-semi-grounded semi-isotonic space. We define the function $f : X \to Y$ as $f(U) = \{0\}$ and $f(X-U) = \{1\}$. Let $A \neq \phi$ and $A \subseteq Y$. If $A = Y$, then $f^{-1}(A) = X$ and hence $sCl(X) = sCl\left(f^{-1}(A)\right) \subseteq X = f^{-1}(A) = f^{-1}(sCl(A))$. If $A = \{0\}$, then $f^{-1}(A) = U$ and hence $sCl(U) = sCl\left[f^{-1}(A)\right] \subseteq U = f^{-1}(A) = f^{-1}\left[sCl(A)\right]$. If $A = \{1\}$, then $f^{-1}(A) = X - U$ and hence $sCl(X-U) = sCl\left[f^{-1}(A)\right] \subseteq X - U = f^{-1}(A) = f^{-1}\left[sCl(A)\right]$. Hence, f is semi-continuous. Since f is not constant, this is acontradiction.

Theorem 3.3. Let $f : (X, sCl) \to (Y, sCl)$ and $g : (Y, sCl) \to (Z, sCl)$ be semi-continuous functions. Then, $gof : (X, sCl) \to (Z, sCl)$ is semi-continuous.

Proof. Supposethat f and g are semi-continuous. Forall $A \subseteq Z$ wehave $sCl\left[(gof)^{-1}(A)\right] = sCl\left[f^{-1}\left(g^{-1}(A)\right)\right] \subseteq f^{-1}\left[sCl\left(g^{-1}(A)\right)\right] \subseteq f^{-1}\left[g^{-1}\left(sCl(A)\right)\right] = (gof)^{-1}\left[sCl(A)\right]$. Hence, $gof : X \to Z$ is semi-continuous.

Theorem 3.4. Let (X, sCl) and (Y, sCl) be generalized semi-closure spaces with semi-grounded semi-isotonic sCl and $f : (X, sCl) \to (Y, sCl)$ be a semi-continuous function onto Y. If X is semi-connected, then Y is semi-connected.

Proof. Supposethat $[0,1]$ isageneralized semi-closure spacewith semi-grounded semi-isotonic sCl and $g : Y \to [0,1]$ is a semi-continuous function. Since f is semi-continuous, by Theorem 3.3, $gof : X \to [0,1]$ is semi-continuous. Since X is semi-connected, gof is constant and hence g is constant. By Theorem 3.2, Y is semi-connected

Definition 9. Let (Y, sCl) be a generalized semi-closure space with semi-grounded semi- isotonic sCl and more than one element. A generalized semi-closure space (X, sCl) with semi-grounded semi-isotonic sCl iscalled Y-semi-connected ifany semi-continuous function $f : X \to Y$ isconstant.

Theorem 3.5. Let (Y, sCl) be a generalized semi-closure space with semi-grounded semi-isotonic semi-enlarging sCl and more than one element. Then every Y-semi-connected generalized semi-closure space with semi-grounded semi-isotonic is semi-connected.

Proof. Let (X, sCl) be a Y-semi-connected generalized semi-closure space with semi-grounded semi-isotonic sCl. Suppose that $f : X \to \{0,1\}$ is a λ-continuous function, where $\{0,1\}$ is T_1-semi-grounded semi-isotonic space.Since Y isageneralized semi-closure space with semi-grounded semi-isotonic semi-enlarging sCl and more than one element, thenthere exists a λ-continuous injection $g : \{0,1\} \to Y$. By Theorem 3.3, $gof : X \to Y$ is semi-continuous. Since X is *Y-semi-connected*, then gof is constant. Thus, f is constant and hence, by Theorem 3.2, X issemi-connected.

Theorem 3.6. Let (X, sCl) and (Y, sCl) begeneralized semi-closure spaceswithsemi-grounded semi-isotonic sCl and $f:(X, sCl)\rightarrow(Y, sCl)$ be a semi-continuous function onto Y. If X is Z-semi-connected, then Y is Z-semi-connected.

Proof. Suppose that $g:Y\rightarrow Z$ is a semi-continous function. Then $gof:X\rightarrow Z$ is semi-continuous. Since X is Z-semi-connected, then gof is constant. This implies that g is constant. Thus, Y is Z-semi-connected.

Definition 10. Ageneralized semi-closure space (X, sCl) isstrongly semi-connected ifthere is no countable collection of pairwise semi-closure-separated sets $\{A_n\}$ such that $X = \cup A_n$.

Theorem 3.7. Everystrongly semi-connected generalized semi-closure spacewith semi-grounded semi-isotonic sCl is semi-connected.

Theorem 3.8. Let (X, sCl) and (Y, sCl) begeneralized semi-closure spaceswith semi-grounded semi-isotonic sCl and $f:(X, sCl)\rightarrow(Y, sCl)$ be a semi-continuous functiononto Y. If X is strongly semi-connected, then Y is strongly semi-connected.

