Fixed Point Theorems forSeveral Contractive Mappings and Expansive Mappings in G-Metric Spaces

Ahmed H. Alwan

Department of Mathematics, College of Education for Pure Sciences, University of Thi-Qar, Thi-Qar, Iraq.

ahha7810@gmail.com

Abstract

The purpose of this paper is to define two types of mappings in G-metric spaces and find the fixed points of these mappings. The first type is called several contractive mappings, where we define the contraction condition on the closure of an orbit, where the orbit is bounded and orbitally complete. In addition, we discuss the uniqueness of a fixed point only in this orbit. We studythe existence of a fixed point for surjective expansive mappings in G-metric spaces.

2010 MSC:47H10, 54H25.

Keywords: G-Metric Space, Contractive map, Expansive map, Fixed point.

1. Introduction

In 2005, a new structure of generalized metric spaces was introduced by Mustafa and Sims [5] called *G*-metric spaces. Mustafa and Sims [5], also Mustafa [4] studies the convergence concept and continuity of G-metric function. Moreover, Mustafa [4] introduced some theorems of fixed point theory.Mustafa [6] and Mustafa et al. [7] gave certain type of contractive mapping in G-metric spaces. Motivated by the previous works on the fixed point of contractive maps, in [4, 5], in this paper, we recall such mapping as several contractive mapping. Our main goal is to study theexistence of a fixed point for contraction and surjective expansive mappings in G-metric spaces.In the third section, we introduce some fixed point theorems for several contractive mappings in the closure of an orbit of the space. Here, the orbit must be bounded and orbitally complete, and the uniqueness of a fixed point will be discussed in this orbit. In the fourth section, we introduce some fixed point theorems for expansive mappings, depending on the convergence of the iterative sequences in G-metric spaces.

2. Preliminaries

Throughout this paper $\mathbb{R}^+$ will represent the set of nonnegative real numbers. In this section, we recall the following definitions and properties concerning the $\boldsymbol{G}$-metric spaces.We begin with the following definition as a recall from [4, 5].

Definition 2.1. Let Xbe a nonempty set, and let $G : X \times X \times X \to \mathbb{R}^+$, be a function satisfying the following:

(G1)$G(x, y, z) = 0$if $x = y = z$,

(G2)$0 < G(x, x, y)$ for all $x, y \in X$, with $x \neq y$,

(G3)$G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$with $z \neq y$,

(G4)$(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$, (symmetry in all three variables), (G5)$G(x,y,z) \leq G(x,a,a) + G(a,y,z)$, for all $x,y,z,a \in X$, (rectangle inequality).

Then the function G is called a generalized metric, or a G-metric on X, and the pair $(X, G)$is a G-metric space.If $(X,d)$is an ordinary metric space, then (X,d)can define G-metrics on $X$by

$G(x,y,z) = d(x,y) + d(y,z) + d(z,x)$, for all $x,y,z \in X$. Or

$G(x,y,z) = \max\{ d(x,y), d(y,z), d(z,x)\}$, for all $x,y,z \in X$.

Proposition 2.2.[7]Every G-metric space (X,G)will define a metric space $(X,d_G)$by

$$d_G(x,y) = G(x,y,y) + G(y,x,x), \text{ for all} x,y \in X.$$

Naidu et. al. [3] introduced the concept of open balls in a D-metric space, while Mustafa et. al. [5] defined the ball in a G-metric space.We recall the following proposition without proof see [3, 5].

Proposition 2.3. Let(X,G)be a G-metric space. Then for $x_0 \in X, r > 0$, the G-ball with center$x_0$and radius$r$, is

$B_G(x_0,r) = \{y \in X : G(x_0,y,y) < r\}$.

Proposition 2.4.[5] Let (X,G)be a G-metric space. Then for any $x_0 \in X, r >$, we have

1. If $G(x_0,x,y) < r$, then $x,y \in B_G(x_0,r)$.
2. If $y \in B_G(x_0,r)$, then there exists a δ > 0 such that $B_G(y,\delta) \subseteq B_G(x_0,r)$.

