ON ξ-CONFORMALLY FLAT TRANS-SASAKIAN MANIFOLDS ADMITTING SEMI-SYMMETRIC NON-METRIC CONNECTION

Abhishek Singh
Department of Mathematics and Statistics
Dr. Rammanohar Lohia Avadh University, Ayodhya, U.P., India
Email: abhi.rmlau@gmail.com

Abstract:
We studied a ξ-conformally flat trans-Sasakian manifold admitting a semi-symmetric non-metric connection. Some interesting results on a β-Kenmotsu manifold admitting the semi-symmetric non-metric connection concluded as well.
Keywords: Semi-symmetric non-metric connection, ξ-conformally flat, trans-Sasakian manifold.

1. Introduction

The study of semi-symmetric connection in a Riemannian manifold was introduced by Yano [16]. Agashe and Chafle [1] introduced the notion of semi-symmetric non-metric connection. Later on it was studied by several geometers (see [5, 2, 15] and their references).
On the other, a class of almost contact metric manifold namely trans-Sasakian manifold [11] established as a generalization of α-Sasakian [14] and β-Kenmotsu [10] manifold. A trans-Sasakian structure of type (0, 0), (α, 0) and (0, β) are cosymplectic, α-Sasakian and β-Kenmotsu respectively. For detail study of trans-Sasakian manifold, we refer to [6, 9, 12]. In this paper, we study some properties of conformal curvature tensor on a trans-Sasakian manifold admitting the semi-symmetric non-metric connection. The conformal curvature tensor C on a $(2n+1)$-dimensional Riemannian manifold is defined as under [7].

$$C(X,Y)Z = R(X,Y)Z - \frac{1}{(2n-1)}[S(Y,Z)X - S(X,Z)Y + \{g(Y,Z)QX - g(X,Z)QY]$$
$$+\frac{r}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{1.1}$$

where S and Q are Ricci-tensor and Ricci-operator respectively.
The paper is organized as under. Section-**2** contains some preliminaries. In Section-**3**, it is proved that a β-Kenmotsu manifold is ξ-conformally flat with respect to semi-symmetric non-metric connection if and only if it is ξ-conformally flat with respect to the Levi-civita connection. We also found the Ricci tensor with respect to the Levi-civita connectionin a ξ-conformally flat trans-Sasakian manifold admitting semi-symmetric non-metric connection. Here we deduce that a ξ-conformally flat β-Kenmotsu manifold admitting semi-symmetric non-metric connection is an $\eta-$Einstein manifold. It is proved that in a ξ-conformally flat trans-Sasakian manifold admitting semi-symmetric non-metric connection, $\xi\beta=0$.

2. Preliminaries

Let M be a $(2n+1)$-dimensional almost contact metric manifold (see [3, 4, 7, 8]) equipped with almost contact metric structure φ, ξ, η, g, where φ is (1,1) tensor field, ξ is a vector field, η is 1-form and g is Riemannian metric such that

$$\varphi^2 = -I + \eta\otimes\xi, \qquad \eta(\xi)=1, \quad \varphi\xi=0, \quad \eta o\varphi=0 \tag{2.1}$$
$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(\varphi X,Y)=-g(X,\varphi Y),\qquad g(X,\xi)=\eta(X), \tag{2.3}$$

for all $X,Y\in TM$.An almost contact metric manifold M is called trans-Sasakian manifold if

$$(\nabla_X\varphi)Y=\alpha\{g(X,Y)\xi-\eta(Y)X\}+\beta\{g(\varphi X,Y)\xi-\eta(Y)\varphi X\} \tag{2.4}$$

where ∇ is Levi-civita connection of Riemannian metric g and α and β are smooth functions on M.The equation (2.4) together with equations (2.1), (2.2) and (2.3), we have

$$\nabla_X\xi=-\alpha\varphi X+\beta[X-\eta(X)\xi], \tag{2.5}$$

$$(\nabla_X\eta)Y=-\alpha g(\varphi X,Y)+\beta g(\varphi X,\varphi Y) \tag{2.6}$$

