Comparison between the standard and Expected Bayes method to estimator for the survival function for the aThree Parameter Lindely distribution by using simulation

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Abstract: This paper deals with the comparison between the standard Bayes method and the Bayesian prediction method in estimating the survival function of a three-parameter Lindley distribution using an asymmetric loss function, which is a general entropy loss function. (IMSE) and the advantage of the standard Bayes method for estimating the survival function has been reached for all sample sizes.

Keywords: standard Bayes method, Bayes Expected Bayesian method,Three Parameter Lindely distribution, general entropy loss function

1. Introduction

The study of the functions of survival analysis has taken a great deal of space among researchers, especially in recent years, because it plays a major role in many areas of life such as medicine, engineering, and vital fields, and because of its great importance in studying the survival of living organisms, after a specified period of time (t), the analysis of survival functions plays an important role in determining risk factors in the biomedical and industrial fields. Survival theory and reliability theory share in measuring the length of survival, as the first is used for living organisms, and the second is used for systems and machines. Fast processing and processing, in contrast to the theory of survival, which lies in its difficulty in dealing with living organisms, and the arrangement of its parts, which makes them lack optimization. Statistics sheds light on the study of biological phenomena with random behavior that are of importance in the life of the organism and society, in general. Therefore, we must study these phenomena and know the probability distributions that they follow in order to study their random behavior. The study of probability distributions is related to the probabilistic results, and there are a number of problems that the researcher must study, and one of the most important of these problems is those related to estimating parameters. Statistics sheds light on the study of biological phenomena with random behavior that are of importance in the life of the organism and society, in general. Therefore, we must study these phenomena and know the probability distributions that they follow in order to study their random behavior. The study of probability distributions is related to the probabilistic results, and there are a number of problems that the researcher must study, and one of the most important of these problems is those related to estimating parameters. Statistical distributions and the accuracy of the methods used in estimation. The methods of estimating parameters go in two main directions, the first trend represented by the traditional methods, which are characterized by the fact that their estimations are unique and assume that the parameters to be estimated are fixed quantities, in contrast to the second trend represented by Bayesian methods. Which assumes that the parameters to be estimated are random variables, where the Bayesian method depends on the prior information available to the researcher, which can be formulated in the form of an initial probability distribution, which in turn is combined with the sample information. In order to obtain the subsequent distribution, the Bayesian school depends mainly on the initial probability density function, and the loss function. In this thesis, we will focus on the second axis of comparing some Bayesian methods to estimate the survival function and the associated functions of the three-parameter Lindley distribution.

2.Significance of The Study

The main objective of this research is to obtain good estimates of the survival function of the three-parameter Lindley distribution using the standard Bayesian method, the Bayesian expectation estimator method, and by employing the simulation method (Monte-Carlo method) for the purpose of comparing the Bayesian estimation methods by using the statistical scale least mean of error squares. integrative (IMSE).

3.Review Of Related Studies

D.V.Lindley(1958), Lindley presented a one-parameter distribution named Lindley distribution in relation to the scientist who proposed it through conducting a number of studies on the construction of Bayesian statistics. Wait times as an alternative to exponential distribution and gamma distribution.**M. Sankaran(1970)He** presented the additive model Lindley-Poisson, as the discontinuous addition was derived from the Poisson distribution and the Lindley distribution, he studied some of its statistical properties, including the functions of density, risk, moments, statistics ranked m and survival, and the researcher employed the proposed model in building a model of counting operations**.J.Mazucheli& J. A. Achcar (2011)** He presented a study dealing with the use of the general Lindley formula and its role in the study of competing risk to analyze life-time data taken from the mortality rate function of Lindley distribution for a group of people with cancer. One of the characteristics that make it a successful alternative to the exponential distribution and the Whipple distribution for the study of competing risk**.ShankerR. K&et al(2017)**He proposed a new formula for the two-parameter Lindley distribution (TPLD) and discussed some of its properties such as the moment generating function, mean deviations, ordered statistics, Lorenze *

Bonferroni curves, Renyi entropy and stress-toughness reliability. They estimated the model parameters using the greatest possibility method. They compared the two-parameter Lindley distribution with the distributions (Lindley general - gamma - Whipple - log-normal distribution - one parameter Lindley distribution - one parameter exponential) by using criteria (Collmcroft-Smirnov - AIC-2lnL) for two real data sets, and they concluded that Better fit than the rest of the distributions in the life time model..**M .Mesfioui&A.Abouammoh,N.M(2018)** He proposed the additive model (Lindley-Pareto), which is a distribution with two parameters. This complex distribution was presented using the Pareto distribution family with a parameter with the Lindley distribution family with a parameter, and the statistical properties of the new model were found, such as the functions of probability density, moments and ordered statistics, and the researchers used The method of greatest possibility and moments in estimating the parameters, survival function and risk function of the model. A good fit test was conducted to show the preference of the distribution performance on three sets of real data. The researchers described the Lindley-Pareto distribution as a successful and flexible distribution, which is considered a more suitable model for modelinglife times.. **E. H. Hafez** & *et al***(2020)** By studying simulation and applying real data that represent control data of the second type for an industrial experiment on lamps, as the (Colum Crove - Samir Nof) test was used for the purpose of fitting the data and it was found that it has a Lindley distribution. Under the same loss function, for each of the parameters of the distribution. They concluded that the Bayesian method of estimating is better than the maximum possibility method for estimating the parameters of the Lindley distribution with two parameters**Aryuyuen S& et al(2020)** The separate addition presented a (Lindley-Weibull) distribution, properties and applications, which is a distribution with four parameters. Such as the probability function, the cumulative function, the moment-generating function, as well as ordered statistics and some properties of reliability such as the survival function and the risk rate function and its applications. Different distribution performance was compared with other composite distributions shared by the Lindley family. The researchers concluded that the new model is useful for modeling data sets and achieves more flexibility.

4. Survival Function[9][12]

One of the methods in statistics is survival analysis, which describes death in living organisms and failures in systems and machines in addition to their uses in the biological and medical aspects. Or death, survival analysis focuses mainly on prediction in determining the probability of risks and is symbolized by the symbol S(t), and it can be expressed mathematically as follows:

$$S(t) = 1 - F(t) \qquad \dots \quad (1)$$

Since:

F(t): Aggregate density function of the random variable t.

T: Represents the time required for failure to occur and is a random variable that represents the time an organism survives to death.

5.Three Parameter Lindley Distribution[4]

The Lindley distribution is one of the mixed continuous statistical distributions widely used in the modeling of life data and survival function. Failure models, and that Lindley distribution has many uses in different fields, including in dependency studies, as well as in population studies represented by life table expectations, and that the probability density function of the distribution is in the following formula:

$$f(x,\theta,\alpha,\beta) = \frac{\theta^2}{\alpha\theta+\beta}(\alpha+\beta x)e^{-\theta x} \quad ,\theta > 0\beta \quad > 0 \ ,x > 0 \quad ,\beta\theta+\alpha > 0 \ \dots \ (2)$$

whereas: -

α: Shape parameters.

β: Shape parameters.

θ : Scale parameter.

The aggregate density function for the distribution (C·D·F) can be given the following:

$$F(x.\theta.\alpha.\beta) = 1 - \left[1 + \frac{\theta\beta x}{\theta\alpha+\beta}\right]e^{-\theta x} \qquad \dots \ (3)$$

As for the survival function and the risk function, they can be formulated according to the following relationship:

$$s(t) = \left[1 + \frac{\theta\beta x}{\theta\alpha+\beta}\right]e^{-\theta x} \qquad \dots \ (4)$$

$$h(x) = \frac{\theta^2(\alpha+\beta x)}{(\beta\theta x+\theta\alpha)+\beta} \qquad \dots \ (5)$$

6.Standard Informative Bayesian Estimator[5][6]

The Bayesian method or the Bayes method is named after Thomas Bayes, as there are two schools in the first estimation called the classical school, which assumes that parameters are fixed quantities that are estimated by classical methods such as the method of greatest possibility, method of least squares, moment method and others, and the other is called the Bayesian school Which assumes that the parameters are random variables with a probability distribution called the priority distribution, and this distribution is usually inappropriate, meaning that its integration for each field is not equal to one and is recognized by previous experiments or from data or by the theory that governs This phenomenon, and the data of the current sample has a role in the estimation process where the possibility function is combined from the data of the current sample with the prior distribution using the inverse Bayes formulaTo get the Posterior distribution. The Bayes method differs from the classical methods in that in the Bayes method, a decision is made to either reduce the loss or maximize the benefit.

