On $(1,2)^{\star}$-$\breve{g}$-closed and open functions

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Abstract Main aim of this paper, we introduce and study the concepts of functions namely $(1,2)^{\star}$-$\breve{g}$-closed functions, $(1,2)^{\star}$-$\breve{g}$-open functions, strongly $(1,2)^{\star}$-$\breve{g}$-closed functions and strongly $(1,2)^{\star}$-$\breve{g}$-open functions in bitopological spaces and we obtain certain characterizations of these class of functions with seen from the special case of normality theorems are preserved under continuous of $(1,2)^{\star}$-$\breve{g}$-closed functions.

Keywords: 1,2)*-$\breve{g}$-closed functions, $(1,2)^{\star}$-$\breve{g}$-open functions, strongly $(1,2)^{\star}$-$\breve{g}$-closed functions and strongly $(1,2)^{\star}$-$\breve{g}$-open functions.

1. Introduction

J.C.Kelly [1] was uttered the geometrical continuation of bitopological space that is a non empty set X together with two arbitrary topologies defined on X at the stage of significant study the shapes of objects. N. Levine [2] was initiated the study of generalizations of closed sets in topological spaces. This concepts was very useful. Recently, several topologist have introduced and investigated the various types generalized closed sets with closed and open functions in bitopological spaces and so on. In this paper, we introduce and study the concepts of functions namely $(1,2)^{\star}$-$\breve{g}$-closed functions, $(1,2)^{\star}$-$\breve{g}$-open functions, strongly $(1,2)^{\star}$-$\breve{g}$-closed functions and strongly $(1,2)^{\star}$-$\breve{g}$-open functions in bitopological spaces and we obtain certain characterizations of these class of functions.

2. Preliminaries

Throughout this paper $(\boldsymbol{X},\boldsymbol{\tau_1},\boldsymbol{\tau_2})$ and $(\boldsymbol{Y},\boldsymbol{\sigma_1},\boldsymbol{\sigma_2})$ (or simply $\boldsymbol{X}$ and $\boldsymbol{Y}$) represents the non-empty bitopological spaces on which no separation axiom are assumed, unless otherwise mentioned. For a subset $\boldsymbol{H}$ of $\boldsymbol{X}$, $\boldsymbol{\tau_{1,2}}$-$\boldsymbol{cl}(\boldsymbol{H})$ and $\boldsymbol{\tau_{1,2}}$-$\boldsymbol{int}(\boldsymbol{H})$ represents the closure of $\boldsymbol{H}$ and interior of $\boldsymbol{H}$ respectively.

Definition 2.1 *Let $\boldsymbol{S}$ be a subset of $\boldsymbol{X}$. Then $\boldsymbol{S}$ is said to be $\boldsymbol{\tau_{1,2}}$-open [3] if $\boldsymbol{S}=\boldsymbol{A}\cup\boldsymbol{B}$ where $\boldsymbol{A}\in\boldsymbol{\tau_1}$ and $\boldsymbol{B}\in\boldsymbol{\tau_2}$.* The complement of $\boldsymbol{\tau_{1,2}}$-open set is called $\boldsymbol{\tau_{1,2}}$-closed.

Notice that $\tau_{1,2}$-open sets need not necessarily form a topology.

Definition 2.2 *[3] Let S be a subset of a bitopological space X. Then*

1. the $\tau_{1,2}$-closure of S, denoted by $\tau_{1,2}$-$cl(S)$, is defined as $\cap\{F: S\subseteq F$ and F is $\tau_{1,2}$-closed$\}$.
2. the $\tau_{1,2}$-interior of S, denoted by $\tau_{1,2}$-$int(S)$, is defined as $\cup\{F: F\subseteq S$ and F is $\tau_{1,2}$-open$\}$.

Definition 2.3 *[3] A subset A of a bitopological space (X,τ_1,τ_2) or X is said to be a $(1,2)^{\star}$-semi open set if $A\subseteq\tau_{1,2}$-$cl(\tau_{1,2}$-$int(A))$.* The complement of the above mentioned set is called a closed set.

Definition 2.4 *[11] A subset A of a bitopological space (X,τ_1,τ_2) or X is said to be a $(1,2)^{\star}$-semi closure of A, denoted by $(1,2)^{\star}$-$scl(A)$, is defined as $\cap\{F: S\subseteq F$ and F is $(1,2)^{\star}$-semi closed$\}$.*

Definition 2.5 *A subset H of a bitopological space (X,τ_1,τ_2) or X is said to be*

1.a $(1,2)^{\star}$-generalized closed set (briefly, $(1,2)^{\star}$-g-closed) [12] if $\tau_{1,2}$-$cl(H)\subseteq U$ whenever $H\subseteq U$ and U is $\tau_{1,2}$-open.

2. a $(1,2)^{\star}$-sg-closed set [10] if $(1,2)^{\star}$-$scl(A)\subseteq U$ whenever $A\subseteq U$ and U is $(1,2)^{\star}$-semi-open.
3. a $(1,2)^{\star}$-gs-closed set [10] if $(1,2)^{\star}$-$scl(A)\subseteq U$ whenever $A\subseteq U$ and U is $\tau_{1,2}$-open.

