Turkish Journal of Computer and Mathematics Education *Vol.12 No.14(2021), 5115- 5125*
Research Article

Stability of homogenous linear system of differential equations – A Discussion

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ABSTRACT

System of ordinary linear differential equations play an important role in physical systems such as double pendulum, oscillating spring, home heating, in the economic systems of priceinventory dynamic and another physical phenomenon. The stability of solutions of these systems also an essential part in deriving the conclusions. In this paper, author presents the different methods to find the solutions of linear systems of ordinary differential equations with constant coefficients and compares the difficulties of each method. It was observed that the application of matrix methods are very useful in discussing the stability of a dynamical system with constant coefficients.

Introduction: The system of homogeneous linear ordinary differential plays an important role in studying the dynamical system. Mathematicians define the dynamical system as a functional representation of a point in geometrical space that is dependent on time. Oscillations of a pendulum, the flow of water in a pipe and the number of fish each springtime in a lake are examples of dynamical system.

For a physicist, a dynamical system is variations of a particle with respect to time, that is expressed as differential equations involving the time derivatives. The study of dynamical systems has wide range of applications in the field of mathematics, physics[1][2], biology [3], chemistry, engineering[4], economics[5], history and medicine.

Mixing problem involving two tanks, current I_1 *and* I_2 in the network containing L,C, R_1, and R_2, motion of two masses m_1 and m_2 joined end to end to two springs are examples of systems of linear differential equations with constant coefficients.

The homogeneous systems of linear differential equations can be solved by elimination method, matrix methods and eigenvalues and eigenvectors. The stability of the system will analysed

The study of systems of ordinary differential equations has theoretical and practical importance, as it allows the study and solution of single n th order ordinary differential equation by the methods of systems and also the inclusion of theory of higher order equations into a first order system[4].

The first order system

$$u_1^{'} = u_2,$$

$$u_2^{'} = u_3,$$

$$\dots \dots$$

$$u_{n-1}^{'} = u_n,$$

$u_n^{'} = g(t, u_1, u_2, \dots, u_n)$ ……….(1)

equivalent to n^{th}order differential equation

$$u^{(n)} = g\left(t, u, u^{'}, u^{''}, \dots, u^{(n-1)}\right) \dots \dots (1)$$

The lessening the order is also a main reason for an attention on first order systems of ordinary differential equations.

Methodology:

Basic Concepts of Systems of ODE a review:

The general form of the system of ordinary differential equations is

$$u_1^{'} = g_1(t, u_1, u_2, \dots, u_n) + h_1(t)$$

$$u_2^{'} = g_2(t, u_1, u_2, \dots, u_n) + h_2(t)$$

$$u_3^{'} = g_3(t, u_1, u_2, \dots, u_n) + h_3(t)$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$u_n^{'} = g_n(t, u_1, u_2, \dots, u_n) + h_n(t)$$

The vectorial form of the system is $\overline{u'} = \bar{g}(t, \bar{u})$.

The solution of the linear differential equations with constant coefficients exists by the theorems Existence and Uniqueness theorem, which were discussed and presented [6].Generalising the theory of linear differential equations with constant coefficients for the homogeneous systems of linear differential equations, we have the following results for the systems of linear differential equations with constant coefficients $\overline{u'} = A\bar{u}$.

$$A = [a_{ij}], i, j = 1,2,3 \dots, n.$$

The solutions of $\overline{u'} = A\bar{u}$leads to eigenvalue problem of A.