The standard Bayes estimator depends on the suffix distribution function which includes the previous information about the parameter (θ) and sample observations of the current_2…..x_n). Under the Bayes theorem, the suffix distribution of the parameter (θ) can be obtained by using The Inverse Bayesian formula is as follows:

$$h\left(\underline{\theta}|x_1.x_2.\dots.x_n\right) = \frac{\pi(\underline{\theta})\prod_{i=1}^{n} f\left(x_1.x_2.\dots.x_n|\underline{\theta}\right)}{\int_{\forall\underline{\theta}}\pi(\underline{\theta})\prod_{i=1}^{n} f\left(x_1.x_2.\dots.x_n|\underline{\theta}\right)d\underline{\theta}} \quad \dots(6)$$

whereas:

$\pi(\underline{\theta})$: The initial distribution of the parameter.

$f\left(x_1.x_2.\dots.x_n|\underline{\theta}\right)$: Possibility function for sample observations of size n.

$h\left(\underline{\theta}|x_1.x_2.\dots.x_n\right)$: Post − distribution of parameters θ.

In order to find the standard Bayesian estimator, one of the Loss Functions must be used. In this message, the General Entropy Loss (EL) function will be used to find the informational standard Bayesian estimator, as follows

6.1. Informative standard Bayes estimator under the general entropy loss function[8][6]

The general asymmetric entropy loss (EL) function is defined by the following formula:

$$L_2(\underline{\hat{\theta}}.\underline{\theta}) = \left(\frac{\underline{\hat{\theta}}}{\underline{\theta}}\right)^q - q\log\left(\frac{\underline{\hat{\theta}}}{\underline{\theta}}\right) - 1 \quad ; q \neq 0 \quad \dots (7)$$

Therefore, the Bayesian estimator under the general entropy loss function, which makes the risk function as little as possible, which represents the expectation of the loss function after finding the first derivative with respect to the parameter to be estimated and equalizing it to zero, we get:

$$\text{Risk} = E\left(\left(\frac{\underline{\hat{\theta}}}{\underline{\theta}}\right)^q - q\log\left(\frac{\underline{\hat{\theta}}}{\underline{\theta}}\right) - 1\right)$$

$$= \int_{\forall\theta}\left(\left(\frac{\underline{\hat{\theta}}}{\underline{\theta}}\right)^q - q\log\left(\frac{\underline{\hat{\theta}}}{\underline{\theta}}\right) - 1\right) h\,(\theta.\alpha.\beta|x_1.x_2.\dots.x_n)d\theta$$

$$= \int_{\forall\theta}\left(\left(\frac{\underline{\hat{\theta}}}{\underline{\theta}}\right)^q - q\log(\hat{\theta}) + q\log(\theta) - 1\right) h\,(\theta.\alpha.\beta|x_1.x_2.\dots.x_n)d\theta$$

$$= \underline{\hat{\theta}}^q\hat{\theta}E\left(\underline{\theta}^{-q}|\underline{x}\right) - q\log(\underline{\hat{\theta}}) + q\,E\left(\log(\underline{\theta})\,|\underline{x}\right) - 1 \quad \dots(8)$$

By partially differentiating equation (8) with respect to $(\hat{\theta})$ and setting the derivative equal to zero, we get:

$$\frac{\partial E\left(\left(\frac{\hat{\theta}}{\underline{\theta}}\right)^q - q\log\left(\frac{\hat{\theta}}{\theta}\right) - 1\right)}{\partial\underline{\hat{\theta}}} = 0$$

$$= q\underline{\hat{\theta}}^{q-1}qE\left(\underline{\hat{\theta}}^{-q}\right) - q\underline{\hat{\theta}}^{-1} = 0 \quad \dots(9)$$

Therefore, the Bayes estimator under the general entropy loss function, after taking the root of the force q for both sides of equation (9), we get the following:

$$\underline{\hat{\theta}}_{EL} = E\left(\underline{\hat{\theta}}^{-q}|\underline{x}\right)^{-\frac{1}{q}} \quad \dots (10)$$

It is noticeable that the Bayes estimator according to the general entropy loss function when

(q = −1) equals the Bayes estimator using the quadratic loss function.

Now we need to give the initial distributions of the parameters to be estimated θ, α, β, and according to what information is available to the researcher about the initial distributions of the parameters, suppose that the initial distributions of those parameters will be as follows:

$\theta \sim$ Gamma (a_1, b_1)

$\alpha \sim$ Gamma $(a_2, b_2$

$\beta \sim$ Beta (c, d)

Thus, the priority distribution function for each parameter is formed as follows:

$$\pi_1(\theta) \propto \frac{b_1^{a_1}}{\Gamma(a_1)}\theta^{a_1-1}e^{-b1\theta} \quad ; \theta > 0 \quad \dots(11)$$

$$\pi_2(\alpha) \propto \frac{b_2^{a_2}}{\Gamma(a_2)}\alpha^{a_2-1}e^{-b_2\alpha} \quad ; \alpha > 0 \quad \dots(12)$$

$$\pi_3(\beta) \propto \frac{\Gamma(c+d)}{\Gamma(c)\Gamma(d)}\beta^{c-1}(1-\beta)^{d-1}; 0 < \beta < 1 \quad \dots (13)$$

Therefore, the joint priority is as follows:

$$\pi(\theta,\alpha,\beta) \propto \frac{b_1^{a_1}}{\Gamma(a_1)}\theta^{a_1-1}e^{-b_1\theta}\frac{b_2^{a_2}}{\Gamma(a_2)}\alpha^{a_2-1}e^{-b_2\alpha}\frac{\Gamma(c+d)}{\Gamma(c)\Gamma(d)}\beta^{c-1}(1-\beta)^{d-1}$$

$$\pi(\theta,\alpha,\beta)\ L\propto A\frac{\theta^{2n+a_1-1}\alpha^{a_2-1}e^{-a_1\theta}e^{-a_2\alpha}}{(\alpha\theta+\beta)^n}\beta^{c-1}(1-\beta)^{d-1}\prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\bar{x}} \quad \dots(14)$$

$$A=\frac{b_1{}^{a_1}b_2{}^{a_2}}{\Gamma(a_1)\Gamma(a_2)}\frac{\Gamma(c+d)}{\Gamma(c)\Gamma(d)}$$

a_1, a_2, c, d, are called meta-parameters that can be chosen in such a way that the initial probability density function is decreasing for the parameters to be estimated, and the probability function for the observations $x_1.x_2.\dots.x_n$ is written as follows:

$$L=\left(\frac{\theta^2}{\alpha\theta+\beta}\right)^{\mathbf{n}}\prod_{\mathbf{i=1}}^{\mathbf{n}}(\alpha+\beta x_i)e^{-n\theta\bar{x}} \quad \dots(15)$$

And the marginal density function of the sample observations is in the following form:

$$f(x_1.x_2.\dots.x_n)=\int_\theta\int_\alpha\int_\beta\pi(\theta.\alpha.\beta)\ L\ d\theta d\alpha d\beta$$

$$=f(x_1.x_2.\dots.x_n)=A\int_\theta\int_\alpha\int_\beta\frac{\theta^{2n+a_1-1}\alpha^{a_2-1}e^{-a_1\theta}e^{-a_2\alpha}\beta^{c-1}(1-\beta)^{d-1}}{(\alpha\theta+\beta)^n}\prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\bar{x}}\dots(16)$$