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Key words: $(1,2)^{\star}$-$\breve{g}$-closed functions, $(1,2)^{\star}$-$\breve{g}$-open functions, strongly $(1,2)^{\star}$-$\breve{g}$-closed functions and strongly $(1,2)^{\star}$-$\breve{g}$-open functions

4. a $(1,2)^\star$-ω-closed set [13] if $\tau_{1,2}$-$cl(H) \subseteq U$ whenever $H \subseteq U$ and U is $(1,2)^\star$-semi open.
5. a $(1,2)^\star$-ψ-closed set [9] if $(1,2)^\star$scl(A) ⊆ U whenever A ⊆ U and U is $(1,2)^\star$-sg-open.
6. a $(1,2)^\star$-$\hat{\hat{g}}$-closed set [6] if $\tau_{1,2}$-$cl(H) \subseteq G$ whenever $H \subseteq G$ and G is $(1,2)^\star$-sg-open.
7. a $(1,2)^\star$-$\hat{\hat{g}}_1$-closed set [5] if $\tau_{1,2}$-$cl(H) \subseteq G$ whenever $H \subseteq G$ and G is $(1,2)^\star$-$\hat{\hat{g}}$-open.
8. a $(1,2)^\star$-$\mathcal{G}$-closed set [5] if $(1,2)^\star$-$scl(H) \subseteq G$ whenever $H \subseteq G$ and G is $(1,2)^\star$-$\hat{\hat{g}}_1$-open.
9. a $(1,2)^\star$-$\breve{g}$-closed set [5] if $\tau_{1,2}$-$cl(H) \subseteq G$ whenever $H \subseteq G$ and G is $(1,2)^\star$-$\mathcal{G}$-open.

The complements of the above mentioned closed sets are called their respective open sets.

Definition 2.6 *A function $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be*

1. a $(1,2)^\star$-continuous [4] if the inverse image of every $\sigma_{1,2}$-closed set of (Y, σ_1, σ_2) is $\tau_{1,2}$-closed set in (X, τ_1, τ_2).
2. a $(1,2)^\star$-$\breve{g}$-continuous [7] if the inverse image of every $\tau_{1,2}$-closed set in (Y, σ_1, σ_2) is $(1,2)^\star$-$\breve{g}$-closed set in (X, τ_1, τ_2).
3. a $(1,2)^\star$-$\mathcal{G}$-irresolute [7] if the inverse image of every $(1,2)^\star$-$\mathcal{G}$-closed in (Y, σ_1, σ_2) is $(1,2)^\star$-$\mathcal{G}$-closed set in (X, τ_1, τ_2).
4. a $(1,2)^\star$-$\mathcal{G}$-continuous [13] if the inverse image of every $\sigma_{1,2}$-closed set of (Y, σ_1, σ_2) is $(1,2)^\star$-semi closed set in (X, τ_1, τ_2).

Definition 2.7 *A space (X, τ_1, τ_2) is called a*

1. $(1,2)^\star$-$T_{\breve{g}}$-space [8] if every $(1,2)^\star$-$\breve{g}$-closed set in it is $\tau_{1,2}$-closed.
2. $(1,2)^\star$-T_ω-space if every $(1,2)^\star$-ω-closed set in it is $\tau_{1,2}$-closed.

3. $(1,2)^\star$-$\breve{g}$-Closed functions

Definition 3.1 *A function $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called $(1,2)^\star$-$\breve{g}$-closed if the image of every $\tau_{1,2}$-closed set in (X, τ_1, τ_2) is $(1,2)^\star$-$\breve{g}$-closed in (Y, σ_1, σ_2) .*

Example 3.2 *Let $X = Y = \{a, b, c\}$, $\tau_1 = \{\Phi, \{a\}, X\}$ and $\tau_2 = \{\Phi, \{b\}, \{a, b\}, X\}$ then $\tau_{1,2} = \{\Phi, \{a\}, \{b\}, \{a, b\}, X\}$ with $\sigma_1 = \{\Phi, \{a, b\}, X\}$ and $\sigma_2 = \{\phi, X\}$ then $\sigma_{1,2} = \{\Phi, \{a, b\}, X\}$. Let $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be the identity function. Then f is $(1,2)^\star$-$\breve{g}$-closed function.*

Definition 3.3 *A function $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called strongly $(1,2)^\star$-$\breve{g}$-continuous if inverse image of every $(1,2)^\star$-g-closed set in (X, τ_1, τ_2) is $(1,2)^\star$-$\breve{g}$-closed in (Y, σ_1, σ_2).*

Proposition 3.4 *A function $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is $(1,2)^\star$-$\breve{g}$-closed <=> $(1,2)^\star$-$\breve{g}$-$cl(f(A)) \subseteq f(\tau_{1,2}$-$cl(A))$ for every subset A of (X, τ_1, τ_2).*

Proof. Suppose that f is $(1,2)^\star$-$\breve{g}$-closed and $A \subseteq X$. Then $\tau_{1,2}$-$cl(A)$ is $\tau_{1,2}$-closed in X and so $f(\tau_{1,2}$-$cl(A))$ is $(1,2)^\star$-$\breve{g}$-closed in (Y, σ_1, σ_2). We have $f(A) \subseteq f(\tau_{1,2}$-$cl(A))$ and $(1,2)^\star$-$\breve{g}$-$cl(f(A)) \subseteq (1,2)^\star$-$\breve{g}$-$cl(f(\tau_{1,2}$-$cl(A))) = f(\tau_{1,2}$-$cl(A))$.

