PROPERTIES OF STRONG COLORINGFOR SOME STANDARD GRAPHS

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Abstract: In graph theory coloring plays a major role. The vertex coloring is an assignment of colors to the vertices of a graph 'G' such that no two adjacent vertices have the same color. That is, no two vertices of an edge should be of the same color. In this study, strong coloring for various standard graphs are derived and further using suitable examples the results are examined.

Keywords: Graphs, coloring, strongcoloring and strong chromatic number .

1. Introduction:

A graph G consists of a finite nonempty set $V = V(G)$ of n vertices together with a prescribed set E of m unordered pairs of distinct vertices of V. Each pair $e = \{u, v\}$ of vertices in E is an edge of G, u and v joined by e. The order of G, denoted by $|V(G)| = n$, is the number of vertices in G. The size of G, denoted by $|E(G)| = m$, is the number of edges in G. If vertex set and edge set of G are finite, then G is finite. AfinitegraphGhavingnoloopsormultipleedgesiscalledasimplegraph.[3]

The complete graph on n vertices, denoted by K_n, is a graph on n vertices such that every pair of vertices is connected by an edge. The complete bipartite graph $K_{m,n}$ *on* $n+m$ vertices as the(unlabelled) graph, isomorphic to $(A \cup B = \{xy : x \in A,\ y \in B\})$, where $|A| = m$ and $|B| = n,\ A \cap B = \varnothing$.

A walk of G is an alternating sequence of vertices and edges $v_0, e_1, v_1, e_2, v_2, e_3, \ldots, v_{r-1}, e_r, v_r$, beginning and ending with vertices, in which each edge is incident with the two vertices immediately preceding and following it. This walk is also called $v_0 - v_r$ walk. A $v_0 - v_r$ walk is closed if $v_0 = v_r$ and is open otherwise. A walk is a trail if all the edges in it are distinct, also it is called a path if all its vertices are distinct. A path on n vertices is denoted by P_n. A circuit is a path which ends at the vertex it begins.[3]

The degree of a vertex v of G, denoted by $d(v)$ or $\deg(v)$, is the number of edges incident to v. A vertex of degree 1 in G is called a leaf or pendant vertex, and a vertex of degree 0 in G is called an isolated vertex. The minimum degree of G, denoted by $\delta(G)$, is the smallest vertex degree in G. The maximum degree of G, denoted by $\Delta(G)$, is the largest vertex degree in G. AgraphG issaidtoberegularifeveryvertexin$G$ hasthesame degree. More precisely, G is said to be k-regularif d(v)= k for each vertex vin G, where k $\geq$ 0. [2, 6] .Let G_1 and G_2 be two graphs with disjoint vertex sets V_1 and V_2, and edge sets E_1 and E_2, respectively. Then their union $G_1 \cup G_2$ is the graph having vertex set $V_1 \cup V_2$ and the edge set $E_1 \cup E_2$. [3]

The vertex coloring is an assignment of colors to the vertices of a graph ‘G’ such that no two adjacent vertices have the same color. In otherwords, no two vertices of an edge should be of the same color.

2. PROPERTIES OF STRONG COLORING FOR SOME STANDARD GRAPHS

In this section,strong coloringis investigated for the complete graph, and complete bipartite graph, regular graph, circuit graph and union of graphs. Also strong coloring s derived for some special graphs.

Definition 2.1 [5]: In graph theory, a strong coloring, with respect to a partition of the vertices into (disjoint) subsets of equal sizes, is a (proper) vertex coloring in which every color appears exactly once in every part.

Definition 2.2 [5]: A graph is strongly k-colorable if, for each partition of the vertices into sets of size k, it admits a strong coloring.

Definition 2.3[1]: The strong chromatic number $s_{\chi}(G)$ of a graph G is the least k such that G is strongly k-colorable. A graph is strongly k-chromatic if it has strong chromatic number k.

Theorem 2.1:

For a complete graph K_n, the strong chromatic number $s_{\chi}(G)$ is equal to the number of vertices n.