According to Bayes' theorem, the posterior distribution function (Posterior pdf) for the parameters (θ, α, β) can be obtained by using the inverse Bayes formula by dividing the joint distribution function by the marginal density function, as follows:

$$h\ (\theta,\alpha,\beta|x_1.x_2.\dots.x_n)=\frac{(\alpha\theta+\beta)^{-\mathbf{n}}\theta^{2n+a_1-1}\alpha^{a_2-1}e^{-a_1\theta}e^{-a_2\alpha}\beta^{c-1}}{\int_\theta\int_\alpha\int_\beta(\alpha\theta+\beta)^{-\mathbf{n}}\ \theta^{2n+a_1-1}\alpha^{a_2-1}e^{-a_1\theta}e^{-a_2\alpha}\beta^{c-1}}.$$
$$\frac{(1-\beta)^{d-1}\prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\bar{x}}}{(1-\beta)^{d-1}\prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\bar{x}}\,d\theta d\alpha d\beta} \quad \dots(17)$$

By using a general entropy loss function, the standard Bayes estimator of the survival function can be obtained, as follows:

$$\hat{S}_{SBSEL}\ (x_1.x_2.\dots.x_n)\ =E\big(S(x)^{-q}|\underline{x}\big)^{-\frac{1}{q}}$$
$$=\left(\int_\theta\int_\alpha\int_\beta S(x)^{-q}h\ (\theta.\alpha.\beta|x_1.x_2.\dots.x_n)d\theta d\alpha d\beta\right)^{-\frac{1}{q}}$$
$$=\left[\int_\theta\int_\alpha\int_\beta\left(\left(\left[1+\frac{\theta\beta x}{\theta\alpha+\beta}\right]e^{-\theta x}\right)^{-q}\left(\frac{(\alpha\theta+\beta)^{-\mathbf{n}}\theta^{2n+a_1-1}e^{-a_1\theta}e^{-a_2\alpha}\beta^{c-1}}{\int_\theta\int_\alpha\int_\beta(\alpha\theta+\beta)^{-\mathbf{n}}\ \theta^{2n+a_1-1}e^{-a_1\theta}e^{-a_2\alpha}\beta^{c-1}}.\right.\right.\right.$$
$$\left.\left.\left.\frac{\alpha^{a_2-1}(1-\beta)^{d-1}\prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\bar{x}}}{\alpha^{a_2-1}(1-\beta)^{d-1}\prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\bar{x}}\,d\theta d\alpha d\beta}\right)\right)d\theta d\alpha d\beta\right]^{-\frac{1}{q}} \quad \dots(18)$$

We note that equation (18) represents a non-linear equation system that is not theoretically tight and cannot be solved by ordinary analytical methods. Therefore, an approximate method must be used to calculate these complex integrals. Therefore, Lindely Approximation will be used to find the standard Bayesian estimator for the survival function. under the general entropy function.

6.Expected Bayesian Estimator [7][2]

This method was presented by the researcher (Han) in (2006), which assumes that the hyper-parameters in the initial distribution are random variables that have a probability density function where the problem of assigning default values to them is eliminated as in the standard Bayes method The estimation is carried out using the base predictor method according to the following steps:

1-Choosing an initial probability density function that includes hyper parameters to be chosen in such a way that the initial probability density function is decreasing with respect to the parameter to be estimated. The probability density functions for the hyper parameters are as follows:

1. $\pi(a_1)\propto\frac{1}{c}$; $0<a_1<c$ … (19)
2. $\pi(a_2)\propto\frac{1}{c_1}$; $0<a_2<c_1$ … (20)
3. $\pi(b_1)\propto\frac{1}{c_2}$; $0<b_1<c_2$ … (21)
4. $\pi(b_2)\propto\frac{1}{c_3}$; $0<b_2<c_3$ … (22)
5. $\pi(c)\propto\frac{1}{c_4}$; $0<c<c_4$ … (23)
6. $\pi(d)\propto\frac{1}{c_5}$; $0<d<c_5$ … (24)

2-Therefore, the common initial distribution of the meta-parameters is as follows:

$$\pi^*(a_1, a_2, b_1, b_2, c, d) \propto \frac{1}{cc_1 c_2 c_3 c_4 c_5} \quad \dots (25)$$

3-Find the pessimist expectation estimator as follows:

$$\underline{\hat{S}}_{EB} = \int_{a_1 a_2 cd\ b_1, b_2} \underline{\hat{S}}_{SEL\ or\ EL}\, \pi^*(a_1. a_2. b_1. b_2. c. d) da_1 da_2 dd\ d\ db_1 db_2 \quad \dots (26)$$

$\pi^*(a_1, a_2, b_1, b_2, c, d)$: Common probability density function for metaparameters.

$\underline{\hat{S}}_{EL}$: The standard piez estimator under the general entropy loss function.

6-1Expected Bayesian Estimator Under General Entropy Loss

According to the initial probability density functions in equation (19), (20), (21), (22) (23), (24) and using the Bayesian prediction formula in equation No. (26), we get the Bayes prediction estimates for the parameters of the Lindley distribution as follows:

$$\underline{\hat{S}}_{EBEL} = \int_{a_1 a_2 cd\ b_1, b_2} \underline{\hat{S}}_{EL}\, \pi^*(a_1, a_2, b_1, b_2, c, d) da_1 da_2, cd\ d\ db_1 db_2$$

$$= \int_{a_1 a_2 cd\ b_1, b_2} \left(\left[\int_\theta \int_\alpha \int_\beta \left[1 + \frac{\theta\beta x}{\theta\alpha + \beta} \right]^{-q} e^{-\theta x} \left[\frac{(\alpha\theta + \beta)^{-n} \theta^{2n+a_1-1} \alpha^{a_2-1} e^{-a_1\theta} e^{-a_2\alpha}}{\int_\theta \int_\alpha \int_\beta (\alpha\theta + \beta)^{-n} \theta^{2n+a_1-1} \alpha^{a_2-1} e^{-a_1\theta} e^{-a_2\alpha}} \right. \right. \right.$$

$$\left. \left. \frac{\beta^{c-1} (1-\beta)^{d-1} \prod_{i=1}^n (\alpha + \beta x_i) e^{-n\theta\bar{x}}}{\beta^{c-1} (1-\beta)^{d-1} \prod_{i=1}^n (\alpha + \beta x_i) e^{-n\theta\bar{x}}\, d\theta d\alpha d\beta} \right] d\theta d\alpha d\beta \right]^{\frac{-1}{q}} . \frac{1}{c_1 c_2 c_3 c_4 c_5 c} dda_1 da_2 d\ b_1 db_2 dcd \dots (27)$$

We note that equation (27) represents a system of non-linear equations, and its estimations cannot be found theoretically, and it cannot be solved by the usual analytical methods. Therefore, an approximate method must be used to calculate these complex integrals. Therefore, Lindely Approximation will be used to find the Bayesian expectation estimator. The survival function under the general entropy function.

7-Approximation [15[16]

The researcher (Lindley) in the year (1980) put an approximate solution to the integration resulting from the use of the Bayesian estimator method.

$$E[u(\underline{\theta}) \backslash x] = \frac{\int_\Omega u(\underline{\theta}) e^{L(\underline{\theta}) + \rho(\underline{\theta})} d\underline{\theta}}{\int_\Omega e^{L(\underline{\theta}) + \rho(\underline{\theta})} d\underline{\theta}} \quad \dots (28)$$

Since:

$L(\underline{\theta})$: The logarithm of the maximum possibility function.

$\rho(\underline{\theta})$: The logarithm of the previous distribution function for the parameter (θ) .

$u(\underline{\theta})$: Any function of the parameter (θ).

The researcher proposed to Lindley the following formula for solving the integrals resulting from the Bayes formula, as follows:

$$E[u(\underline{\theta}) / x] = u(\hat{\theta}) + \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^m [u_{ij} + 2u_i \rho_j] \sigma_{ij} + \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^m \sum_{k=1}^m \sum_{l=1}^m L_{ijk} u_l \sigma_{ij} \sigma_{kl} \dots (29)$$

Since:

m:represents the number of parameters, (m=3).