Proof.(1)Followsaccording to G3, while (2)follows according to(G5)with $\delta = r - G(x_0,y,z)$. □

It follows from (2)of the above proposition that the family of all G-balls:

$$\mathcal{B} = \{B_G(x_0,r) : x \in X, r > 0\}$$

is the base of a topology $\tau(G)$on X, the G-metric topology.

In our main work we will use the following definitions which can be found in [5] or [4].

Definition 2.5. Let (X,G)be a G-metric space, and let $\{x_n\}$be sequence of a pointsof X. If there exists apoint $x \in X$, such that $\lim_{n,m\to\infty} G(x,x_n,x_m) = 0$, then the sequence $x_n$is G-convergentto x, and x is said to be the limit point of the sequence.
Or, for any $\epsilon > 0$, there exists $k \in \mathbb{N}$, such that $G(x,x_n,x_m) < \epsilon$for all $n,m \geq k$. Throughout the paper$\mathbb{N}$is the set ofnatural numbers.

Definition 2.6. Let (X,G) be a G-metric space. Thesequence $\{x_n\} \subseteq X$ is said to be G-Cauchysequenceif for every $\epsilon > 0$, there exists $k \in \mathbb{N}$, such that $G(x_n,x_m,x_p) < \epsilon$for all $n,m,p \geq k$.

Definition 2.7. A G-metric space $(X,G)$is said to be G-complete(or complete G-metric space)if every G-Cauchy sequence in$(X,G)$is G-convergent in (X,G).

Definition 2.8. Let $(X,G)$and $(X^{'},G^{'})$be G-metric spaces and let $f:X\longrightarrow X^{'}$be a mapping, then f is said to be G-continuous at a point $a\in X$ if and only if, given $\epsilon>0$, there exists $\delta>0$ such that for all$x,y\in X$; and $G(a,x,y)<\delta$ implies $G(f(a),f(x),f(y))<0$. A mapping$f$ is G-continuous at X if and only if it is G-continuous at all $a\in X$.

Definition 2.9. [4]Let (X,G)be aG-metric space, the function G is jointly continuous in all three of its variables, if for any convergent sequences $\{u_n\},\{v_n\},\{w_n\}$ in G-metric space X, where $\{u_n\}$ converges to $u\in X$, $\{v_n\}$ converges to $v\in X$ and $\{w_n\}$ converges to $w\in X$. Then, $\{G(u_n,v_n,w_n)\}$ converges to $G(u,v,w)$.

Mustafa and Sims [5] introduced the following propositions, here we mention to their without proof.

Proposition 2.10. Let (X,G) and $(X^{'},G^{'})$ be G-metric spaces. Then a mapping $f:X\longrightarrow X^{'}$ is G-continuous at a point $x\in X$if and only if it is G-sequentially continuous at x, that is, whenever $\{x_n\}$ is G-convergent to x one has $\{f(x_n)\}$ is G-convergent to $f(x)$.

Proposition 2.11. Let (X,G)be G-metric space. Then the function Gisjointly continuous in all three of its variables.

Analogues to [2] and [1], we define the following concepts in the G-metric space.

Definition 2.12. Let (X,G) be a G-metric space and let$f:X\to X$be a function. Then

1. An *orbit* of f at the point $x\in X$ is the set:

$o(x)=\{x,fx,f^2x,\dots\}$, and the *closure* of an orbitis $\overline{o(x)}$ denotes the set of all $a\in X$ such thatthere is a sequence in $o(x)$ which converges to x.

2. An orbit $o(x)$ of$x$ in X is said to be *G-bounded* (or bounded). Ifthere exists a constant $k>0$, such that $G(u,v,w)\le k$ for all $u,v,w\in o(x)$.The constant k is calledbound.
3. A G-metric space X(an orbit $o(x)$) is called an$f$*-orbitally complete*if every Cauchy sequence in$o(x)$converges to a point in X for all $x\in X$.

We introduce the next definition in G-metric spaces.