Proof: Let K_n be a complete graph of n vertices, such that every pair of vertices is connected by an edge. Therefore, K_n is n colorable this implies each color class contains a vertex.The strong coloring subsets are a singleton disjoints subsets of vertex set V. Hence the strong coloring subsets are colored by a different color. This implies, the strong chromatic number $s_{\chi}(G) = n.$

Example 2.1:

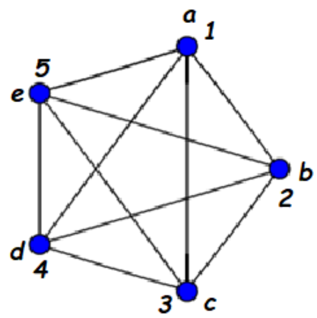


Figure 2.1: Complete graph K_5

In the above example complete graph K_5 is considered which is 5 colorable. Whose strong chromatic number $s_{\chi}(G) = \{\{a\},\{b\},\{c\},\{d\},\{e\}\} = 5.$ The singletons in the set are disjoint subset of V. This implieseach color class contains a vertex. The strong coloring subsets is colored by a different color. Therefore, $s_{\chi}(G) = 5$.

Theorem 2.2:

The complete bipartite graph $K_{m,n}$ is strong colorable if and only if the number of vertices in vertex partitions are same.

Proof: Let $K_{m,n}$ be a complete bipartite graph with two vertex partitions V_1 & V_2 which are disjoint sets whose order is m and n respectively. Assume complete bipartite graph $K_{m,n}$ is strong colorable. This implies a strong coloring, with respect to a partition of the vertices into disjoint subsets of equal sizes, is a proper vertex coloring in which every color appears exactly once in every part. In complete bipartite graph $K_{m,n}$ every vertex of V_1 is joined to every vertex of V_2. So, V_1 & V_2 are colored by two different colors . Clearly V_1 & V_2 are color partitionsof $K_{m,n}$. This implies the number of vertices in vertex partition is same. Hence, $m = n$.

Conversely, we assume the number of vertices in vertex partition with equal number of vertices, $m = n$. The vertex set partition into two disjoint vertex sets V_1 & V_2 having $m = n$ vertices respectively. The complete bipartite graph is 2 colorable. Clearly two disjoint vertex sets V_1 & V_2 is 2 colorable and it contains equal number of vertices. Hence the complete bipartite graph is strong 2-colorable.

Example 2.2:

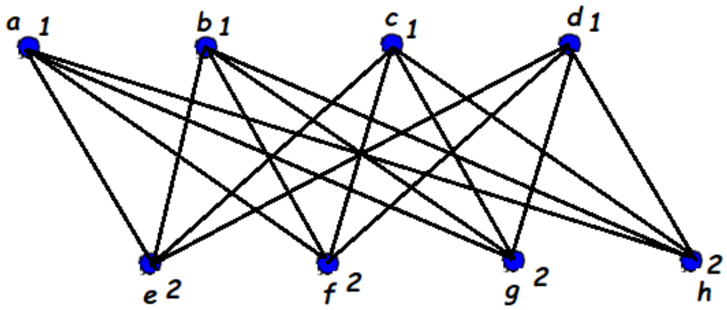


Figure 2.2: Complete bipartite graph $K_{4,4}$

In above example complete bipartite graph $K_{4,4,}$ is considered, where thedisjoint vertex sets V_1 & V_2 haveequal number of vertices. The vertex set partition of $K_{4,4}$ are $\{\{a,b,c,d\},\{e,f,g,h\}\}$.

Theorem 2.3:

If a graph $G(V,E)$ is a k-regular graph,then the strong chromatic number $s_\chi(G) = \left(\frac{n}{k}\right)$.

Proof: Let $G(V,E)$ be a k regular graph. This implies the degree of every vertices in $G(V,E)$ is equal to k. Clearly every vertex is adjacent to k number of vertices in $G(V,E)$. Therefore $G(V,E)$ is $\left(\frac{n}{k}\right)$-colorable and $G(V,E)$ contains k number of disjoint vertex sets.

Since n is divisible by k, the strong chromatic number $s_{\chi}(G)$ is equal to degree of vertices. Therefore, $s_{\chi}(G) = \left(\frac{n}{k}\right)$.