$u(\hat{\theta})$: Estimated maximum survival function.

$$L_{ijk} = \frac{\partial^3 L(\underline{\theta})}{\partial\underline{\theta}_1\, \partial\underline{\theta}_2\, \partial\underline{\theta}_3} |_{\underline{\theta} = \underline{\hat{\theta}}} \qquad i. j. k = 1.2.3 \quad (30)$$

$\partial\underline{\theta}_1 = \partial\theta$

$\partial\underline{\theta}_2 = \partial\alpha$

$\partial\underline{\theta}_3 = \partial\beta$

$$\sigma_{ij} = -\left(\frac{\partial^2 L(\underline{\theta})}{\partial\underline{\theta}_i\, \partial\underline{\theta}_j} |_{\underline{\theta}} \right.$$

$$\rho = \mathrm{Log}\left(\pi(\underline{\theta})\right)$$
$$= \mathrm{Log}\left(\frac{b_1{}^{a_1} b_2{}^{a_2}}{\Gamma(a_1)\Gamma(a_2)}\right.$$

$$\rho_i = \frac{\partial \log(\rho)}{\partial \underline{\theta}_i} \qquad i = 1,2,3 \qquad \dots (33)$$

$$u_i = \frac{\partial u(\underline{\theta})}{\partial \underline{\theta}_i}; \qquad i = 1,2,3 \qquad \dots (34)$$

$$u_{ij} = \frac{\partial u^2}{\partial \underline{\theta}_i \, \partial \underline{\theta}_j} \qquad i.j = 1,2,3 \qquad \dots (35)$$

And to get the standard Bayes estimator for the survival function under a general entropy function and assuming that:

$$u(\theta;\alpha;\beta) = \left[\int_\theta\int_\alpha\int_\beta \left[\left[1+\frac{\theta\beta x}{\theta\alpha+\beta}\right]e^{-\theta x}\right]^{-q} \frac{(\alpha\theta+\beta)^{-n}\theta^{2n+a_1-1}\alpha^{a_2-1}e^{-a_1\theta}e^{-a_2\alpha}}{\int_\theta\int_\alpha\int_\beta(\alpha\theta+\beta)^{-n}\theta^{2n+a_1-1}\alpha^{a_2-1}e^{-a_1\theta}e^{-a_2\alpha}} \frac{\beta^{c-1}(1-\beta)^{d-1}\prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\overline{x}}}{\beta^{c-1}(1-\beta)^{d-1}\prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\overline{x}}\,d\theta d\alpha d\beta} d\theta d\alpha d\beta\right]^{\frac{-1}{q}} \qquad \dots (36)$$

by deriving the equation (34):

$$u_1 = \frac{\partial u\left[\int_\theta\int_\alpha\int_\beta \left[\left[1+\frac{\theta\beta x}{\theta\alpha+\beta}\right]e^{-\theta x}\right]^{-q} \frac{(\alpha\theta+\beta)^{-n}\theta^{2n+a_1-1}\alpha^{a_2-1}e^{-a_1\theta}e^{-a_2\alpha}}{\int_\theta\int_\alpha\int_\beta(\alpha\theta+\beta)^{-n}\theta^{2n+a_1-1}\alpha^{a_2-1}e^{-a_1\theta}e^{-a_2\alpha}} \frac{\beta^{c-1}(1-\beta)^{d-1}\prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\overline{x}}}{\beta^{c-1}(1-\beta)^{d-1}\prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\overline{x}}d\theta d\alpha d\beta} d\theta d\alpha d\beta\right]^{\frac{-1}{q}}}{\partial\theta}$$

$$u_2 = \frac{\partial u\left[\int_\theta\int_\alpha\int_\beta \left[\left[1+\frac{\theta\beta x}{\theta\alpha+\beta}\right]e^{-\theta x}\right]^{-q} \frac{(\alpha\theta+\beta)^{-n}\theta^{2n+a_1-1}\alpha^{a_2-1}e^{-a_1\theta}e^{-a_2\alpha}}{\int_\theta\int_\alpha\int_\beta(\alpha\theta+\beta)^{-n}\theta^{2n+a_1-1}\alpha^{a_2-1}e^{-a_1\theta}e^{-a_2\alpha}} \frac{\beta^{c-1}(1-\beta)^{d-1}\prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\overline{x}}}{\beta^{c-1}(1-\beta)^{d-1}\prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\overline{x}}d\theta d\alpha d\beta} d\theta d\alpha d\beta\right]^{\frac{-1}{q}}}{\partial\alpha}$$

$$u_3 = \frac{\partial u\left[\int_\theta\int_\alpha\int_\beta \left[\left[1+\frac{\theta\beta x}{\theta\alpha+\beta}\right]e^{-\theta x}\right]^{-q} \frac{(\alpha\theta+\beta)^{-n}\theta^{2n+a_1-1}\alpha^{a_2-1}e^{-a_1\theta}e^{-a_2\alpha}}{\int_\theta\int_\alpha\int_\beta(\alpha\theta+\beta)^{-n}\theta^{2n+a_1-1}\alpha^{a_2-1}e^{-a_1\theta}e^{-a_2\alpha}} \frac{\beta^{c-1}(1-\beta)^{d-1}\prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\overline{x}}}{\beta^{c-1}(1-\beta)^{d-1}\prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\overline{x}}d\theta d\alpha d\beta} d\theta d\alpha d\beta\right]^{\frac{-1}{q}}}{\partial\beta}$$

$$u_{12} = \frac{\partial^2 u\left[\int_\theta\int_\alpha\int_\beta \left[\left[1+\frac{\theta\beta x}{\theta\alpha+\beta}\right]e^{-\theta x}\right]^{-q} \frac{(\alpha\theta+\beta)^{-n}\theta^{2n+a_1-1}\alpha^{a_2-1}e^{-a_1\theta}e^{-a_2\alpha}}{\int_\theta\int_\alpha\int_\beta(\alpha\theta+\beta)^{n}\theta^{2n+a_1-1}\alpha^{a_2-1}e^{-a_1\theta}e^{-a_2\alpha}} \frac{\beta^{c-1}(1-\beta)^{d-1}\prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\overline{x}}}{\beta^{c-1}(1-\beta)^{d-1}\prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\overline{x}}d\theta d\alpha d\beta} d\theta d\alpha d\beta\right]^{\frac{-1}{q}}}{\partial\theta\,\partial\alpha}$$

$$u_{21} = u_{12}$$

$$u_{12} = \frac{\partial^2 u\left[\int_\theta\int_\alpha\int_\beta \left[\left[1+\frac{\theta\beta x}{\theta\alpha+\beta}\right]e^{-\theta x}\right]^{-q} \frac{(\alpha\theta+\beta)^{-n}\theta^{2n+a_1-1}\alpha^{a_2-1}e^{-a_1\theta}e^{-a_2\alpha}}{\int_\theta\int_\alpha\int_\beta(\alpha\theta+\beta)^{-n}\theta^{2n+a_1-1}\alpha^{a_2-1}e^{-a_1\theta}e^{-a_2\alpha}} \frac{\beta^{c-1}(1-\beta)^{d-1}\prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\overline{x}}}{\beta^{c-1}(1-\beta)^{d-1}\prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\overline{x}}d\theta d\alpha d\beta} d\theta d\alpha d\beta\right]^{\frac{-1}{q}}}{(\partial\alpha)^2}$$

$$u_{13} = \frac{\partial^2 u\left[\int_\theta\int_\alpha\int_\beta\left[\left[1+\frac{\theta\beta x}{\theta\alpha+\beta}\right]e^{-\theta x}\right]^{-q}\frac{(\alpha\theta+\beta)^{-n}\theta^{2n+a_1-1}\alpha^{a_2-1}e^{-a_1\theta}e^{-a_2\alpha}}{\int_\theta\int_\alpha\int_\beta(\alpha\theta+\beta)^{-n}\theta^{2n+a_1-1}\alpha^{a_2-1}e^{-a_1\theta}e^{-a_2\alpha}}\right.}{\partial\theta\,\partial\beta}$$

$$\frac{\left.\frac{\beta^{c-1}(1-\beta)^{d-1}\prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\bar{x}}}{\beta^{c-1}(1-\beta)^{d-1}\prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\bar{x}}d\theta d\alpha d\beta}d\theta d\alpha d\beta\right]^{\frac{-1}{q}}}{}$$