Conversely, let A be any $\tau_{1,2}$-closed set in (X, τ_1, τ_2). Then $A = \tau_{1,2}$-$cl(A)$ and so $f(A) = f(\tau_{1,2}$-$cl(A)) \supseteq (1,2)^\star$-$\breve{g}$-$cl(f(A))$, by hypothesis. We have $f(A) \subseteq (1,2)^\star$-$\breve{g}$-$cl(f(A))$. Therefore $f(A) = (1,2)^\star$-$\breve{g}$-$cl(f(A))$. i.e., $f(A)$ is $(1,2)^\star$-$\breve{g}$-closed. Thus f is $(1,2)^\star$-$\breve{g}$-closed.

Proposition 3.5 *Let $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$be a function such that $(1,2)^\star$-$\breve{g}$-$cl(f(A)) \subseteq f(\tau_{1,2}$-$cl(A))$ for every subset $A \subseteq X$. Then the image $f(A)$ of a $\tau_{1,2}$-closed set A in (X, τ_1, τ_2) is $(1,2)^\star$-$\breve{g}$-closed in (Y, σ_1, σ_2).*

Proof. Let A be a $\tau_{1,2}$-closed set in (X, τ_1, τ). Then by hypothesis $(1,2)^\star$-$\breve{g}$-$cl(f(A)) \subseteq f(\tau_{1,2}$-$cl(A)) = f(A)$ and so $(1,2)^\star$-$\breve{g}$-$cl(f(A)) = f(A)$. Therefore $f(A)$ is $(1,2)^\star$-$\breve{g}$-closed in (Y, σ_1, σ_2).

Theorem 3.6 *A function $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is $(1,2)^\star$-$\breve{g}$-closed <=> for each subset B of (Y, σ_1, σ_2) and each $\tau_{1,2}$-open set U containing $f^{-1}(B)$ there is an $(1,2)^\star$-$\breve{g}$-open set V of (Y, σ_1, σ_2) such that $B \subseteq V$ and $f^{-1}(V) \subseteq U$.*

Proof. Suppose that f is $(1,2)^\star$-$\breve{g}$-closed. Let $B \subseteq Y$ and U be an $\tau_{1,2}$-open set of (X, τ_1, τ_2)such that $f^{-1}(B) \subseteq U$. Then $V = (f(U^c))^c$ is $(1,2)^\star$-$\breve{g}$-open set containing B such that $f^{-1}(V) \subseteq U$.

For the converse, let H be a $\tau_{1,2}$-closed set of (X,τ_1,τ_2). Then $f^{-1}((f(H))^c) \subseteq H^c$ and H^c is $\tau_{1,2}$-open. By assumption, there exists $(1,2)^\star$-$\breve{g}$-open set V in (Y,σ_1,σ_2) such that $(f(H))^c \subseteq V$ and $f^{-1}(V) \subseteq H^c$ and so $H \subseteq (f^{-1}(V))^c$. Hence $V^c \subseteq f(H) \subseteq f((f^{-1}(V))^c) \subseteq V^c$ which implies $f(H) = V^c$. Since V^c is $(1,2)^\star$-$\breve{g}$-closed, $f(H)$ is $(1,2)^\star$-$\breve{g}$-closed and therefore f is $(1,2)^\star$-$\breve{g}$-closed.

Theorem 3.7 *If $f:(X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is $(1,2)^\star$-$\mathcal{G}$-irresolute, $(1,2)^\star$-$\breve{g}$-closed and A is $(1,2)^\star$-$\breve{g}$-closed subset of (X,τ_1,τ_2), then $f(A)$ is $(1,2)^\star$-$\breve{g}$-closed in (Y,σ_1,σ_2).*

Proof. Let U be an $(1,2)^\star$-$\mathcal{G}$-open set in (Y,σ_1,σ_2)such that $f(A) \subseteq U$. Since f is $(1,2)^\star$-$\mathcal{G}$-irresolute, $f^{-1}(U)$ is $(1,2)^\star$-$\mathcal{G}$-open set containing A. Hence $\tau_{1,2}$-$cl(A) \subseteq f^{-1}(U)$ as A is $(1,2)^\star$-$\breve{g}$-closed in (X,τ_1,τ_2). Since f is $(1,2)^\star$-$\breve{g}$-closed, $f(\tau_{1,2}$-$cl(A))$ is $(1,2)^\star$-$\breve{g}$-closed set contained in $(1,2)^\star$-$\mathcal{G}$-open set U, which implies that $\tau_{1,2}$-$cl(f(\tau_{1,2}$-$cl(A))) \subseteq U$ and hence $\tau_{1,2}$-$cl(f(A)) \subseteq U$. Therefore, $f(A)$ is $(1,2)^\star$-$\breve{g}$-closed set in (Y,σ_1,σ_2).