Definition 2.13.

1. Let $x\in X$, a sequence $\{x_n\}$ of points in X is said to be an *iterative* sequence of f at x if $x_{n+1}=f^nx=fx_n, n=1,2,\dots$
2. Let $f:X\to X$be a mapping on a G-metric space$X$.Then$x$ is called a *unique* fixed pointof$f$in X, if x is the unique element of Xsatisfies$fx=x$.

Definition 2.14. Let (X,G) be a G-metric space and$S$ be a nonempty subset of X. Define the diameter of S, as:

$$\delta_G(S)=\sup\{G(x,y,z):x,y,z\in S\}.$$

3. Fixed Point Theorems for Several Contractive Mapping in G-Metric Spaces

In this section we will proved some theorems of fixed point for several contractive maps. Mustafa Z. et al [6], introduce the following definition, here, we will recall that by "several contractive map".

Definition 3.1. Let (X, G) be a G-metric space and let $T: X \to X$ be a mapping, then$T$ is said to be several contractive mapping on satisfies G-metric space if

$$G(Tx, Ty, Tz) \leq aG(x, Tx, Tx) + bG(y, Ty, Ty) + cG(z, Tz, Tz) \qquad (3.1)$$

for all $x, y, z \in X$ where$0 < a + b + c < 1$,

Also, if the contractive condition (3.1) restricted on all $x, y, z \in o(x_0)$ then we say T is several contractive on the orbit $o(x_0)$.

Now,we need to prove the following proposition.

Proposition 3.2. Let (X, G) be a G-metric space, and let $x \in X$, such that $x_n = f^n x, n \in \mathbb{N}$. If the orbit $\{x_n\}$ is bounded. Define:

$$\gamma_i = \delta_G\{x_i, x_{i+1}, x_{i+2}, \dots\}, i = 1,2, \dots$$

Then,

1. $\gamma_n$is finite for all $n \in \mathbb{N}$.
2. $\{\gamma_i\}$is non-increasing sequence, for all $n \in \mathbb{N}$.
   Moreover, $\gamma_n \to \gamma \geq 0$, as $n \to \infty$.

Proof. For (1) Since$\{x_n\}$ is bounded, then the diameter of $\{x_n\}$ is finite.

Therefore, $\gamma_n$is finite for all $n \in \mathbb{N}$.

For (1)Let $\gamma_r, \gamma_{r+1} \in \{\gamma_n\}$.

$$\gamma_r = \delta_G\{x_r, x_{r+1}, x_{r+2}, \dots\} \geq \delta_G\{x_{r+1}, x_{i+2}, x_{i+3}, \dots\} = \gamma_{r+1} \text{for all } n \in \mathbb{N}.$$

Therefore,from (1) and (2), we have $\gamma_n \to \gamma \geq 0$, as $n \to \infty$. □

Theorem 3.3. Let (X, G) be a G-metric spaceand $f: X \longrightarrow X$ be a mapping. If there exists $x_0 \in X$, such that $O(x_0)$ is G-bounded, and f is several contractive mapping on the orbit $O(x_0)$. Then, $\{f^n x_0\}$is a G-Cauchy sequence in $O(x_0)$.

Proof. Let $x_n = f^n x_0, n \in \mathbb{N}$.

Since f is several contractive mapping on the orbit $O(x_0)$, we have:

$$G(x_n, x_{n+p}, x_{n+p+t}) = G(x_{n-1}, x_{n+p-1}, x_{n+p+t-1})$$

$$\leq aG(x_{n-1}, fx_{n-1}, fx_{n-1}) + bG(x_{n+p-1}, fx_{n+p-1}, fx_{n+p-1}) +$$

$cG(x_{n+p+t-1}, fx_{n+p+t-1}, fx_{n+p+t-1})$.

$= aG(x_{n-1}, x_n, x_n) + bG(x_{n+p-1}, x_{n+p}, x_{n+p}) + cG(x_{n+p+t-1}, x_{n+p+t}, x_{n+p+t})$.