In a trans-Sasakian manifold, we also have [9, 12]

$$R(X,Y)\xi=(\alpha^2-\beta^2)(\eta(Y)X-\eta(X)Y)+2\alpha\beta(\eta(Y)\varphi X-\eta(X)\varphi Y)+(Y\alpha)\varphi X$$
$$-(X\alpha)\varphi Y+(Y\beta)\varphi^2X-(X\beta)\varphi^2Y \tag{2.7}$$

$$R(\xi,Y)X=(\alpha^2-\beta^2)(g(X,Y)\xi-\eta(X)Y)+2\alpha\beta(g(\varphi X,Y)\xi+\eta(X)\varphi Y)+(X\alpha)\varphi Y$$
$$+g(\varphi X,Y)(grad\,\alpha)+X\beta(Y-\eta(Y)\xi)-g(\varphi X,\varphi Y)(grad\beta), \tag{2.8}$$

$$R(\xi,X)\xi=(\alpha^2-\beta^2-\xi\beta)(\eta X\xi-X) \tag{2.9}$$

and $2\alpha\beta+\xi\alpha=0$, (2.10)

where R is the curvature tensor.

$$S(X,\xi)=(2n(\alpha^2-\beta^2)-\xi\beta)\eta(X)-(2n-1)X\beta-(\varphi X)\alpha, \tag{2.11}$$

$$Q\xi=(2n(\alpha^2-\beta^2)-\xi\beta)\xi-(2n-1)\,grad\beta+\varphi(grad\alpha), \tag{2.12}$$

Where S is the Ricci-curvature and Q is the Ricci-operator of trans-Sasakian manifold of type (α,β). S and Q are related to each other by

$$S(X,Y)=g(QX,Y).$$

Under the condition $\varphi(grad\alpha)=(2n-1)(grad\beta)$,we have

$$\xi\beta=0. \tag{2.13}$$

Hence

$$S(X,\xi)=(2n(\alpha^2-\beta^2)-\xi\beta)\eta X, \tag{2.14}$$

$$Q\xi=(2n(\alpha^2-\beta^2)-\xi)\xi. \tag{2.15}$$

In an almost contact metric manifold M, $\eta-$Einstein characterized as under:

$$S(X,Y)=ag(X,Y)+b\eta(X)\eta(Y),$$

where a and b are smooth functions on M. A $\eta-$Einstein manifold becomes Einstein if $b=0$.

Let $\{e_1, e_, ..., e_n = \xi\}$ is a local orthonormal basis of vector fields in an n-dimensional almost contact manifold M. Definitely, then $\{\varphi e_1, \varphi e_2, ..., \varphi e_{n-1}, \xi\}$ is also a local orthonormal basis. Hence, we have

$$\sum_{i=1}^{n} g(e_i, e_i) = \sum_{i=1}^{n-1} g(\varphi e_i, \varphi e_i) + g(\xi, \xi) = n,$$

A linear connection $\tilde{\nabla}$ in an almost contact metric manifold M is said to be

- semi-symmetric connection [16] if its torsion tensor satisfies
$$T(X, Y) = \eta(Y)X - \eta(X)Y$$
- non-metric connection [1] if
$$(\tilde{\nabla})g \neq 0.$$

A semi-symmetric non-metric connection $\tilde{\nabla}$ [1] in an almost contact metric manifold M is defined as

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)X. \quad (2.16)$$

Let $\tilde{R}$ and R be the curvature tensors of the semi-symmetric non-metric connection $\tilde{\nabla}$ and the Levi-civita connection ∇ respectively. Then it is well known that

$$\tilde{R}(X, Y)Z = R(X, Y)Z + A(X, Z)Y - A(Y, Z)X, \quad (2.17)$$

where A is a tensor field of type (0,2) given by

$$A(X, Y) = (\tilde{\nabla}_X \eta)Y = (\nabla_X \eta)Y - \eta(X)\eta(Y) \quad (2.18)$$

From (2.17), we deduce that

$$\tilde{S}(X, Y) = S(X, Y) - 2nA(X, Y), \quad (2.19)$$
$$\tilde{r} = r - 2n\, traceA, \quad (2.20)$$

where $\tilde{S}$ and S are Ricci-tensors and $\tilde{r}$ and r are scalar curvatures of the semi-symmetric non-metric connection $\tilde{\nabla}$ and the Levi-civita connection ∇ respectively.
On a trans-Sasakian manifold with respect to semi symmetric non-metric connection, we have [13]