Remark 3.8 *The composition of two $(1,2)^\star$-$\breve{g}$-closed functions is but not a $(1,2)^\star$-$\breve{g}$-closed as shown in the following Example.*

Example 3.9 *Let : (X,τ_1,τ_2), $((Y,\sigma_1,\sigma_2)$ and f be as in Example 3.2. Let $Z = \{a,b,c\}$ with $\eta_1 = \{\Phi,\{a,b\},Z\}$ and $\eta_2 = \{\Phi,\{a\},Z\}$ then $\eta_{1,2} = \{\Phi,\{a\},\{a,b\},Z\}$. Let $g:(Y,\sigma_1,\sigma_2) \to (Z,\eta_1,\eta_2)$ be the identity function. Then both f and g are $(1,2)^\star$-$\breve{g}$-closed functions but their composition $g \circ f$: : $(X,\tau_1,\tau_2) \to (Z,\eta_1,\eta_2)$ is not $(1,2)^\star$-$\breve{g}$-closed function, since for the $\tau_{1,2}$-closed set $\{a,c\}$ in : (X,τ_1,τ_2), $(g \circ f)(\{a,c\}) = \{a,c\}$, which is not $(1,2)^\star$-$\breve{g}$-closed set in (Z,η_1,η_2).*

Corollary 3.10 *Let $f:(X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ be $(1,2)^\star$-$\breve{g}$-closed and $g:(Y,\sigma_1,\sigma_2) \to (Z,\eta_1,\eta_2)$ be $(1,2)^\star$-$\breve{g}$-closed and $(1,2)^\star$-$\mathcal{G}$-irresolute, then their composition $g \circ f:(X,\tau_1,\tau_2) \to (Z,\eta_1,\eta_2)$ is $(1,2)^\star$-$\breve{g}$-closed.*

Proof. Let A be a $\tau_{1,2}$-closed set of (X,τ_1,τ_2). Then by hypothesis $f(A)$ is $(1,2)^\star$-$\breve{g}$-closed set in (Y,σ_1,σ_2). Since g is both $(1,2)^\star$-$\breve{g}$-closed and $(1,2)^\star$-$\mathcal{G}$-irresolute by Theorem 3.7, $g(f(A)) = (g \circ f)(A)$ is $(1,2)^\star$-$\breve{g}$-closed in (Z,η_1,η_2) and therefore $g \circ f$ is $(1,2)^\star$-$\breve{g}$-closed.

Theorem 3.11 *Let $f:(X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ and $g:(Y,\sigma_1,\sigma_2) \to (Z,\eta_1,\eta_2)$ be $(1,2)^\star$-$\breve{g}$-closed functions where (Y,σ_1,σ_2) is a $(1,2)^\star$-$T_{\breve{g}}$-space. Then their composition $g \circ f:(X,\tau_1,\tau_2) \to (Z,\eta_1,\eta_2)$ is $(1,2)^\star$-$\breve{g}$-closed.*

Proof. Let A be a $\tau_{1,2}$-closed set of (X,τ_1,τ_2). Then by assumption $f(A)$ is $(1,2)^\star$-$\breve{g}$-closed in (Y,σ_1,σ_2). Since (Y,σ_1,σ_2) is a $(1,2)^\star$-$T_{\breve{g}}$-space, $f(A)$ is $\tau_{1,2}$-closed in (Y,σ_1,σ_2) and again by assumption $g(f(A))$ is $(1,2)^\star$-$\breve{g}$-closed in (Z,η_1,η_2). i.e., $(g \circ f)(A)$ is $(1,2)^\star$-$\breve{g}$-closed in (Z,η_1,η_2) and so $g \circ f$ is $(1,2)^\star$-$\breve{g}$-closed.

Theorem 3.12 *If $f:(X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is $(1,2)^\star$-$\breve{g}$-closed, $g:(Y,\sigma_1,\sigma_2) \to (Z,\eta_1,\eta_2)$ is $(1,2)^\star$-g-closed (resp. $(1,2)^\star$-Ψ-closed, $(1,2)^\star$-sg-closed and $(1,2)^\star$-gs-closed) and (Y,σ_1,σ_2) is a $(1,2)^\star$-$T_{\breve{g}}$-space, then their composition $g \circ f:(X,\tau_1,\tau_2) \to (Z,\eta_1,\eta_2)$ is $(1,2)^\star$-g-closed (resp. $(1,2)^\star$-Ψ-closed, $(1,2)^\star$-sg-closed and $(1,2)^\star$-gs-closed).*