Taking the nonincreasing sequence (by Proposition 3.2). Hence,

$$\gamma_n \leq a\gamma_{n-1} + b\gamma_{n-1} + c\gamma_{n-1}.$$

$$\leq (a + b + c)\gamma_{n-1}.$$

But, $0 < a + b + c < 1$, we have:

$$(a + b + c)\gamma_{n-1} < \gamma_{n-1}$$

Thus, $\gamma_n \leq \gamma_{n-1}$.

Taking the limit as $n \longrightarrow \infty$, we get $\gamma < \gamma$ and if $\gamma > 0$, which is a contradiction.

Hence, $\gamma = 0$, that is $\gamma_n \longrightarrow 0$, as $n \longrightarrow \infty$. Then,

$G(x_n, x_{n+p}, x_{n+p+t}) \leq (a + b + c)\gamma_{n-1} \longrightarrow 0$, as $n \longrightarrow \infty$.

Hence,$\{f^n x_0\}$is a G-Cauchy sequence in $O(x_0)$.□

Theorem 3.4. Let (X, G) be a G-metric spaceand $O(x_0)$ be bounded and orbitally complete for some $x_0 \in X$. If $f: X \longrightarrow X$ is a several contractive mapping on the orbit $\overline{O(x_0)}$. Then f has a unique fixed point in $\overline{O(x_0)}$.

Proof. For existence, since $O(x_0)$ is bounded and $\{x_n\}$ is a sequence in $O(x_0)$, then from Theorem 3.3, we have $\{x_n\}$ is a Cauchy sequence.

Since $O(x_0)$ is orbitally complete, so there exists $p \in \overline{O(x_0)}$ such that $\{x_n\}$ converges to p.

For $n \in \mathbb{N}$, and since f is several contractive mapping on the orbit $\overline{O(x_0)}$, we have:

$$G(x_n, fp, fp) \leq aG(x_{n-1}, x_n, x_n) + bG(x_{n-1}, x_n, x_n) + cG(p, fp, fp) \quad (3.2)$$

Since G jointly continuous in three variables.

Taking the limit into both sides of the inequality (3.2) as $n \longrightarrow \infty$, we have:

$$G(p, fp, fp) \leq aG(p, p, p) + bG(p, p, p) + cG(p, fp, fp)$$

Thus,

$$G(p, fp, fp) \leq cG(p, fp, fp) < G(p, fp, fp).$$

If $G(p, fp, fp) > 0$, which is not true. Thus, $G(p, fp, fp) = 0$, and then $fp = p$, and p is a fixed point of f in $\overline{O(x_0)}$.

To prove the uniqueness, suppose that q is another fixed point of f in $\overline{O(x_0)}$, i.e., $fp = p$, $fq = q$.

By using the property of several contraction mapping, we have:

$$G(p, q, q) \leq G(fp, fq, fq) \leq aG(p, fp, fp) + bG(q, fq, fq) + cG(q, fq, fq).$$

$$\leq aG(p, fp, fp) + (b + c)G(q, fq, fq).$$

Hence, $p = q$.

Therefore, p is the unique fixed point of $f$in $\overline{O(x_0)}$. □

Corollary 3.5 Let (X, G) be a G-metric spaceand $f: X \longrightarrow X$ be a mapping. If there exists $x_0 \in X$ such that $O(x_0)$ is bounded and orbitally complete, where:

$G(fx, fy, fy) \leq aG(x, fx, fx) + bG(y, fy, fy)$
(3.3)

for all $x, y \in \overline{O(x_0)}$, where $0 < a + b < 1$. Then f has a unique fixed point in $\overline{O(x_0)}$.

Proof. The proof follows directly from Theorem 3.4, by putting $z = y$ in inequality (3.1), then we see that every mapping satisfies inequality (3.3) satisfies the inequality (3.1) on the orbit $\overline{O(x_0)}$.□

Corollary 3.6 Let (X, G) be a G-metric spaceand $f: X \longrightarrow X$ be a mapping. If there exists $x_0 \in X$ such that $O(x_0)$ is bounded and orbitally complete, where:

$G(fx, fy, fy) \leq dG(x, y, y)$ (3.4)

for all $x, y \in \overline{O(x_0)}$, where $0 < d < 1/4$. Then f has a unique fixed point in $\overline{O(x_0)}$.