Theorem1: If a constant coefficient matrix $A = [a_{ij}]$ has a linearly independent set of n eigenvectors, then the corresponding solutions $u^{(1)} = xe^{\alpha_1 t}, u^{(2)} = xe^{\alpha_2 t}, \dots, u^{(n)} = xe^{\alpha_n t}$

are solutions of $\overline{u'} = A\bar{u}$ and corresponding general solution is

$$u = k_1 xe^{\alpha_1 t} + k_2 xe^{\alpha_2 t} + \cdots + k_n xe^{\alpha_n t}, where\ k_1, k_2, \dots\dots, k_n\ are\ constants.$$

Let $\overline{u'} = A\bar{u}$ be homogeneous system with constant coefficients consisting of n – equations

The components of this system are

$$u_1^{'} = a_{11}u_1 + a_{12}u_2 + \cdots + a_{1n}u_n$$

$$u_2^{'} = a_{21}u_1 + a_{22}u_2 + \cdots + a_{2n}u_n$$

$$u_3^{'} = a_{31}u_1 + a_{32}u_2 + \cdots + a_{3n}u_n$$

$$\dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots.\ \dots$$

$$u_n^{'} = a_{n1}u_1 + a_{n2}u_2 + \cdots + a_{nn}u_n$$

The solutions of the homogenous system of linear differential equations can be plotted on $u_1 u_2 \dots u_n$ space, as a single parametric curve, with parameter 't'. These curves are called trajectories or orbit or path. And the space $u_1 u_2 \dots u_n$ is called the phase space of the systems of equations. The space containing the trajectories of the system is called phase portrait.

A phase portrait gives a general qualitative impression of the complete family of solutions. As a result, the studies of solutions in the phase plane have become pretty significant, with the development of computers and computer graphics.

The critical point of a system of ordinary differential equations is a pointP_0, at which $\frac{u_i^{'}}{u_j^{'}}\ is\ undetermined\ for\ i \neq j$

For example, for a system of two differential equations,

$$u_1^{'} = a_{11}u_1 + a_{12}u_2$$

$$u_2^{'} = a_{21}u_1 + a_{22}u_2$$

The critical point is a point, where $\frac{u_2^{'}}{u_1^{'}} = \frac{a_{21}u_1 + a_{22}u_2}{a_{11}u_1 + a_{12}u_2}$, which connections every point A, a unique tangent direction of the trajectories passing through A, except at A(0,0).

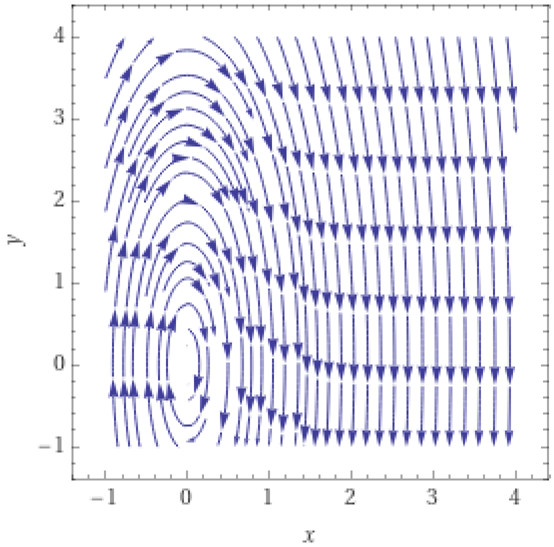
The point A (0,0) is called a critical point of the system, because where $\frac{u_2^{'}}{u_1^{'}} = \frac{a_{21}u_1 + a_{22}u_2}{a_{11}u_1 + a_{12}u_2}$is undetermined at A (0,0).

Classification of critical points through trajectories: The critical points of the system are classified into five types, depending upon the geometrical shape of the trajectories near the critical point. The five types of critical points are - improper nodes, proper nodes, saddle points, centres and spiral point. Each type of critical points has been defined and explained with an example.

A) **Improper nodes:** A critical point C_0 is an improper node, if only two different trajectories have the limiting direction of the tangent at C_0. All other trajectories at C_0 may have different limiting directions.
For example: The system of differential equations