Proof. Let A be a $\tau_{1,2}$-closed set of (X,τ_1,τ_2). Then by assumption $f(A)$ is $(1,2)^\star$-$\breve{g}$-closed in (Y,σ_1,σ_2) . Since (Y,σ_1,σ_2) is a $(1,2)^\star$-$T_{\breve{g}}$-space, $f(A)$ is $\tau_{1,2}$-closed in (Y,σ_1,σ_2) and again by assumption $g(f(A))$ is $(1,2)^\star$-g-closed (resp. $(1,2)^\star$-Ψ-closed, $(1,2)^\star$-sg-closed and $(1,2)^\star$-gs-closed) in (Z,η_1,η_2) . i.e., $(g \circ f)(A)$ is $(1,2)^\star$-g-closed (resp. $(1,2)^\star$-Ψ-closed, $(1,2)^\star$-sg-closed and $(1,2)^\star$-gs-closed) in (Z,η_1,η_2) and so $g \circ f$ is $(1,2)^\star$-g-closed (resp. $(1,2)^\star$-Ψ-closed, $(1,2)^\star$-sg-closed and $(1,2)^\star$-gs-closed).

Theorem 3.13 *Let $f:(X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ be a $(1,2)^\star$-closed function and $g:(Y,\sigma_1,\sigma_2) \to (Z,\eta_1,\eta_2)$ be $(1,2)^\star$-$\breve{g}$-closed function, then their composition $g \circ f:(X,\tau_1,\tau_2) \to (Z,\eta_1,\eta_2)$ is $(1,2)^\star$-$\breve{g}$-closed.*

Proof. Let A be a $\tau_{1,2}$-closed set of (X,τ_1,τ_2). Then by assumption $f(A)$ is $\tau_{1,2}$-closed in (Y,σ_1,σ_2) and again by assumption $g(f(A))$ is $(1,2)^\star$-$\breve{g}$-closed in (Z,η_1,η_2). i.e., $(g \circ f)(A)$ is $(1,2)^\star$-$\breve{g}$-closed in (Z,η_1,η_2) and so $g \circ f$ is $(1,2)^\star$-$\breve{g}$-closed.

Remark 3.14 *If $f:(X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is $(1,2)^\star$-$\breve{g}$-closed and $g:(Y,\sigma_1,\sigma_2) \to (Z,\eta_1,\eta_2)$ is $\tau_{1,2}$-closed, then their composition need not be $(1,2)^\star$-$\breve{g}$-closed function as shown in the following Example.*

Example 3.15 *Let (X,τ_1,τ_2), (Y,σ_1,σ_2) and f be as in Example 3.2. Let $Z = \{a,b,c,d\}$ with $\eta_1 = \{\Phi,\{a\},\{a,b\},Z\}$ and $\eta_2 = \{\Phi,Z\}$ then $\eta_{1,2} = \{\Phi,\{a\},\{a,b\},Z\}$. Let $g:(Y,\sigma_1,\sigma_2) \to ((Z,\eta_1,\eta_2)$ be the identity function. Then f is $(1,2)^\star$-$\breve{g}$-closed function and g is a $(1,2)^\star$-closed function. But their composition $g \circ f:((X,\tau_1,\tau_2) \to (Z,\eta_1,\eta_2)$ is not $(1,2)^\star$-$\breve{g}$-closed function, since for the $\tau_{1,2}$-closed set $\{a,c\}$ in (X,τ_1,τ_2), $(g \circ f)(\{a,c\}) = \{a,c\}$, which is not $(1,2)^\star$-$\breve{g}$-closed set in (Z,η_1,η_2).*

Theorem 3.16 *Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ and $g:(Y,\sigma_1,\sigma_2)\to(Z,\eta_1,\eta_2)$ be two functions such that their composition $g\circ f:(X,\tau_1,\tau_2)\to(Z,\eta_1,\eta_2)$ is (1,2)*-$\breve{g}$-closed function. Then the following statements are true.*

1. f is (1,2)*-continuous and surjective $\Rightarrow g$ is (1,2)*-$\breve{g}$-closed.

2. g is (1,2)*-$\breve{g}$-irresolute and injective $\Rightarrow f$ is (1,2)*-$\breve{g}$-closed.

3. f is (1,2)*-ω-continuous, surjective and (X,τ_1,τ_2) is a (1,2)*-T_ω-space $\Rightarrow g$ is (1,2)*-$\breve{g}$-closed.

4. g is strongly (1,2)*-$\breve{g}$-continuous and injective $\Rightarrow f$ is $\tau_{1,2}$-closed.

Proof.

1. Let A be a $\tau_{1,2}$-closed set of (Y,σ_1,σ_2). Since f is (1,2)*-continuous, $f^{-1}(A)$ is $\tau_{1,2}$-closed in (X,τ_1,τ_2) and since $g\circ f$ is (1,2)*-$\breve{g}$-closed, $(g\circ f)(f^{-1}(A))$ is (1,2)*-$\breve{g}$-closed in (Z,η_1,η_2). i,e., $g(A)$ is (1,2)*-$\breve{g}$-closed in (Z,η_1,η_2), since f is surjective. Therefore g is (1,2)*-$\breve{g}$-closed function.