Proof. By using property (G5) of G-metric function, we have:

$G(x, y, y) \leq G(x, fx, fx) + G(fx, y, y)$
(3.5)

$G(fx, y, y) \leq G(fx, fy, fy) + G(fy, y, y)$
(3.6)

$G(fy, y, y) \leq G(y, fy, fy) + G(fy, y, y)$
(3.7)

Hence, from inequalities (3.5)-(3.7), we see that inequality (3.4) becomes:

$$G(fx, fy, fy) \leq dG(x, y, y)$$

$\leq dG(x, fx, fx) + dG(fx, fy, fy) + 2dG(y, fy, fy)$ (3.8)

Then, f will satisfy the following inequality:

$G(fx, fy, fy) \leq aG(x, fx, fx) + bG(y, fy, fy)$ (3.9) for all $x, y \in \overline{O(x_0)}$, where $a = \frac{d}{1-d}$ and $b = \frac{2d}{1-d}$.

Since $d \leq 1/4$, so $a + b < 1$.

Therefore, inequality (3.4) is satisfied and the proof follows from Corollary 3.5. □

Now, we prove the following theorem by supposing that the iterative sequence has a convergent subsequence.

Theorem 3.7. Let (X, G) be a G-metric spaceand $f: X \longrightarrow X$ be a mapping. If there exists $x_0 \in X$ such that the sequence $\{f^{n_i}x_0\}$ is a convergent sequence in X, where:

$$G(fx, fy, fz) \leq qG(x, y, z) \qquad (3.10)$$

for all $x, y, z \in X$ and for some $0 \leq q < 1$. Then f has a unique fixed point in X.

Proof. For the existence, suppose that $\{f^{n_i}x_0\}$ is a convergent sequence in X.

Then, there exists a point $t \in X$, and $\lim_{i\to\infty} f^{n_i}x_0 = t$.

To show $\lim_{i\to\infty} f^{n_i+1}x_0 = ft$, by using inequality (4.1)

$$G(f^{n_i+1}x_0, f^{n_i+1}x_0, ft) \leq qG(f^{n_i}x_0, f^{n_i}x_0, t) \qquad (3.11)$$

By taking the limit to the both sides of inequality (3.11) as $n \to \infty$, we get $\lim_{i\to\infty} f^{n_i+1}x_0 = ft$.

If $ft \neq t$, there exists $k \in \mathbb{N}$, such that if $i > k$, then there exist two G-open balls $B_1 = (t, \epsilon)$ and

$B_2 = (ft, \epsilon)$, where:

$$\epsilon < \min\{G(t, ft, ft), G(ft, t, t)\}$$

And

$$G(f^{n_i}x_0, f^{n_i+1}x_0, f^{n_i+1}x_0) > \epsilon, \text{ for all } i > k \qquad (3.12)$$

From inequality (3.10), we have:

$$G(f^{n_i+1}x_0, f^{n_i+2}x_0, f^{n_i+2}x_0) \leq qG(f^{n_i}x_0, f^{n_i+1}x_0, f^{n_i+1}x_0) \qquad (3.13)$$

for all $\ell > j > k$, and by inequality (3.13), we get

$$G(f^{n_\ell}x_0, f^{n_\ell+1}x_0, f^{n_\ell+1}x_0) \leq qG(f^{n_\ell-1}x_0, f^{n_\ell}x_0, f^{n_\ell}x_0)$$

$$\leq q^2G(f^{n_\ell-2}x_0, f^{n_\ell-1}x_0, f^{n_\ell-1}x_0)$$

$$\vdots$$

$$\leq \boldsymbol{q^{n_\ell-n_j}G(f^{n_j}x_0, f^{n_j+1}x_0, f^{n_j+1}x_0)}$$

Taking $\boldsymbol{\ell} \to \infty$, we get:

$\lim_{\boldsymbol{\ell}\to\infty} G(f^{n_\ell}x_0, f^{n_\ell+1}x_0, f^{n_\ell+1}x_0) \leq 0.$
This is a contradiction with (3.12).
Therefore, $ft = t$ and t is a fixed point of f in X.
For uniqueness, suppose that r is another fixed point of f in X.
This means that $ft = t$ and $fr = r$ and $t \neq r$.
By inequality (3.10), we have:

$$G(t,t,r) = G(ft,ft,fr) \leq G(t,t,r) > 0$$

Which is a contradiction if $G(t,t,r) > 0$
Hence, $G(t,t,r) = 0$ and so, $t = r$.
Thus, t is a unique fixed point of f.
Therefore, f has a unique fixed point in X.□

Corollary 3.8.Let (X,G) be a G-metric spaceand $f: X \longrightarrow X$ be a mapping. If there exists $x_0 \in X$ such that the sequence $\{f^{n_i}x_0\}$ is a convergent sequence in X, where:

$$G(fx, fz, fz) \leq qG(x, z, z) \qquad (3.14)$$

for all $x, z \in X$ and for some $0 \leq q < 1$. Then f has a unique fixed point in X.

Proof. The proof follows from Theorem 3.7, by taking $y = z$ in inequality (3.10), consequently f has a unique fixed poit in X.□

4. Fixed Point Theorems for Expansive Mapping in G-Metric Spaces

In this section we will proved some theorems of fixed point for expansive mappings in $\boldsymbol{G}$-metric space. Analogous to [2, 1], we define the expansive mappings which defined on $\boldsymbol{G}$-metric space.

Definition 4.1. Let (X, G) be a G-metric space and let $f: X \to X$ be a mapping, then f is called expansive mapping if there exists a constant $q > 1$, such that:

$$G(fx, fy, fz) \geq qG(x, y, z) \qquad (4.1)$$

for all $x, y, z \in X$.

Theorem 4.2. Let (X, G) be a G-metric spaceand $f: X \to X$ be an expansive and surjective mapping. If there exists $x_0 \in X$ such that $\{f^{n_i}x_0\}$ is a convergent sequence in X. Then f has a unique fixed point in X.

Proof. Suppose that f is a surjective on X.
To show that f is injective mapping on X.
Let $x, y \in X$, such that $fx = fy$.
Then, $G(fx, fx, fy) = 0$.
Since f is an expansive mapping, we have:
$G(fx, fx, fy) \geq qG(x, x, y)$.
Thus, $qG(x, x, y) \leq 0$.
Hence, $G(x, x, y) = 0$and $x = y$.
Then, f is an injective mapping, but f is an surjective mapping.
Thus, f is a bijective mapping.
Therefore, f is an invertible mapping.
Suppose that g is the inverse mapping of f.
Hence:
$G(x, y, z) = G(f(gx), f(gy), f(gz)) \geq qG(hx, hy, hz)$.
Then, we get:

$$G(hx, hy, hz) \leq pG(x, y, z)$$

for all $x, y, z \in X$, where $p = \frac{1}{q} < 1$.
Now, we have the inverse mapping g satisfies all these conditions in Theorem 3.7.
By Theorem 3.7, g has a unique fixed point in X, $gu = u$.
But, $u = f(gu) = fu$.
Thus, u is also a fixed point of f in X.

Uniqueness. Suppose that v is another fixed poit of in X, $f(v) = v$, $v \neq u$.
Then,
$fv = v = f\big(g(v)\big) = g(f(v))$.
Thus, fv is another fixed point of g in X.
By uniqueness of fixed poit, we get:

$v = fv = u$.
Thus, u is a unique fixed point f in X.
Therefore, f has a unique fixed point in X.□

Corollary 4.3. Let (X, G) be a G-metric spaceand $f: X \to X$ be a surjective mapping. If there exists $x_0 \in X$ such that $\{f^{n_i}x_0\}$ is a convergent sequence in X, where:

$G(fx, fz, fz) \geq qG(x, z, z)$ (4.2)
for all $x, z \in X$ and for some $q > 1$. Then f has a unique fixed point in X.