Example 2.3:

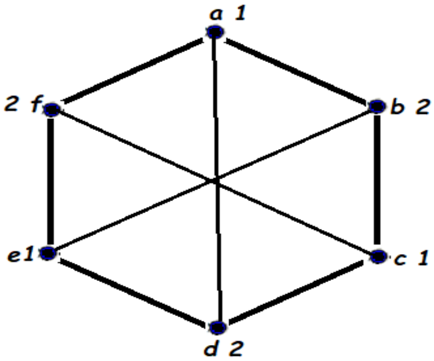


Figure 2.3: 3-Regular graph

In the above example 3-regular graph having $O(G) = 6$ is considered. The disjoint strong vertex sets of $G$are $\{\{a,c,e\},\{b,d,f\}\}$. The number of strong color classes is equal to two.

Theorem 2.4:

For a circuit graph $G(V,E)$ with$n$ number of even vertices, thestrong chromatic number $s_{\chi}(G)$ is equal to degree of vertices. $s_{\chi}(G) = 2$.

Proof: Let $G(V,E)$ be a circuit graph. This implies the degree of every vertices in $G(V,E)$ is two. That is, $d_N(u) = 2, \forall u \in G$. Clearly every vertex adjacent to pair of vertices in $G(V,E)$. Therefore $G(V,E)$ is 2-colorable and $G(V,E)$ contains two disjoint vertex color sets X_1 and X_2. Therefore strong chromatic number $s_{\chi}(G) = 2$

Example 2.4:

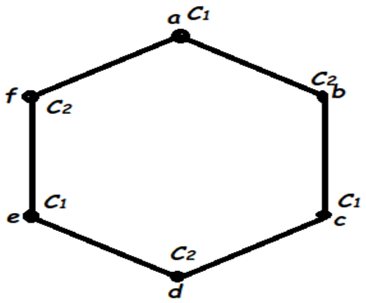


Figure 2.4: Circuit graphs with 6 vertices

Consider the circuit graph$G(V,E)$ with $O(G) = 6$, whose partition of the vertices into disjoint subsets of equal sizes are $\{\{a,c,e\},\{b,d,f\}\}$. Therefore, thestrong chromatic number $s_{\chi}(G) = 2$.

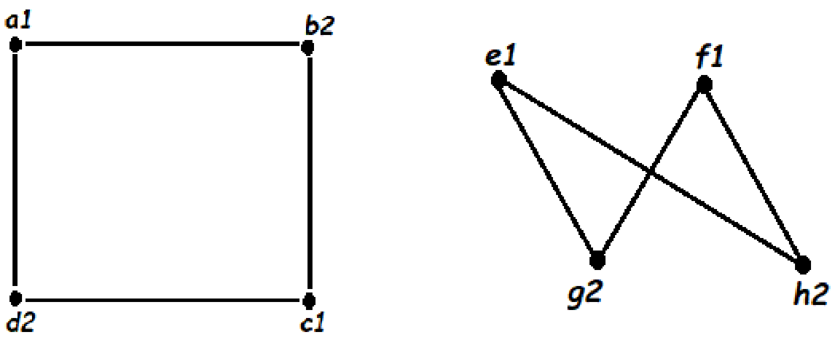
Theorem 2.5:

For the disjoint strong coloring graphs G_1 and G_2having same strong chromatic number, the union of graphs $S_{\chi}(G_1 \cup G_2) = S_{\chi}(G_1) + S_{\chi}(G_2)$.

Proof: Let G_1 and G_2 be two disjoint strong coloring graphs having equal number of strong coloring classes this implies $S_\chi(G_1)=n$ & $S_\chi(G_2)=n$. In $G_1 \cup G_2$ the edge set $E=\{uv \in E_1 \cup E_2\}$. Clearly $G_1 \cup G_2$ is a disconnected graph this implies $X_i \subseteq V_1$ *and* $Y_i \subseteq V_2 . i=1,2,\ldots n$ is the strong coloring class of $G_1 \cup G_2$. This implies $Z_i = X_i \cup Y_i \;\; i \in 1,2,\ldots n$ in $G_1 \cup G_2$ is strong coloring, since X_i & $Y_i \;\; i \in N$ are strong coloring classes in G_1 & G_2. Hence $G_1 \cup G_2$ is a strong coloring graph.