Lemma 2.1 *Let M be a trans-Sasakian manifold with respect to semi-symmetric non-metric connection, then*

$$(\tilde{\nabla}_X \varphi)(Y) = \alpha\{g(X, Y)\xi - \eta(Y)X\} + \beta\{g(\varphi X, Y)\xi - \eta(Y)\varphi X\} - \eta(Y)\varphi X, \quad (2.21)$$
$$\tilde{\nabla}_X \xi = X - \alpha\varphi X + \beta\{X - \eta(X)\xi\}, \quad (2.22)$$

$$(\tilde{\nabla}_X \eta)Y = -\alpha g(\varphi X, Y) + \beta g(\varphi X, \varphi Y) - \eta(X)\eta(Y), \quad (2.23)$$

$$\tilde{R}(X, Y)Z = R(X, Y)Z + \alpha\{g(\varphi Y, Z)X - g(\varphi X, Z)Y\} - \beta\{g(Y, Z)X - g(X, Z)Y\} + (\beta + 1)\eta(Z)\{\eta(Y)X - \eta(X)Y\} \quad (2.24)$$

We also have the following theorem [13].

Theorem 2.2 *In an $(2n+1)$-dimensional trans-Sasakian manifold, the Ricci-tensor $\tilde{S}$ and the scalar curvature $\tilde{r}$ with respect to semi-symmetric non-metric connection $\tilde{\nabla}$ are given by*

$$S(X,\ Y) = S\ (X, Y\ ) + 2n[\alpha g\ (\varphi X, Y\ )] - \beta g\ (X,\ Y) + (\beta + 1)\eta(X)\ \eta(Y)], \qquad (2.25)$$
$$\tilde{r} = r - 2n(2n\beta -\ 1). \qquad (2.26)$$

3. ξ−conformally flat trans-Sasakian manifolds admitting semi-symmetric non-metric connection

The relation between the conformal curvature tensor with respect to semi-symmetric non-metric connection and the conformal curvature tensor with respect to Levi-civita connection on a trans-Sasakian manifold is as follows[13]

$$\tilde{C}(X,Y)Z = C(X,Y)Z - \frac{\alpha}{(2n-1)}[\{g(\varphi Y,Z)X - g(\varphi X,Z)Y\}$$
$$+2n\{g(Y,Z)\varphi X - g(X,Z)\varphi Y\}] + \frac{(1+\beta)}{(2n-1)}[\{g(Y,Z)X - g(X,Z)Y\}$$
$$+\eta(Z)\{\eta(X)Y - \eta(Y)X\} + 2n\{\eta(Y)g(X,Z) - \eta(X)g(Y,Z)\}\xi], \qquad (3.1)$$

where $\tilde{C}$ and C are the conformal curvature tensor admitting semi-symmetric non-metric connection and the conformal curvature tensor admitting Levi-civita connection respectively. Taking Z=ξ in the equation (5.3.1),we get

$$\tilde{C}(X,Y)\ \xi = C(X,Y)\ \xi - \frac{2n\alpha}{(2n-1)}[\eta(Y)\ \varphi X - \eta(X)\ \varphi Y]. \qquad (3.2)$$

On taking α=0in the equation (5.3.2), we have

$$\tilde{C}(X,Y)\xi = C(X,Y)\xi, \qquad (3.3)$$

which leads to the following theorem:

Theorem 3.1 A β-kenmotsu manifold is ξ-conformally flat admitting semi-symmetric non-metric connection if and only if it is ξ-conformally flat with respect to the Levi-civita connection.

On taking Z=ξ in the equation (1.1),we get

$$C(X,\ Y)\xi = R(X,Y)\ \xi - \frac{1}{(2n-1)}[S(Y,\ \xi)\ X - S(X,\ \xi)\ Y + \eta(Y)\ QX - \eta(X)\ QY]$$
$$+\frac{r}{2n(2n-1)}[\eta(Y)\ X - \eta(X)\ Y], \qquad (3.4)$$

and taking account of equations (2.7) and (2.11),we get

$$C(X,Y)\xi = (\alpha^2 - \beta^2)\ (\eta(Y)\ X - \eta(X)\ Y) + 2\alpha\beta(\eta(Y)\ \varphi X - \eta(X)\ \varphi Y)$$
$$+ (Y\alpha)\ \varphi X - (X\alpha)\ \varphi Y + (Y\beta)\varphi^2 X - (X\beta)\ \varphi^2 Y$$
$$-\frac{1}{(2n-1)}[\{(2n(\alpha^2 - \beta^2) - \xi\beta)\ \eta(Y) - ((2n-1)Y\beta + (\varphi Y)\alpha)\}X$$
$$- \{(2n\ (\alpha^2 - \beta^2) - \xi\beta)\ \eta(X)\ - ((2n-1)X\beta + (\varphi X)\alpha)\}Y$$
$$+ (\eta(Y)QX - \eta(X)QY)] + \frac{r}{2n(2n-1)}[\eta(Y)X - \eta(X)Y].$$