Proof. Suppose that f is a surjective mapping on X.
To show that f is an injective mapping on X.
Let $x, y \in X$ such that $fx = fy$.
Then,$G(fx, fy, fy) = 0$.
By using inequality (4.2), we have:
$G(fx, fy, fy) \geq qG(x, y, y)$.
Thus, $qG(x, y, y) \leq 0$.
Hence, $G(x, y, y) = 0$ and $x = y$.
Then, f is an injective mapping, but f is a surjective mapping.
Thus, f is a bijective mapping.
Therefore, f is invertible mapping, i.e., f has inverse mapping, say, g is inverse mapping of f.
By using inequality (4.2), we see:
$G(x, z, z) = G(f(gx), f(gz), f(gz)) \geq qG(gx, gz, gz)$.
Thus:

$$G(gx, gz, gz) \leq pG(x, z, z)$$

for all $x, z \in X$, where $p = \frac{1}{q} < 1$.
By using Corollary 3.8, g has a unique fixed point, say w in X, $g(w) = w$.
But, $w = f\big(g(w)\big) = fw$.
Thus, w is also a fixed point of f in X.
For the uniqueness, suppose that v another fixed point of f in X, such that $f(v) = v$, $v \neq w$.
Then, $fv = v = f\big(g(v)\big) = g(f(v))$.
Thus, fv is another fixed point of g in X.
By uniqueness of a fixed point, we have $v = fv = w$.
Thus, w is a unique fixed point of f in X.
Therefore, f has a unique fixed point in X. □

Corollary 4.4. Let (X, G) be a G-metric spaceand $f: X \to X$ be a surjective mapping. If there exists $x_0 \in X$ such that $\{f^{n_i}x_0\}$ is a convergent sequence in X, where:

$G(fx, fy, fz) \geq q\{G(x, y, y) + G(z, y, y)\}$(4.3)

for all $x, y, z \in X$ and for some $q > 1$. Then f has a unique fixed point in X.

Proof. The proof follows from Corollary 4.3, by taking $y = z$ in inequality (4.3).□

Remark 4.5. In Corollary 4.4, if the contraction condition restriction on the *closure* of some orbit $\overline{o(x_0)}$ of f at $x_0 \in X$. Then f has a unique fixed point in $\overline{o(x_0)}$.

Acknowledgment

I would like to thank the referee for valuable comments.

References

1. B. Ahmed and M. Ashraf, Some fixed point theorems, Southeast Asian Bulletin of Math., 27. pp. 769-780.
2. B. C. Dhage, A. M. Pathan and B. E. Rhoades, A general existence principle for fixed point theorems in D-metric spaces, Internet Journal Math. and Math. Sci., 23, pp. 329–336, 1992.
3. S. V. R. Naidu, K. P. R. Rao, and N. Srinivasa Rao, "On the concepts of balls in a D-metric space," International Journal of Mathematics and Mathematical Sciences, no. 1, pp. 133–141, 2005.
4. Z. Mustafa, A new structure for generalized metric spaces - with applications to fixed point theory, PhD Thesis, the University of Newcastle, Australia, 2005.
5. 2. Z. Mustafa and B. Sims, "A new approach to generalized metric spaces," Journal of Nonlinear and Convex Analysis, vol. 7, no. 2, pp. 289–297, 2006.
6. Z. Mustafa, H. Obiedat and F. Awawdeh, Some fixed point theorem for mapping on complete G-metric spaces, Fixed Point Theory and applications, article ID 189870, 12 pages, 2008.
7. Z. Mustafa, W. Shatanawi, and M. Bataineh, Existence of fixed point results in G-metric spaces, International Journal of Mathematics and Mathematical Sciences, 2009, Article ID 283028, 10 pages.