2. Let B be a $\tau_{1,2}$-closed set of (X,τ_1,τ_2). Since $g\circ f$ is (1,2)*-$\breve{g}$-closed, $(g\circ f)(B)$ is (1,2)*-$\breve{g}$-closed in (Z,η_1,η_2). Since g is (1,2)*-$\breve{g}$-irresolute, $g^{-1}((g\circ f)(B))$ is (1,2)*-$\breve{g}$-closed set in (Y,σ_1,σ_2). i,e., $f(B)$ is (1,2)*-$\breve{g}$-closed in (Y,σ_1,σ_2), since g is injective. Thus f is (1,2)*-$\breve{g}$-closed function.

3. Let C be a $\tau_{1,2}$-closed set of (Y,σ_1,σ_2). Since f is (1,2)*-ω-continuous, $f^{-1}(C)$ is (1,2)*-ω-closed in (X,τ_1,τ_2). Since (X,τ_1,τ_2) is a (1,2)*-T_ω-space, $f^{-1}(C)$ is $\tau_{1,2}$-closed in (X,τ_1,τ_2) and so as in (1), g is (1,2)*-$\breve{g}$-closed function.

4. Let D be a $\tau_{1,2}$-closed set of (X,τ_1,τ_2). Since $g\circ f$ is (1,2)*-$\breve{g}$-closed, $(g\circ f)(D)$ is (1,2)*-$\breve{g}$-closed in (Z,η_1,η_2). Since g is strongly (1,2)*-$\breve{g}$-continuous, $g^{-1}((g\circ f)(D))$ is $\tau_{1,2}$-closed in (Y,σ_1,σ_2). i,e., $f(D)$ is $\tau_{1,2}$-closed set in (Y,σ_1,σ_2), since g is injective. Therefore f is a $\tau_{1,2}$-closed function.

As shown in the following Theorem of normality is preserved under continuous (1,2)*-$\breve{g}$-closed functions.

Theorem 3.17 *If $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is a (1,2)*-continuous, (1,2)*-$\breve{g}$-closed function from a normal space (X,τ_1,τ_2) onto a space (Y,σ_1,σ_2), then (Y,σ_1,σ_2) is normal.*

Proof. Let A and B be two disjoint $\tau_{1,2}$-closed subsets of (Y,σ_1,σ_2). Since f is (1,2)*-continuous, $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint $\tau_{1,2}$-closed sets of (X,τ_1,τ_2). Since (X,τ_1,τ_2) is normal, there exist disjoint $\tau_{1,2}$-open sets U and V of (X,τ_1,τ_2) such that $f^{-1}(A)\subseteq U$ and $f^{-1}(B)\subseteq V$. Since f is (1,2)*-$\breve{g}$-closed, by Theorem 3.6, there exist disjoint (1,2)*-$\breve{g}$-open sets G and H in (Y,σ_1,σ_2) such that $A\subseteq G, B\subseteq H, f^{-1}(G)\subseteq U$ and $f^{-1}(H)\subseteq V$. Since U and V are disjoint, $\tau_{1,2}$-$int(G)$ and $\tau_{1,2}$-$int(H)$ are disjoint $\tau_{1,2}$-open sets in (Y,σ_1,σ_2). Therefore $A\subseteq \tau_{1,2}$-$int(G)$. Similarly $B\subseteq\tau_{1,2}$-$int(H)$ and hence (Y,σ_1,σ_2) is normal.

4. $(1,2)^\star$-$\breve{g}$-Open functions and it's characterizations

Definition 4.1 *A function $\boldsymbol{f}:(\boldsymbol{X},\boldsymbol{\tau_1},\boldsymbol{\tau_2})\to(\boldsymbol{Y},\boldsymbol{\sigma_1},\boldsymbol{\sigma_2})$ is called $(\mathbf{1},\mathbf{2})^\star$-$\breve{\boldsymbol{g}}$-open function if the image $\boldsymbol{f}(\boldsymbol{A})$ is $(\mathbf{1},\mathbf{2})^\star$-$\breve{\boldsymbol{g}}$-open in $(\boldsymbol{Y},\boldsymbol{\sigma_1},\boldsymbol{\sigma_2})$ for each $\boldsymbol{\tau_{1,2}}$-open set $\boldsymbol{A}$ in $(\boldsymbol{X},\boldsymbol{\tau_1},\boldsymbol{\tau_2})$.*

Theorem 4.2 *For any bijection $\boldsymbol{f}:(\boldsymbol{X},\boldsymbol{\tau_1},\boldsymbol{\tau_2})\to(\boldsymbol{Y},\boldsymbol{\sigma_1},\boldsymbol{\sigma_2})$, the following statements are equivalent:*

1. $\boldsymbol{f^{-1}}:(\boldsymbol{Y},\boldsymbol{\sigma_1},\boldsymbol{\sigma_2})\to(\boldsymbol{X},\boldsymbol{\tau_1},\boldsymbol{\tau_2})$ is $(\mathbf{1},\mathbf{2})^\star$-$\breve{\boldsymbol{g}}$-continuous.