$u_{13} = u_{31}$

$$u_{23} = \frac{\partial^2 u\left[\int_\theta\int_\alpha\int_\beta\left[\left[1+\frac{\theta\beta x}{\theta\alpha+\beta}\right]e^{-\theta x}\right]^{-q}\frac{(\alpha\theta+\beta)^{-n}\theta^{2n+a_1-1}\alpha^{a_2-1}e^{-a_1\theta}e^{-a_2\alpha}}{\int_\theta\int_\alpha\int_\beta(\alpha\theta+\beta)^{-n}\theta^{2n+a_1-1}\alpha^{a_2-1}e^{-a_1\theta}e^{-a_2\alpha}}\right.}{\partial\alpha\,\partial\beta}$$

$$\frac{\left.\frac{\beta^{c-1}(1-\beta)^{d-1}\prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\bar{x}}}{\beta^{c-1}(1-\beta)^{d-1}\prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\bar{x}}d\theta d\alpha d\beta}d\theta d\alpha d\beta\right]^{\frac{-1}{q}}}{}$$

$u_{23} = u_{32}$

$$u_{33} = \frac{\partial^2\left[\int_\theta\int_\alpha\int_\beta\left[\left[1+\frac{\theta\beta x}{\theta\alpha+\beta}\right]e^{-\theta x}\right]^{-q}\frac{(\alpha\theta+\beta)^{-n}\theta^{2n+a_1-1}\alpha^{a_2-1}e^{-a_1\theta}e^{-a_2\alpha}}{\int_\theta\int_\alpha\int_\beta(\alpha\theta+\beta)^{-n}\theta^{2n+a_1-1}\alpha^{a_2-1}e^{-a_1\theta}e^{-a_2\alpha}}\right.}{(\partial\beta)^2}$$

$$\frac{\left.\frac{\beta^{c-1}(1-\beta)^{d-1}\prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\bar{x}}}{\beta^{c-1}(1-\beta)^{d-1}\prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\bar{x}}d\theta d\alpha d\beta}d\theta d\alpha d\beta\right]^{\frac{-1}{q}}}{}$$

$$L_{123} = \frac{\partial^3\left(\frac{\theta^2}{\alpha\theta+\beta}\right)^n\prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\bar{x}}}{\partial\theta\,\partial\alpha\,\partial\beta}$$

$$L_{132} = \frac{\partial^3\left(\frac{\theta^2}{\alpha\theta+\beta}\right)^n\prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\bar{x}}}{\partial\theta\,\partial\beta\,\partial\alpha}$$

$$L_{213} = \frac{\partial^3\left(\frac{\theta^2}{\alpha\theta+\beta}\right)^n\prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\bar{x}}}{\partial\alpha\,\partial\theta\,\partial\beta}$$

$$L_{231} = \frac{\partial^3\left(\frac{\theta^2}{\alpha\theta+\beta}\right)^n\prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\bar{x}}}{\partial\alpha\,\partial\beta\,\partial\theta}$$

$$L_{312} = \frac{\partial^3\left(\frac{\theta^2}{\alpha\theta+\beta}\right)^n\prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\bar{x}}}{\partial\beta\,\partial\theta\,\partial\alpha}$$

$$L_{321} = \frac{\partial^3\left(\frac{\theta^2}{\alpha\theta+\beta}\right)^n\prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\bar{x}}}{\partial\beta\,\partial\alpha\,\partial\theta}$$

$$L_{112} = \frac{\partial^3\left(\frac{\theta^2}{\alpha\theta+\beta}\right)^n\prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\bar{x}}}{\partial\theta\,\partial\theta\,\partial\alpha}$$

$$L_{332} = \frac{\partial^3\left(\frac{\theta^2}{\alpha\theta+\beta}\right)^n\prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\bar{x}}}{\partial\theta\,\partial\theta\,\partial\beta}$$

$$L_{323} = \frac{\partial^3\left(\frac{\theta^2}{\alpha\theta+\beta}\right)^n\prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\bar{x}}}{\partial\beta\,\partial\alpha\,\partial\beta}$$

$$L_{233} = \frac{\partial^3\left(\frac{\theta^2}{\alpha\theta+\beta}\right)^n\prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\bar{x}}}{\partial\alpha\,\partial\beta\,\partial\beta}$$

$$L_{113} = \frac{\partial^3\left(\frac{\theta^2}{\alpha\theta+\beta}\right)^n\prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\bar{x}}}{\partial\theta\,\partial\theta\,\partial\beta}$$

$$L_{131} = \frac{\partial^3 \left(\frac{\theta^2}{\alpha\theta+\beta}\right)^n \prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\overline{x}}}{\partial\theta\,\partial\beta\,\partial\theta}$$

$$L_{311} = \frac{\partial^3 \left(\frac{\theta^2}{\alpha\theta+\beta}\right)^n \prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\overline{x}}}{\partial\beta\,\partial\theta\,\partial\theta}$$

$$L_{233} = \frac{\partial^3 \left(\frac{\theta^2}{\alpha\theta+\beta}\right)^n \prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\overline{x}}}{\partial\alpha\,\partial\beta\,\partial\beta}$$

$$L_{323} = \frac{\partial^3 \left(\frac{\theta^2}{\alpha\theta+\beta}\right)^n \prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\overline{x}}}{\partial\beta\,\partial\alpha\,\partial\beta}$$

$$L_{322} = \frac{\partial^3 \left(\frac{\theta^2}{\alpha\theta+\beta}\right)^n \prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\overline{x}}}{\partial\beta\,\partial\alpha\,\partial\alpha}$$

$$L_{333} = \frac{\partial^3 \left(\frac{\theta^2}{\alpha\theta+\beta}\right)^n \prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\overline{x}}}{\partial\beta\,\partial\beta\,\partial\beta}$$

$$\rho_1 = \frac{\partial \mathrm{Log}\left(\frac{{b_1}^{a_1}{b_2}^{a_2}}{\Gamma(a_1)\Gamma(a_2)}\theta^{a_1-1}\alpha^{a_2-1}e^{-b_1\theta}e^{-b_2\alpha}\frac{\Gamma(c+d)}{\Gamma(c)\Gamma(d)}\beta^{c-1}(1-\beta)^{d-1}\right)}{\partial\theta}$$

$$\rho_2 = \frac{\partial \mathrm{Log}\left(\frac{{b_1}^{a_1}{b_2}^{a_2}}{\Gamma(a_1)\Gamma(a_2)}\theta^{a_1-1}\alpha^{a_2-1}e^{-b_1\theta}e^{-b_2\alpha}\frac{\Gamma(c+d)}{\Gamma(c)\Gamma(d)}\beta^{c-1}(1-\beta)^{d-1}\right)}{\partial\alpha}$$

$$\rho_2 = \frac{\partial \mathrm{Log}\left(\frac{{b_1}^{a_1}{b_2}^{a_2}}{\Gamma(a_1)\Gamma(a_2)}\theta^{a_1-1}\alpha^{a_2-1}e^{-b_1\theta}e^{-b_2\alpha}\frac{\Gamma(c+d)}{\Gamma(c)\Gamma(d)}\beta^{c-1}(1-\beta)^{d-1}\right)}{\partial\beta}$$

$$\sigma_{11} = -\left(\frac{\partial^2 \left(\frac{\theta^2}{\alpha\theta+\beta}\right)^n \prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\overline{x}}}{\partial\theta\,\partial\theta}\right)^{-1}$$