Proof. Suppose that Y is not strongly semi-connected. Then, there exists a countable collection of pairwise semi-closure-separated sets $\{A_n\}$ such that $Y=\bigcup A_n$. Since $f^{-1}(A_n)\cap sCl\left[f^{-1}(A_m)\right]\subseteq f^{-1}(A_n)\cap f^{-1}\left[sCl(A_m)\right]=\phi$ for all $n\neq m$, then the collection $\left\{f^{-1}(A_n)\right\}$ is pairwise semi-closure-separated. This is a contradiction. Hence, Y isstrongly semi-connected.

Theorem 3.9. Let $\left(X,(sCl)_X\right)$ and $\left(Y,(sCl)_Y\right)$ aregeneralizedsemi-closurespaces.Then the following statements are equivalent for a function $f:X\rightarrow Y$:

(1) f is semi-continuous.

(2) $f^{-1}\left[sInt(B)\right]\subseteq sInt\left[f^{-1}(B)\right]$ foreach $B\subseteq Y$.

Theorem 3.10. Let (X, sCl) be a generalized semi-closure space with semi-grounded semi-isotonic sCl. Then (X, sCl) is strongly semi-connected if and only if (X, sCl) is Y-semi-connected for any countable T_1-semi-grounded semi-isotonic space (Y, sCl).

Proof. $(\Rightarrow)$: Let (X, sCl) be strongly semi-connected. Suppose that (X, sCl) is not *Y-semi-connected* forsomecountable T_1-semi-grounded semi-isotonic space (Y, sCl). There exists a semi-continuous function $f:X\rightarrow Y$ which is not constant and hence $K=f(X)$ is a countable set with more than one element. For each $y_n\in K$, there exists $U_n\subseteq X$ such that $U_n=f^{-1}(\{y_n\})$ and hence $Y=\bigcup U_n$. Since f is semi-continuous and Y is semi-grounded, thenforeach $n\neq m$, $U_n\cap sCl(U_m)= f^{-1}(\{y_n\})\cap sCl\left(f^{-1}(\{y_m\})\right)\subseteq f^{-1}(\{y_n\})\cap f^{-1}\left(sCl(\{y_m\})\right)\subseteq f^{-1}(\{y_n\})\cap f^{-1}(\{y_m\})=\phi$. This contradicts with the strong semi-connectedness of X. Thus, X is Y-semi-connected.

$(\Leftarrow)$: Let X be Y-semi-connected for any countable T_1-semi-grounded semi-isotonic space (Y, sCl). Suppose that X is not strongly semi-connected. There exists a countable collection ofpairwise semi-closure-separated sets $\{U_n\}$ suchthat $X=\bigcup U_n$. Wetakethespace (Z, sCl), where Z isthesetofintegersand $sCl:P(Z)\rightarrow P(Z)$ isdefinedby $sCl(K)=K$ for each $K\subseteq Z$. Clearly (Z, sCl) is a countable T_1-semi-grounded semi-isotonic space. Put $U_k\in\{U_n\}$. Wedefineafunction $f:X\rightarrow Z$ by

$f(U_k)=\{x\}$ and $f(X-U_k)=\{y\}$ where $x, y\in Z$ and $x\neq y$. Since $sCl(U_k)\cap U_n=\phi$ forall $n\neq k$, then $sCl(U_k)\cap(\bigcup_{n\neq k}U_n)=\phi$ and hence $sCl(U_k)\subseteq U_k$. Let $\phi\neq K\subseteq Z$. If $x, y\in K$, then $f^{-1}(K)=X$ and $sCl[f^{-1}(K)]= sCl(X)\subseteq X=f^{-1}(K)=f^{-1}[sCl(K)]$. If $x,\in K$ and $y\notin K$, then $f^{-1}(K)=U_k$ and $sCl[f^{-1}(K)]=sCl(U_k)\subseteq U_k=f^{-1}(K)=f^{-1}[sCl(K)]$. If $y\in K$ and $x\notin K$, then $f^{-1}(K)=X-U_k$. Since $sCl(K)=K$ foreach $K\subseteq Z$, then $sInt(K)=K$ foreach $K\subseteq Z$. Also, $X-U_k\subseteq\bigcup_{n\neq k}U_n\subseteq X-sCl(U_k)=sInt(X-U_k)$. Thus, $f^{-1}[sInt(K)]=X-U_k= f^{-1}(K)\subseteq sInt(X-U_k)$ $=sInt[f^{-1}(K)]$. Henceweobtainthat f is semi-continuous. Since f is not constant, this is a contradiction with the Z-semi-connectedness of X. Hence, X is strongly semi-connected.

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CONFLICT OF INTERESTS

The authors declare that there is no conflict of interest.

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