$$x' = y,$$
$$y' = -9x$$

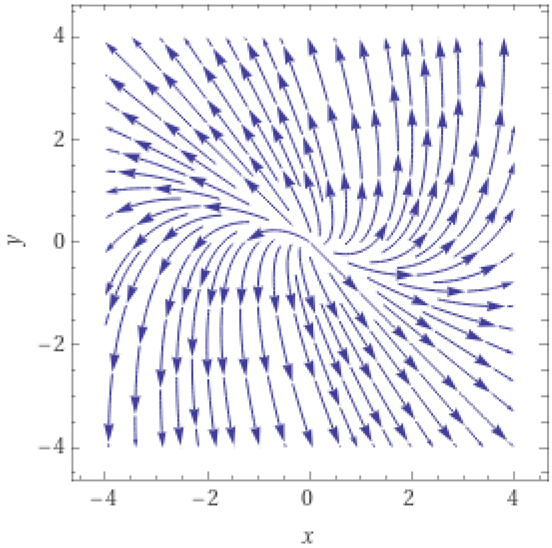
Has an improper node at (0,0), since all the trajectories, except two have the limiting direction to the tangents at (0,0).



B) **Proper node:** A critical point C_0 is a proper node, if for any given direction d at C_0, there is a trajectory having 'd' as its limiting direction.
For example: The dynamical system

$$x' = x - y,$$
$$y' = x + 3y$$

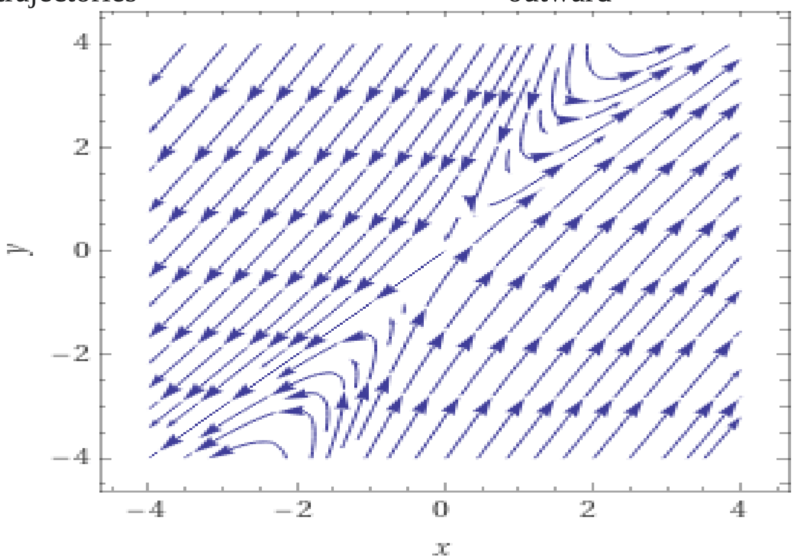
has a proper node at (0,0) since the solutions are proportional to eachother.



C) **Saddle point:** A critical point C_0 is said to be a saddle point if there are two incoming, two outgoing trajectories and the rest of the trajectories in the neighbourhood of C_0, sidestep C_0.
For example: For the given dynamical system of differential equations

$$x' = 2x - y,$$
$$y' = 3x - 2y$$

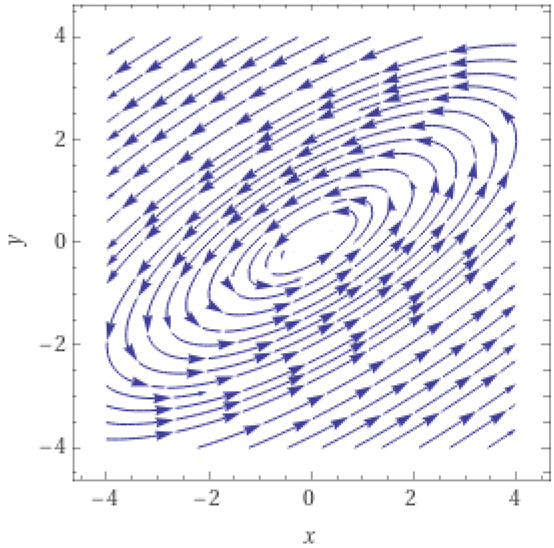
(0,0) is a saddle point, since there are only two trajectories inward and two trajectories outward direction.