$$\sigma_{12} = -\left(\frac{\partial^2 \left(\frac{\theta^2}{\alpha\theta+\beta}\right)^n \prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\overline{x}}}{\partial\theta\,\partial\alpha}\right)^{-1}$$

$$\sigma_{13} = -\left(\frac{\partial^2 \left(\frac{\theta^2}{\alpha\theta+\beta}\right)^n \prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\overline{x}}}{\partial\theta\,\partial\beta}\right)^{-1}$$

$$\sigma_{21} = -\left(\frac{\partial^2 \left(\frac{\theta^2}{\alpha\theta+\beta}\right)^n \prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\overline{x}}}{\partial\beta\,\partial\theta}\right)^{-1}$$

$$\sigma_{22} = -\left(\frac{\partial^2 \left(\frac{\theta^2}{\alpha\theta+\beta}\right)^n \prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\overline{x}}}{\partial\alpha\,\partial\alpha}\right)^{-1}$$

$$\sigma_{23} = -\left(\frac{\partial^2 \left(\frac{\theta^2}{\alpha\theta+\beta}\right)^n \prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\overline{x}}}{\partial\alpha\,\partial\beta}\right)^{-1}$$

$$\sigma_{32} = -\left(\frac{\partial^2 \left(\frac{\theta^2}{\alpha\theta+\beta}\right)^n \prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\overline{x}}}{\partial\beta\,\partial\alpha}\right)^{-1}$$

$$\sigma_{31}=-\left(\frac{\partial^2\left(\frac{\theta^2}{\alpha\theta+\beta}\right)^n\prod_{i=1}^n(\alpha+\beta x_i)e^{-n\theta\overline{x}}}{\partial\beta\,\partial\theta}\right)^{-1}$$

$$\sigma_{33}=-\left(\frac{\partial^2\left(\frac{\theta^2}{\alpha\theta+\beta}\right)^n\prod_{i=1}^n(\alpha+\beta x_i)e^{-n\theta\overline{x}}}{\partial\beta\,\partial\beta}\right)^{-1}$$

Equation (29) is as follows:

$$\begin{aligned}\hat{S}_{SBSEL}=\hat{S}_{mle}&+u_1\rho_1\sigma_{11}+u_1\rho_2\sigma_{12}++u_1\rho_3\sigma_{13}+0.5L_{231}u_1\sigma_{23}\sigma_{11}+0.5L_{233}u_1\sigma_{23}\sigma_{31}+0.5L_{311}u_1\sigma_{31}\sigma_{11}\\&+0.5L_{312}u_1\sigma_{31}\sigma_{21}+0.5L_{313}u_1\sigma_{31}^2+0.5L_{321}u_1\sigma_{32}\sigma_{11}+0.5L_{323}u_1\sigma_{32}\sigma_{31}+0.5L_{331}u_1\sigma_{33}\sigma_{13}\\&+0.5L_{332}u_1\sigma_{33}\sigma_{21}+0.5L_{333}u_1\sigma_{33}\sigma_{31}+0.5L_{111}u_1\sigma_{11}^2+0.5L_{112}u_1\sigma_{11}\sigma_{21}\\&+0.5L_{113}u_1\sigma_{11}\sigma_{31}\qquad\qquad ...(37)\end{aligned}$$

And to get the Expectation Bayes estimator for the survival function under a general entropy function and assuming that

$$u(\theta;\alpha;\beta)=\left[\int_{a_1a_2cd\ b_1,b_2}\left[1+\frac{\theta\beta x}{\theta\alpha+\beta}\right]^{-q}e^{-\theta x}\left[\frac{(\alpha\theta+\beta)^{-n}\theta^{2n+a_1-1}}{\int_\theta\int_\alpha\int_\beta(\alpha\theta+\beta)^{-n}\theta^{2n+a_1-1}}\right.\right.$$

$$\left.\left.\frac{\alpha^{a_2-1}e^{-a_1\theta}e^{-a_2\alpha}\beta^{c-1}(1-\beta)^{d-1}\prod_{i=1}^n(\alpha+\beta x_i)e^{-n\theta\overline{x}}}{\alpha^{a_2-1}e^{-a_1\theta}e^{-a_2\alpha}\beta^{c-1}(1-\beta)^{d-1}\prod_{i=1}^n(\alpha+\beta x_i)e^{-n\theta\overline{x}}\,d\theta d\alpha d\beta}\right]\right)d\theta d\alpha d\beta\right]^{\frac{-1}{q}}.$$

$$\left.\frac{1}{cc_1c_2c_3c_4c_5}\right)da_1da_2d\ b_1db_2dcd\ d\quad\Bigg]\qquad ...(38)$$

$$u1=\frac{\partial u\left[\int_{a_1a_2cd\ b_1,b_2}\left(\int_\theta\int_\alpha\int_\beta\left[\left(1+\frac{\theta\beta x}{\theta\alpha+\beta}\right)e^{-\theta x}\right]^{-q}\left[\frac{(\alpha\theta+\beta)^{-n}\theta^{2n+a_1-1}\alpha^{a_2-1}e^{-a_1\theta}}{\int_\theta\int_\alpha\int_\beta(\alpha\theta+\beta)^{-n}\theta^{2n+a_1-1}\alpha^{a_2-1}e^{-a_1\theta}}\right.\right.\right.}{\partial\theta}$$

$$\underline{\left.\left.\left.\frac{e^{-a_2\alpha}\beta^{c-1}(1-\beta)^{d-1}\prod_{i=1}^n(\alpha+\beta x_i)e^{-n\theta\overline{x}}}{\beta^{c-1}(1-\beta)^{d-1}\prod_{i=1}^n(\alpha+\beta x_i)e^{-n\theta\overline{x}}d\theta d\alpha d\beta}\right]\right)d\theta d\alpha d\beta\right]^{\frac{-1}{q}}\frac{1}{cc_1c_2c_3c_4c_5}\right]da_1da_2d\ b_1db_2dcd\ d}$$

$$u2=\frac{\partial u\left[\int_{a_1a_2cd\ b_1,b_2}\left(\int_\theta\int_\alpha\int_\beta\left[\left(1+\frac{\theta\beta x}{\theta\alpha+\beta}\right)e^{-\theta x}\right]^{-q}\left[\frac{(\alpha\theta+\beta)^{-n}\theta^{2n+a_1-1}\alpha^{a_2-1}e^{-a_1\theta}e^{-a_2\alpha}}{\int_\theta\int_\alpha\int_\beta(\alpha\theta+\beta)^{-n}\theta^{2n+a_1-1}\alpha^{a_2-1}e^{-a_1\theta}e^{-a_2\alpha}}\right.\right.\right.}{\partial\alpha}$$

$$\underline{\left.\left.\left.\frac{\beta^{c-1}(1-\beta)^{d-1}\prod_{i=1}^n(\alpha+\beta x_i)e^{-n\theta\overline{x}}}{\beta^{c-1}(1-\beta)^{d-1}\prod_{i=1}^n(\alpha+\beta x_i)e^{-n\theta\overline{x}}d\theta d\alpha d\beta}\right]\right)d\theta d\alpha d\beta\right]^{\frac{-1}{q}}\frac{1}{cc_1c_2c_3c_4c_5}\right]da_1da_2d\ b_1db_2dcd\ d}$$

$$u3=\frac{\partial u\left[\int_{a_1a_2cd\ b_1,b_2}\left(\int_\theta\int_\alpha\int_\beta\left[\left(1+\frac{\theta\beta x}{\theta\alpha+\beta}\right)e^{-\theta x}\right]^{-q}\left[\frac{(\alpha\theta+\beta)^{-n}\theta^{2n+a_1-1}\alpha^{a_2-1}e^{-a_1\theta}e^{-a_2\alpha}\beta^{c-1}}{\int_\theta\int_\alpha\int_\beta(\alpha\theta+\beta)^{-n}\theta^{2n+a_1-1}\alpha^{a_2-1}e^{-a_1\theta}e^{-a_2\alpha}\beta^{c-1}}\right.\right.\right.}{\partial\beta}$$