On simplifying, we get

$$C(X,Y)\xi = \frac{1}{(2n-1)}\left(\frac{r}{2n} - ((\alpha^2 - \beta^2) - \xi\beta)\right)(\eta(Y)X - \eta(X)Y)$$
$$+2\alpha\beta(\eta(Y)\varphi X - n(X)\varphi Y) + ((Y\alpha)\varphi X - (X\alpha)\varphi Y)$$
$$+\big((Y\beta)\eta(X) - (X\beta)\eta(Y)\big)\xi + \frac{1}{(2n-1)}((\varphi Y)\alpha X - (\varphi X)\alpha Y)$$
$$-\frac{1}{(2n-1)}(\eta(Y)QX - \eta(X)QY). \qquad (3.5)$$

Putting the value of $C(X,Y)\xi$ in the equation (3.2), we get

$$\tilde{C}(X,Y)\xi = \frac{1}{(2n-1)}\left(\frac{r}{2n} - ((\alpha^2-\beta^2)-\xi\beta)\right)(\eta(Y)X-\eta(X)Y)$$
$$+\left(2a\beta - \frac{2n\alpha}{(2n-1)}\right)(\eta(Y)\varphi X - \eta(X)\varphi Y)$$
$$+\big((Y\alpha)\varphi X - (X\alpha)\varphi Y\big) + ((Y\beta)\eta(X) - (X\beta)\eta(Y))\xi$$
$$+\frac{1}{(2n-1)}((\varphi Y)\alpha X - (\varphi X)\alpha Y) - \frac{1}{(2n-1)}(\eta(Y)QX - \eta(X)QY). \quad (3.6)$$

Since the manifold under consideration is ξ-conformally flat with respect to semi-symmetric non-metric connection, hence equation (3.6) yields.

$$\eta(Y)QX = \left(\frac{r}{2n} - ((\alpha^2-\beta^2)-\xi\beta)\right)(\eta(Y)X-\eta(X)Y)$$
$$+(2(2n-1)\alpha\beta - 2n\alpha)(\eta(Y)\varphi X - \eta(X)\varphi Y)$$
$$+(2n-1)((Y\alpha)\varphi X - (X\alpha)\varphi Y)$$
$$+(2n-1)\big((Y\beta)\eta(X) - (X\beta)\eta(Y)\big)\xi$$
$$+\big((\varphi Y)\alpha X - (\varphi X)\alpha Y\big) + \eta(X)QY$$

On taking $Y=\xi$ in the above equation, we get

$$QX = \left(\frac{r}{2n} - ((\alpha^2-\beta^2)-\xi\beta)\right)X + \left\{(2n+1)(\alpha^2-\beta^2) + (2n-1)\xi\beta - \frac{r}{2n}\right\}\eta(X)\xi$$
$$-2n\alpha(\varphi X) - (2n-1)(X\beta)\xi - (\varphi X)\alpha\xi - (2n-1)\eta(X)grad\,\beta +$$
$$\eta(X)\varphi(grad\alpha).$$

Taking account of $S(X,Y) = g(QX,Y)$ in the above equation, we get

$$S(X,Y) = \left(\frac{r}{2n} - ((\alpha^2-\beta^2)-\xi\beta)\right)g(X,Y)$$
$$+\left\{(2n+1)(\alpha^2-\beta^2) + (2n-1)\xi\beta - \frac{r}{2n}\right\}\eta(X)\eta(Y) - 2n\alpha g(\varphi X,Y)$$
$$-\{(2n-1)(X\beta) + (\varphi X)\alpha\}\eta(Y) - \{(2n-1)(Y\beta) + (\varphi Y)\alpha\}\eta(Y). \quad (3.7)$$

Hence we have

Theorem 3.2 In a ξ-conformally flat trans-Sasakian manifold admitting semi-symmetric non-metric connection, Ricci tensor with respect to Levi-civita connection is given by the equation (3.7).