D) **Center:** Center is a critical point that is surrounded by infinitely many closed trajectories.The linear system of differential equations

$$x' = x - 2y,$$
$$y' = x - y$$

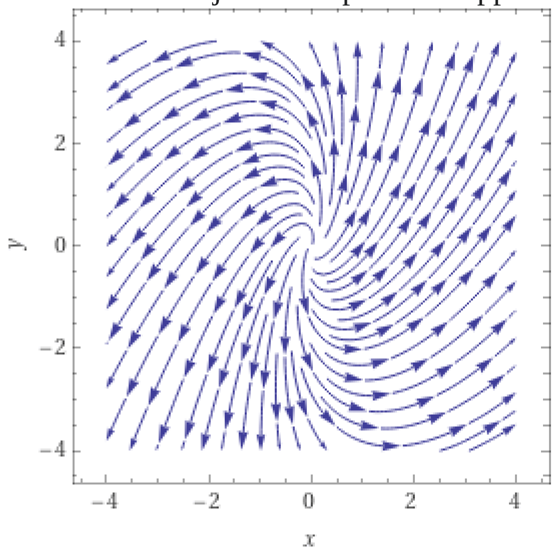
has (0,0) as its center, because the critical point (0,0) is surrounded by infinite number of ellipses.



E) **Spiral:** A critical point C_0 is said to be a spiral if the trajectories spiral and approach C_0 as t approaches infinity.
For example, the system

$$x^{'} = 4x - y,$$
$$y^{'} = 5x + 2y$$

for which the trajectories spiral and approach to (0,0) as t approaches to infinity.



Principles for critical points through characteristic values of the system: The previous discussion, we classified the critical points of the homogenous system through the phase portrait of the solutions of the system. It was observed that system with constant coefficients will lead to acharacteristic value problem. The characteristic values of $\vec{u'} = A\bar{u}$ are the solutions of the characteristic equation

$$|A - \lambda I| = 0$$

Expanding the determinant, we get

$$|A - \lambda I| = \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0$$

Since the roots of the characteristic equation are λ_1, λ_2 the characteristic equation can be written as $(\lambda - \lambda_1)(\lambda - \lambda_2) = 0 \Rightarrow \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 = 0$

Comparing these two equations, we get sum of the roots of the characteristic equation is sum of the characteristic values, product of the characteristic values is equal to the determinant of the coefficient matrix A.

With this observation, we can formulate the principles for the critical points as follows

A critical point C_0 is

i) A **node** if the product of the characteristic values is positive and the discriminant of the characteristic equation is non negative.
ii) A **saddle** point if the product of characteristic values is negative.
iii) A **centre** if sum of the characteristic values is equal to zero and product of the characteristic values is positive.
iv) A **spiral** if sum of the eigenvalues is not equal to zero and the discriminant of the characteristic equation is negative.

For **example, 1**: Consider the following homogeneous dynamical system

$$\frac{dx}{dt} = 4x - y$$

$$\frac{dy}{dt} = 5x + 2y$$

The system can be written in the matrix $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$;

The coefficient matrix $A = \begin{bmatrix} 4 & -1 \\ 5 & 2 \end{bmatrix}$

The eigenvalues of A are roots of characteristic equation $\begin{vmatrix} 4 & -1 \\ 5 & 2 \end{vmatrix} - \lambda I = 0$

$$\begin{vmatrix} 4-\lambda & -1 \\ 5 & 2-\lambda \end{vmatrix} = 0$$

Expanding the determinant, we get $(4-\lambda)(2-\lambda) + 5 = 0$

$$8 - 4\lambda - 2\lambda + \lambda^2 + 5 = 0$$

$$\lambda^2 - 6\lambda + 13 = 0$$

Using the formula for the roots of a quadratic, we get $\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(13)}}{2(1)}$

$$= \frac{6 \pm \sqrt{36-52}}{2(1)}$$

$$= \frac{6 \pm \sqrt{-16}}{2(1)}$$

$$= \frac{6 \pm 4i}{2(1)}$$

$$= 3 \pm 2i$$

Therefore, the roots of the characteristic equation are $\lambda_1 = 3 + 2i \; and \; \lambda_2 = 3 - 2i$