2. $\boldsymbol{f}$ is $(\mathbf{1},\mathbf{2})^\star$-$\breve{\boldsymbol{g}}$-open function.

3. $\boldsymbol{f}$ is $(\mathbf{1},\mathbf{2})^\star$-$\breve{\boldsymbol{g}}$-closed function.

Proof.

(1) $\Rightarrow$ (2). Let U be an $\tau_{1,2}$-open set of (X,τ_1,τ_2). By assumption, $(f^{-1})^{-1}(U)=f(U)$ is (1,2)*-$\breve{g}$-open in (Y,σ_1,σ_2) and so f is (1,2)*-$\breve{g}$-open.

(2) $\Rightarrow$ (3). Let F be a $\tau_{1,2}$-closed set of (X,τ_1,τ_2). Then F^c is $\tau_{1,2}$-open set in (X,τ_1,τ_2). By assumption, $f(F^c)$ is (1,2)*-$\breve{g}$-open in (Y,σ_1,σ_2). That is $f(F^c)=(f(F))^c$ is (1,2)*-$\breve{g}$-open in (Y,σ_1,σ_2) and therefore $f(F)$ is (1,2)*-$\breve{g}$-closed in (Y,σ_1,σ_2). Hence f is (1,2)*-$\breve{g}$-closed.

(3) $\Rightarrow$ (1). Let F be a $\tau_{1,2}$-closed set of (X,τ_1,τ_2). By assumption, $f(F)$ is (1,2)*-$\breve{g}$-closed in (Y,σ_1,σ_2). But $f(F)=(f^{-1})^{-1}(F)$ and therefore f^{-1} is (1,2)*-$\breve{g}$-continuous.

Theorem 4.3 *Assume that the collection of all (1,2)***-*$\breve{g}$*-open sets of* Y *is* $\tau_{1,2}$*-closed under arbitrary union. Let* $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *be a function. Then the following statements are equivalent:*

1. f is an (1,2)*-$\breve{g}$-open function.

2. For a subset A of (X, τ_1, τ_2), $f(\tau_{1,2}\text{-}int(A)) \subseteq$ (1,2)*-$\breve{g}$-$int(f(A))$.

3. For each $x \in X$ and for each neighborhood U of x in (X, τ_1, τ_2), there exists a (1,2)*-$\breve{g}$-neighborhood N of $f(x)$ in (Y, σ_1, σ_2) such that $N \subseteq f(U)$.

Proof. (1) ⇒ (2). Suppose that f is (1,2)*-$\breve{g}$-open. Let $A \subseteq X$. Then $\tau_{1,2}\text{-}int(A)$ is $\tau_{1,2}$-open in (X, τ_1, τ_2) and so $f(\tau_{1,2}\text{-}int(A))$ is (1,2)*-$\breve{g}$-open in (Y, σ_1, σ_2). We have $f(\tau_{1,2}\text{-}int(A)) \subseteq f(A)$. Therefore $f(\tau_{1,2}\text{-}int(A)) \subseteq$ (1,2)*-$\breve{g}$-$int(f(A))$.

(2) ⇒ (3). Suppose that (2) holds. Let $x \in X$ and U be an arbitrary neighborhood of x in (X, τ_1, τ_2). Then there exists $\tau_{2_{1,2}}$-open set G such that $x \in G \subseteq U$. By assumption, $f(G) = f(\tau_{2_{1,2}}\text{-}int(G)) \subseteq$ (1,2)*-$\breve{g}$-$int(f(G))$. This implies $f(G) =$ (1,2)*-$\breve{g}$-$int(f(G))$, we have $f(G)$ is (1,2)*-$\breve{g}$-open in (Y, σ_1, σ_2). Further, $f(X) \in f(G) \subseteq f(U)$ and so (3) holds, by taking $N = f(G)$.

(3) ⇒ (1). Suppose that (3) holds. Let U be any $\tau_{1,2}$-open set in (X, τ_1, τ_2), $x \in U$ and $f(x) = y$. Then $y \in f(U)$ and for each $y \in f(U)$, by assumption there exists an (1,2)*-$\breve{g}$-neighborhood N_y of y in (Y, σ_1, σ_2) such that $N_y \subseteq f(U)$. Since N_y is an (1,2)*-$\breve{g}$-neighborhood of y, there exists an (1,2)*-$\breve{g}$-open set V_y in (Y, σ_1, σ_2) such that $y \in V_y \subseteq N_y$. Therefore, $f(U) = \cup \{V_y : y \in f(U)\}$ is an (1,2)*-$\breve{g}$-open set in (Y, σ_1, σ_2) by the given condition. Thus f is an (1,2)*-$\breve{g}$-open function.