$$Z_i = X_i \cup Y_i$$
$$|Z_i| = |X_i \cup Y_i|$$
$$S_\chi(G_1 \cup G_2) = S_\chi(G_1) + S_\chi(G_2)$$

Example 2.5:



Disjoint graphs G_1 and G_2

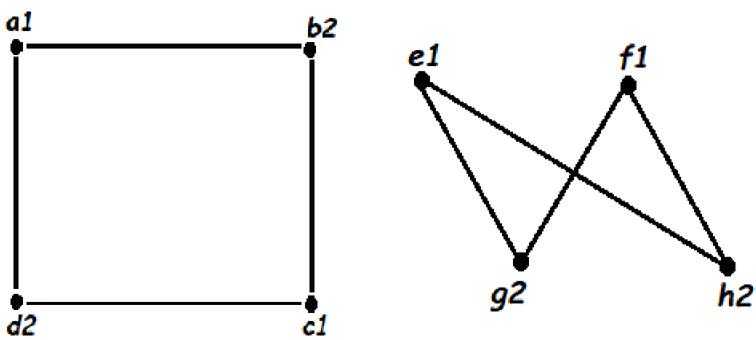


Figure 2.5: Union of disjoint graphs $G_1 \cup G_2$

In the above example the strong coloring classes of G_1 and G_2 are $S_\chi(G_1)=$. $\{\{a,c\},\{b,d\}\}$ and $S_\chi(G_2)=\{\{e,h\},\{f,g\}\}$. The strong coloring classes of $G_1 \cup G_2$ are $\{\{a,c,e,h\},\{b,d,f,g\}\}$.

3. STRONG CHROMATIC NUMBER FOR SOME SPECIAL GRAPHS

In this section, the strong coloring of some special graphs like Bidiakis cube, Durer graph and Petersen graph are identified.

Bidiakis Cube: Bidiakis cubeis a 3-regular graph with 12 vertices and 18 edges.

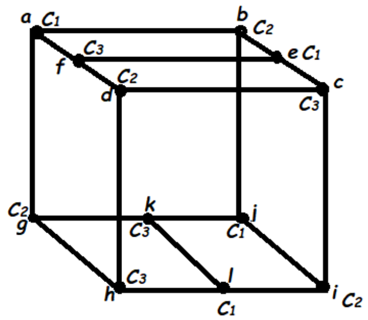


Figure 3.1

The strong coloring classes ofBidiakis cube are $C_1 = \{a, e, j, l\}, C_2 = \{b, d, g, i\}$ and $C_3 = \{c, f, h, k\}$.Therefore thestrong chromatic number $s_\chi(G) = 3$.

Durer graph: The Durer graph is an undirected cubic graph with 12 vertices and 18 edges.

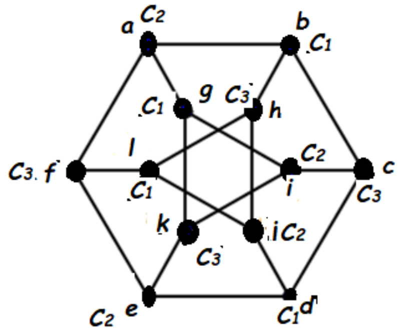


Figure 3.2

Remarks:

1. Durer graph is a 3 regular graph.
2. In the Durer graph 6 vertices forms an outer Hexagon and remaining 6 vertices forms inner 2 triangle region.
3. The 6 vertices of the inner triangles are adjacent to only one vertex on the Hexagon.
4. The strong coloring classes ofDurer graph are $C_1 = \{b, d, g, l\}$, $C_2 = \{a, e, i, j\}$, and $C_3 = \{c, f, h, k\}$.
5. Tthestrong chromatic number $s_\chi(G) = 3$.

Conclusion: In this study, we calculated and analysed a strongcoloring in complete graph, complete bipartite graph, regular graph, circuit graph and union of disjoint graphs and some special graphs are derived. Further explain the theorem using suitable examples.

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