Since the sum of characteristic values is not equal to zero and discriminant of the characteristic equation is negative, the critical point is **a Spiral.**

Example2: The given system is $\bar{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \bar{x}$

$$\Rightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

The coefficient matrix $A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$

The eigen values of A are roots of characteristic equation $\begin{vmatrix} 2 & -1 \\ 3 & -2 \end{vmatrix} - \lambda I = 0$

$$\begin{vmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{vmatrix} = 0$$

Expanding the determinant, we get $(2-\lambda)(-2-\lambda) + 3 = 0$

$$-4 - 2\lambda + 2\lambda + \lambda^2 + 3 = 0$$

$$-1 + \lambda^2 = 0 \Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1$$

Therefore, $\lambda_1 = 1 \; and \; \lambda_2 = -1$ are eigenvalues of the coefficient matrix A. The eigenvalues are real and distinct.

Since the product of characteristic values is negative, the critical point is a **saddle point**.

Example 3: The system is $\begin{bmatrix} {x_1}' \\ {x_2}' \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

The coefficient matrix $A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$

The eigen values of A are roots of characteristic equation $\begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix} - \lambda I = 0$

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

Expanding the determinant, we get $(1-\lambda)(3-\lambda) + 1 = 0$

$$3 - \lambda - 3\lambda + \lambda^2 + 1 = 0$$

$$4 - 4\lambda + \lambda^2 = 0$$

Using the formula for the roots of a quadratic, we get $\lambda = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-(-4)\pm\sqrt{(-4)^2-4(1)(4)}}{2(1)}$

$$= \frac{4 \pm \sqrt{0}}{2(1)}$$

$$= 2$$

Therefore, the roots of the characteristic equation are $\lambda_1 = 2 \; and \; \lambda_2 = 2$ (real and repeated roots)

Since the product of the characteristic roots is positive and discriminant of the characteristic equation is non negative, the critical point is a**propernode**.

Example 4:The given system is $\bar{X}' = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} \bar{X} \ldots (1)$

The coefficient matrix $A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$

The eigenvalues of A are roots of characteristic equation $\begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix} - \lambda I = 0$

$$\begin{vmatrix} 1-\lambda & -2 \\ 1 & -1-\lambda \end{vmatrix} = 0$$

Expanding the determinant, we get $(1-\lambda)(-1-\lambda) + 2 = 0$

$$-1 - \lambda + \lambda + \lambda^2 + 2 = 0$$

$$1 + \lambda^2 = 0 \Rightarrow \lambda^2 = -1 \Rightarrow \lambda = \pm i$$

Therefore, $\lambda_1 = i \; and \; \lambda_2 = -i$ are eigenvalues of the coefficient matrix A. The eigenvalues are complex conjugates & pure imaginary. The critical point is a **Center.**

Stability: The stability of a dynamical system is defined as a change (or disturbance) of the system in all subsequent times, with a minute change at some instant of time. The stability analysis plays a prominent role in studying all dynamical system for engineers and physicists. Hence, an understanding and classifying the types of stability is important. The stability of a critical point of a dynamical system is classified as stable, stable & attractive, unstable depending on the following criteria:

i) Stable and attractive if sum of the eigenvalues of the system is negative and product of the latent roots of the system is positive.

For example: The dynamical system $\begin{bmatrix} {x_1}' \\ {x_2}' \end{bmatrix} = \begin{bmatrix} -6 & -1 \\ -9 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

The latent roots are $\lambda_1 = -3\, and\, \lambda_2 = -6$, which are the roots of the characteristic equation

$$\begin{vmatrix} -6-\lambda & -1 \\ -9 & -6-\lambda \end{vmatrix} = 0 \Rightarrow (-6-\lambda)(-6-\lambda) - 9 = 0 \Rightarrow (6+\lambda)^2 - 3^2 = 0$$
$$\Rightarrow (6+\lambda+3)(6+\lambda-3) = 0 \Rightarrow (9+\lambda)(3+\lambda) = 0 \Rightarrow \lambda = -3\, or -6$$