Theorem 4.4 *A function* $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is (1,2)***-*$\breve{g}$*-open <=> for any subset* T *of* (Y, σ_1, σ_2) *and for any* $\tau_{1,2}$*-closed set* F *containing* $F^{-1}(T)$*, there exists an (1,2)***-*$\breve{g}$*-closed set* K *of* (Y, σ_1, σ_2) *containing* T *such that* $f^{-1}(K) \subseteq F$.

As follows Theorem 3.6.

Corollary 4.5 *A function* $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is (1,2)***-*$\breve{g}$*-open <=>* $f^{-1}((1,2)^\star\text{-}\breve{g}\text{-}cl(B)) \subseteq \tau_{1,2}\text{-}cl(f^{-1}(B))$ *for each subset* B *of* (Y, σ_1, σ_2)*.*

Proof. Suppose that f is (1,2)*-$\breve{g}$-open. Then for any $B \subseteq Y, f^{-1}(B) \subseteq \tau_{1,2}\text{-}cl(f^{-1}(B))$. By Theorem 4.4, there exists (1,2)*-$\breve{g}$-closed set K of (Y, σ_1, σ_2) such that $B \subseteq K$ and $f^{-1}(K) \subseteq \tau_{1,2}\text{-}cl(f^{-1}(B))$. Therefore, $f^{-1}((1,2)^\star\text{-}\breve{g}\text{-}cl(B)) \subseteq (f^{-1}(K)) \subseteq \tau_{1,2}\text{-}cl(f^{-1}(B))$, since K is (1,2)*-$\breve{g}$-closed set in (Y, σ_1, σ_2).

Conversely, let T be any subset of (Y, σ_1, σ_2) and F be any $\tau_{1,2}$-closed set containing $f^{-1}(T)$. Put $K =$ (1,2)*-$\breve{g}$-$cl(T)$. Then K is (1,2)*-$\breve{g}$-closed set and $T \subseteq K$. By assumption, $f^{-1}(K) = f((1,2)^\star\text{-}\breve{g}\text{-}cl(T)) \subseteq \tau_{1,2}\text{-}cl(f^{-1}(T)) \subseteq F$ and therefore by Theorem 4.4, f is (1,2)*-$\breve{g}$-open.

5 Strongly $(1,2)^\star$-$\breve{g}$-closed and open functions

Definition 5.1 *A function* $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is called a*

1. Strongly (1,2)*-$\breve{g}$-closed if the image $f(A)$ is (1,2)*-$\breve{g}$-closed in (Y, σ_1, σ_2) for every (1,2)*-$\breve{g}$-closed set A in (X, τ_1, τ_2).

2. Strongly (1,2)*-$\breve{g}$-open if the image $f(A)$ is (1,2)*-$\breve{g}$-open in (Y, σ_1, σ_2) for every (1,2)*-$\breve{g}$-open set A in (X, τ_1, τ_2).

Example 5.2 *For example the function* f *in Example 3.2 is strongly (1,2)***-*$\breve{g}$*-closed function.*

Proposition 5.3 *Every strongly (1,2)***-*$\breve{g}$*-closed function is (1,2)***-*$\breve{g}$*-closed function.*

Remark 5.4 *The converse part of Proposition 5.3 is not true as seen from the following Example.*

Example 5.5 *Let* $X = Y = \{a, b, c, d, e\}, \tau_1 = \{\Phi, \{a, b\}, X\}$ *and* $\tau_2 = \{\Phi, X\}$ *then* $\tau_{1,2} = \{\Phi, \{a, b\}, X\}$ *with* $\sigma_1 = \{\Phi, \{a\}, Y\}$ *and* $\sigma_2 = \{\Phi, \{a, b\}, Y\}$ *then* $\sigma_{1,2} = \{\Phi, \{a\}, \{a, b\}, Y\}$*. Let* $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *be the identity function. Then* f *is (1,2)***-*$\breve{g}$*-closed but not strongly (1,2)***-*$\breve{g}$*-closed function. Since* $\{a, c\}$ *is (1,2)***-*$\breve{g}$*-closed set in* (X, τ_1, τ_2)*, but its image under* f *is* $\{a, c\}$ *which is not (1,2)***-*$\breve{g}$*-closed set in* (Y, σ_1, σ_2)*.*

Theorem 5.6 *A function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is strongly (1,2)*-$\breve{g}$-closed <=> (1,2)*-$\breve{g}$-$cl(f(A))\subseteq f($(1,2)*-$\breve{g}$-$cl(A))$ for every subset A of (X,τ_1,τ_2).*

Proof. Follows from the Proposition 3.4.

Theorem 5.7 *For any bijection $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$, the following statements are equivalent:*

1. $f^{-1}:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is (1,2)*-$\breve{g}$-irresolute.

2. f is strongly (1,2)*-$\breve{g}$-open function.

3. f is strongly (1,2)*-$\breve{g}$-closed function.

Proof. Follows from the Theorem 4.2.

Theorem 5.8 *If a function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is (1,2)*-$\mathcal{G}$-irresolute and (1,2)*-$\breve{g}$-closed, then it is strongly (1,2)*-$\breve{g}$-closed function.*

Proof. Follows from the Theorem 3.7

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