$$\underline{\left.\left.\left.\frac{(1-\beta)^{d-1}\prod_{i=1}^n(\alpha+\beta x_i)e^{-n\theta\overline{x}}}{(1-\beta)^{d-1}\prod_{i=1}^n(\alpha+\beta x_i)e^{-n\theta\overline{x}}d\theta d\alpha d\beta}\right]\right)d\theta d\alpha d\beta\right]^{\frac{-1}{q}}\frac{1}{cc_1c_2c_3c_4c_5}\right]da_1da_2d\ b_1db_2dcd\ d}$$

$$u12=\frac{\partial^2u\left[\int_{a_1a_2cd\ b_1,b_2}\left(\int_\theta\int_\alpha\int_\beta\left[\left(1+\frac{\theta\beta x}{\theta\alpha+\beta}\right)e^{-\theta x}\right]^{-q}\left[\frac{(\alpha\theta+\beta)^{-n}\theta^{2n+a_1-1}\alpha^{a_2-1}e^{-a_1\theta}e^{-a_2\alpha}\beta^{c-1}}{\int_\theta\int_\alpha\int_\beta(\alpha\theta+\beta)^{-n}\theta^{2n+a_1-1}\alpha^{a_2-1}e^{-a_1\theta}e^{-a_2\alpha}\beta^{c-1}}\right.\right.\right.}{\partial\theta\,\partial\alpha}$$

$$\underline{\left.\left.\left.\frac{(1-\beta)^{d-1}\prod_{i=1}^n(\alpha+\beta x_i)e^{-n\theta\overline{x}}}{(1-\beta)^{d-1}\prod_{i=1}^n(\alpha+\beta x_i)e^{-n\theta\overline{x}}d\theta d\alpha d\beta}\right]\right)d\theta d\alpha d\beta\right]^{\frac{-1}{q}}\frac{1}{cc_1c_2c_3c_4c_5}\right]ddda_1da_2d\ b_1db_2dcd\ d}$$

$u_{12}=u_{21}$

$$u22=\frac{\partial^2u\left[\int_{a_1a_2cd\ b_1,b_2}\left(\int_\theta\int_\alpha\int_\beta\left[\left(1+\frac{\theta\beta x}{\theta\alpha+\beta}\right)e^{-\theta x}\right]^{-q}\left[\frac{(\alpha\theta+\beta)^{-n}\theta^{2n+a_1-1}\alpha^{a_2-1}e^{-a_1\theta}e^{-a_2\alpha}\beta^{c-1}}{\int_\theta\int_\alpha\int_\beta(\alpha\theta+\beta)^{-n}\theta^{2n+a_1-1}\alpha^{a_2-1}e^{-a_1\theta}e^{-a_2\alpha}\beta^{c-1}}\right.\right.\right.}{(\partial\alpha)^2}$$

$$\frac{(1-\beta)^{d-1}\prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\bar{x}}}{(1-\beta)^{d-1}\prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\bar{x}}d\theta d\alpha d\beta}\Bigg]\Bigg)d\theta d\alpha d\beta\Bigg]^{\frac{-1}{q}}\frac{1}{cc_1c_2c_3c_4c_5}\Bigg]da_1da_2d\,b_1db_2dcd\,d$$

$$u13=\frac{\partial^2 u\left[\int_{a_1a_2cd\ b_1,b_2}\left[\int_\theta\int_\alpha\int_\beta\left[\left(1+\frac{\theta\beta x}{\theta\alpha+\beta}\right)e^{-\theta x}\right]^{-q}\left[\frac{(\alpha\theta+\beta)^{-n}\theta^{2n+a_1-1}\alpha^{a_2-1}e^{-a_1\theta}e^{-a_2\alpha}\beta^{c-1}}{\int_\theta\int_\alpha\int_\beta(\alpha\theta+\beta)^{-n}\theta^{2n+a_1-1}\alpha^{a_2-1}e^{-a_1\theta}e^{-a_2\alpha}\beta^{c-1}}\right.\right.\right.}{\partial\theta\,\partial\beta}$$

$$\frac{(1-\beta)^{d-1}\prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\bar{x}}}{(1-\beta)^{d-1}\prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\bar{x}}d\theta d\alpha d\beta}\Bigg]\Bigg)d\theta d\alpha d\beta\Bigg]^{\frac{-1}{q}}\frac{1}{cc_1c_2c_3c_4c_5}\Bigg)da_1da_2d\,b_1db_2dcd\,d$$

$u_{13}=u_{31}$

$$u23=\frac{\partial^2 u\left[\int_{a_1a_2cd\ b_1,b_2}\left[\int_\theta\int_\alpha\int_\beta\left[\left(1+\frac{\theta\beta x}{\theta\alpha+\beta}\right)e^{-\theta x}\right]^{-q}\left[\frac{(\alpha\theta+\beta)^{-n}\theta^{2n+a_1-1}\alpha^{a_2-1}e^{-a_1\theta}e^{-a_2\alpha}\beta^{c-1}}{\int_\theta\int_\alpha\int_\beta(\alpha\theta+\beta)^{-n}\theta^{2n+a_1-1}\alpha^{a_2-1}e^{-a_1\theta}e^{-a_2\alpha}\beta^{c-1}}\right.\right.\right.}{\partial\alpha\,\partial\beta}$$

$$\frac{(1-\beta)^{d-1}\prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\bar{x}}}{(1-\beta)^{d-1}\prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\bar{x}}d\theta d\alpha d\beta}\Bigg]\Bigg)d\theta d\alpha d\beta\Bigg]^{\frac{-1}{q}}\frac{1}{cc_1c_2c_3c_4c_5}\Bigg]ddda_1da_2d\,b_1db_2dcd\,d$$

$u_{23}=u_{32}$

$$u33=\frac{\partial^2 u\left[\int_{a_1a_2cd\ b_1,b_2}\left[\int_\theta\int_\alpha\int_\beta\left[\left(1+\frac{\theta\beta x}{\theta\alpha+\beta}\right)e^{-\theta x}\right]^{-q}\left[\frac{(\alpha\theta+\beta)^{-n}\theta^{2n+a_1-1}\alpha^{a_2-1}e^{-a_1\theta}e^{-a_2\alpha}\beta^{c-1}}{\int_\theta\int_\alpha\int_\beta(\alpha\theta+\beta)^{-n}\theta^{2n+a_1-1}\alpha^{a_2-1}e^{-a_1\theta}e^{-a_2\alpha}\beta^{c-1}}\right.\right.\right.}{(\partial\beta)^2}$$

$$\frac{(1-\beta)^{d-1}\prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\bar{x}}}{(1-\beta)^{d-1}\prod_{i=1}^{n}(\alpha+\beta x_i)e^{-n\theta\bar{x}}d\theta d\alpha d\beta}\Bigg]\Bigg)d\theta d\alpha d\beta\Bigg]^{\frac{-1}{q}}\frac{1}{cc_1c_2c_3c_4c_5}\Bigg]ddda_1da_2d\,b_1db_2dcd\,d$$

Equation (29) is as follows:

$$\begin{aligned}\hat{S}_{SBSEL}=\hat{S}_{mle}&+u_1\rho_1\sigma_{11}+u_1\rho_2\sigma_{12}++u_1\rho_3\sigma_{13}+0.5L_{231}u_1\sigma_{23}\sigma_{11}+0.5L_{233}u_1\sigma_{23}\sigma_{31}+0.5L_{311}u_1\sigma_{31}\sigma_{11}\\&+0.5L_{312}u_1\sigma_{31}\sigma_{21}+0.5L_{313}u_1\sigma_{31}^2+0.5L_{321}u_1\sigma_{32}\sigma_{11}+0.5L_{323}u_1\sigma_{32}\sigma_{31}+0.5L_{331}u_1\sigma_{33}\sigma_{13}\\&+0.5L_{332}u_1\sigma_{33}\sigma_{21}+0.5L_{333}u_1\sigma_{33}\sigma_{31}+0.5L_{111}u_1\sigma_{11}^2+0.5L_{112}u_1\sigma_{11}\sigma_{21}\\&+0.5L_{113}u_1\sigma_{11}\sigma_{31}\qquad\ldots(39)\end{aligned}$$

Note that all the integrals and derivations of the aforementioned equations were made within the functions of the Matlab program because it is difficult to solve them manually.