Since, the sum of latent roots is $\lambda_1 + \lambda_2 = -3 - 6 = -9 < 0$

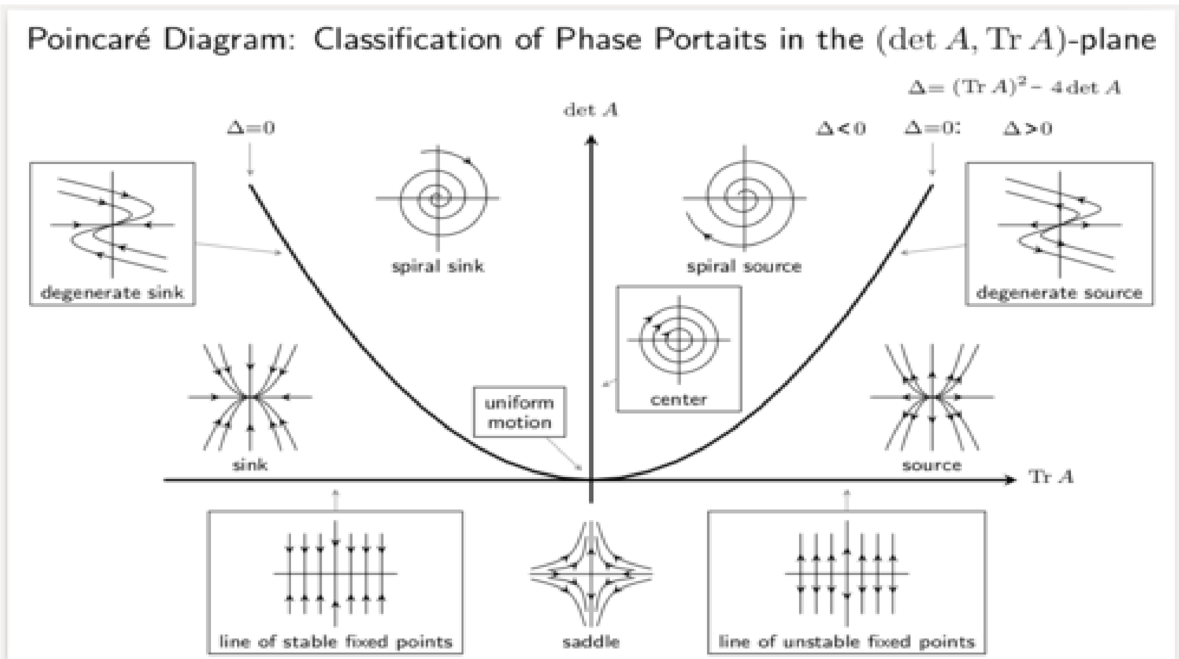
Product of the latent roots is $\lambda_1 . \lambda_2 = (-3)(-6) = 18 > 0$, the critical point is improper node, stable and attractive .

ii) Stable if the sum of the latent roots is negative (including zero) and product of the latent roots is positive.
For example, for the systemis $\begin{bmatrix} x_1{}' \\ x_2{}' \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ the critical point is proper node and stable.

iii) Unstable if the sum of the characteristic values is positive or product of the characteristic roots is negative.
Example $\bar{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \bar{x}$; The critical point is a saddle point and unstable because product of eignevalues is negative.

iv) The critical point of the system matrix $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$; is a Spiral and unstable, since the sum of the eigenvalues is positive and production of the eigenvalues is also positive.

The above classification of critical points is summarized in the following way as astability chart.



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Results& Conclusion: The critical points of a dynamical system have been presented through the phase portrait of the system. The critical points have been classified as node, center, spriral, saddle point. They have been classified through the roots of characteristic equation of the dynamical system and also through the eigenvalues. Through the eigenvalues the stability of the dynamical system we discussed. In conclusion, it is easy to discuss the stability of the system through the eigenvalues than the phase portrait. But, if the system has higher order obtaining the eigenvalues is a big problem.

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