It is known that a trans-Sasakian manifold of kind $(0,\ \beta)$ is a β-Kenmotsu manifold and in a β-Kenmotsu manifold, β is constant. Hence in a β-Kenmotsu manifold equation (3.7) reduces to

$$S(X,Y) = \left(\frac{r}{2n} + \beta^2\right)g(X,Y) - \left\{(2n+1)\beta^2 + \frac{r}{2r}\right\}\eta(X)\eta(Y). \quad (3.8)$$

This leads to the following corollary:

Corollary 3.3 A ξ-conformally flat β-Kenmotsu manifold admitting semi-symmetric non-metric connection is an η-Einstein manifold.

Let $\{e_1, e_2, \ldots\ldots\ldots e_{2n}, e_{2n+1}=\xi\}$ is a local orthonormal basis of vector fields in an n-dimensional almost contact manifold M. Contracting equation (3.7) and using

$$\sum_{i=1}^{2n+1} S(e_i,\ e_i) = r,$$

$$\sum_{i=1}^{2n+1} g(e_i,\ e_i) = 2n + 1$$

and $$\eta(\mathrm{e_i}) = 0,$$

we get $$\xi\beta=0.$$

Hence, we have

Corollary 3.4 In a ξ-conformally flat trans-Sasakian manifold admitting semi-symmetric non-metric connection, ξβ=0.

Acknowledgment:
The author is thankful to Professor Ram Nivas, Department of Mathematics and Astronomy, University of Lucknow providing suggestions for the improvement of this paper.

References:

1. **Agashe, N.S. and Chafle, M. R.,** "*A semi-symmetric non-metric connection,* Indian J. Pure Math. 23(1992), 399-409.

2. **Biswas, S. C. and De, U. C.,** "*On a type of semi-symmetric non-metric connection on a Riemannian manifold",*Ganita 48 (1997), 91-94.

3. **Blair, D. E.,** "*Contact manifolds in Riemannian geometry",* Lecture Notes in Math.509, springer Verlag, 1976.

4. **Blair, D. E.,** "*Riemannian geometry of Contact and Symplectic manifolds",* Birkhauser Boston, 2002.

5. **De, U. C. and Kamilya, D.,** "*Hypersurfaces of a Riemannian manifold with semi-symmetric non-metric connection",* J. Indian Inst. Sci.75 (1995), 707-710.

6. **De, U. C. and Tripathi, M. M.,** "*Ricci tensor in 3-dimensional trans- Sasakian Manifolds",* Kyungpook Math. J., 2, 247-255 (2005).

7. **De, U. C. and Shaikh, A. A.,** "*Differential Geometry of manifolds",* Narosa Publishing House, 2007.

8. **De, U. C. and Shaikh, A. A.,** "*Complex manifolds and contact manifolds",* Narosa Publishing House, 2009.

9. **Jeong-Sik Kim, Prasad, R. and Tripathi, M. M.,** "*On generalized Ricci recurrent trans-Sasakian manifolds",* J. Korean Math. Soc. 39 (2002), No. 6, 953-961.

10. **Kenmotsu, K.,** "*A class of almost contact Riemannian manifolds",* Tohoku Math. J., 24 (1972), 93-103.

11. **Oubiña, J. A.,** "*New Classes of almost contact metric structures",* Pub. Math. Debrecen 32 (1985),187-193.

12. **Prasad, R., Pankaj, Tripathi, M. M. and Shukla, S. S.,** "*On some special type of trans-Sasakian Manifolds",* Rev. Mat. Univ. Parma, Vol.8, 2 (2009), 1-17.

13. **Prasad, R. and Pankaj.,** "*Some curvature tensors on a trans-Sasakian Manifold with respect to semi-symmetric non-metric connections", J. N*at.Acad.Math., Sp.Vol. (2009), 55-64.

14. **Sasaki, S.,** "*On differentiable manifolds with certain structure which are closed related to an almost contact structure",* Tohoku Math. Journal. 12 (1960), 459-476.

15. **Singh, R. N., Pandey, M. K.,** "*Some properties of semi-symmetric non-metric connection on a Riemannian manifold",* Bull. Cal. Math. Soc., 98-5(1997), 443-454.

16. **Yano, K.,** "*Integral Formulas in Riemannian geometry",* Pure and Applied Mathematics, No.1 Marcel Dekker, Inc., New York, 1970.