8-Simulations by Monte-Carlo method[3][10]

In order to compare the efficiency of the Bayesian estimation method and the Bayesian prediction method to obtain good estimates of survival function with the desired characteristics, the simulation method was employed by the (Monte-Carlo) method. 50.75.100), noting that the repetition of the experiment was (1000) where the number of estimated experiments was three experiments applied with the MATLAB program)) and the following is a detailed presentation of the experiments.

Table (1) default values for distribution parameters

Experiment	θ	α	β
1	2.5	3.5	4
2	2.5	4	2

Table (2) the real and estimated values of the survival function by all estimation methods and the mean values of integral error squares (IMSE) at each sample size

Model 1		θ=2.5α=3.5		β=4	Model 2		θ=2.5	α=4	β=2
n	**ti**	Real(R(t))	$\hat{S}(t)_{SBEL}$	$\hat{S}(t)_{EBEL}$	**n**	**ti**	Real(R(t))	$\hat{S}(t)_{SBEL}$	$\hat{S}(t)_{EBEL}$
10	0.1	0.99986	0.95000	0.97500		0.1	0.99990	0.95000	0.97500
	0.2	0.99922	0.90000	0.92500		0.2	0.99944	0.90000	0.92500
	0.3	0.99785	0.85000	0.87500		0.3	0.99846	0.85000	0.87500
	0.4	0.99559	0.80000	0.82500		0.4	0.99684	0.80000	0.82500

	0.5	0.99232	0.75000	0.77500	10	0.5	0.99449	0.75000	0.77500
	0.6	0.98791	0.70000	0.72500		0.6	0.99132	0.70000	0.72500
	0.7	0.98227	0.65000	0.67500		0.7	0.98727	0.65000	0.67500
	0.8	0.97533	0.60000	0.62500		0.8	0.98227	0.60000	0.62500
	0.9	0.96703	0.55000	0.57500		0.9	0.97627	0.55000	0.57500
	1	0.95730	0.50000	0.52500		1	0.96923	0.50000	0.52500
IMSE			0.084837	0.072439	IMSE			0.087958	0.075356
Best			$\hat{S}(t)_{EBEL}$		Best			$\hat{S}(t)_{EBEL}$	
	0.1	0.99986	0.97500	0.98750		0.1	0.99990	0.97500	0.98750
	0.2	0.99922	0.95000	0.96250		0.2	0.99944	0.95000	0.96250
	0.3	0.99785	0.92500	0.93750		0.3	0.99846	0.92500	0.93750
	0.4	0.99559	0.90000	0.91250		0.4	0.99684	0.90000	0.91250
25	0.5	0.99232	0.87500	0.88750	25	0.5	0.99449	0.87500	0.88750
	0.6	0.98791	0.85000	0.86250		0.6	0.99132	0.85000	0.86250
	0.7	0.98227	0.82500	0.83750		0.7	0.98727	0.82500	0.83750
	0.8	0.97533	0.80000	0.81250		0.8	0.98227	0.80000	0.81250
	0.9	0.96703	0.77500	0.78750		0.9	0.97627	0.77500	0.78750
	1	0.95730	0.75000	0.76250		1	0.96923	0.75000	0.76250
IMSE			0.018559	0.015641	IMSE			00.020022	0.017002
Best			$\hat{S}(t)_{EBEL}$		Best			$\hat{S}(t)_{EBEL}$	
	0.1	0.99986	0.99975	0.99167		0.1	0.99990	0.99981	0.99167
	0.2	0.99922	0.99883	0.97500		0.2	0.99944	0.99912	0.97500
	0.3	0.99785	0.99707	0.95833		0.3	0.99846	0.99782	0.95833
	0.4	0.99559	0.99436	0.94167		0.4	0.99684	0.99582	0.94167
50	0.5	0.99232	0.99058	0.92500	50	0.5	0.99449	0.99304	0.92500
	0.6	0.98791	0.98565	0.90833		0.6	0.99132	0.98942	0.90833
	0.7	0.98227	0.97950	0.89167		0.7	0.98727	0.98491	0.89167
	0.8	0.97533	0.97205	0.87500		0.8	0.98227	0.97946	0.87500
	0.9	0.96703	0.96328	0.85833		0.9	0.97627	0.97302	0.85833
	1	0.95730	0.95312	0.84167		1	0.96923	0.96557	0.84167
IMSE			0.000137	0.005945	IMSE			0.000095	0.006787
Best			$\hat{S}(t)_{SBEL}$		Best			$\hat{S}(t)_{SBEL}$	
	0.1	0.99986	0.99978	0.99375		0.1	0.99990	0.99983	0.99375
	0.2	0.99922	0.99895	0.98125		0.2	0.99944	0.99918	0.98125
	0.3	0.99785	0.99732	0.96875		0.3	0.99846	0.99792	0.96875
	0.4	0.99559	0.99475	0.95625		0.4	0.99684	0.99597	0.95625
75	0.5	0.99232	0.99114	0.94375	75	0.5	0.99449	0.99324	0.94375
	0.6	0.98791	0.98640	0.93125		0.6	0.99132	0.98967	0.93125
	0.7	0.98227	0.98045	0.91875		0.7	0.98727	0.98521	0.91875
	0.8	0.97533	0.97322	0.90625		0.8	0.98227	0.97979	0.90625

	0.9	0.96703	0.96466	0.89375		0.9	0.97627	0.97339	0.89375
	1	0.95730	0.95473	0.88125		1	0.96923	0.96597	0.88125
IMSE			0.000136	0.002828	IMSE			0.000055	0.003411
Best			$\hat{S}(t)_{SBEL}$		Best			$\hat{S}(t)_{SBEL}$	
100	0.1	0.99986	0.99968	0.99500	100	0.1	0.99990	0.99984	0.99500
	0.2	0.99922	0.99859	0.98500		0.2	0.99944	0.99920	0.98500
	0.3	0.99785	0.99660	0.97500		0.3	0.99846	0.99797	0.97500
	0.4	0.99559	0.99360	0.96500		0.4	0.99684	0.99604	0.96500
	0.5	0.99232	0.98949	0.95500		0.5	0.99449	0.99334	0.95500
	0.6	0.98791	0.98421	0.94500		0.6	0.99132	0.98980	0.94500
	0.7	0.98227	0.97770	0.93500		0.7	0.98727	0.98535	0.93500
	0.8	0.97533	0.96990	0.92500		0.8	0.98227	0.97995	0.92500
	0.9	0.96703	0.96078	0.91500		0.9	0.97627	0.97356	0.91500
	1	0.95730	0.95031	0.90500		1	0.96923	0.96614	0.90500
IMSE			0.000139	0.001513	IMSE			0.000063	0.001939
Best			$\hat{S}(t)_{SBEL}$		Best			$\hat{S}(t)_{SBEL}$	

9-Conclusions

- The standard Bayes estimator under the general entropy loss function is more efficient because it has the lowest value (IMSE) for most sample sizes and default.
- The standard Bayes estimator under the general entropy loss function is more efficient because it has the lowest IMSE value for most sample sizes and default parameter values.
- The standard Bayes method proved its efficiency in estimating the survival function of a three-parameter Lindley distribution model.
- We note that the estimated values of the survival function decrease as time (t) increases, and this matches the theory about the behavior of the survival function as it is a decreasing function.

10-Recommendations

- The researcher recommends relying on the standard Bayes method to estimate the survival function instead of the Bayes prediction method.
- The researcher recommends expanding the scope of the study to include the use of real data based on the Bayesian methods used in the research.
- The researcher recommends suggesting loss functions with the used loss functions.

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