Bayesian estimation of the beta distribution parameter (α) when the parameter (β) is known

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Abstract: In this paper, the Bayesian estimation was used to estimate the parameter of the beta distribution (α) when (β = 1), (2 = β) and (3 = β). By employing the Monte Carlo simulation method, the simulation was tested with different sample sizes (10, 20, 25, 40, 75, 100) and based on the MSE statistical standard), where the best estimator for parameter (α) was found when (β = 1)
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1. Introduction

One of the commonly used Bayesian methods is the Standard Bayes Method (SB), which assumes that the parameters to be estimated are random variables.) where the probability function of this distribution is combined with the possibility function of the observations using the inverse Bayes rule and the resulting function is called the subsequent conditional probability density function (Posterior pdf) for the random parameter and using the loss function because these estimations can be obtained by reducing the risk function (Risk). Function) to the least possible, which in turn represents the expectation of the loss function, since the Bayesian problem is to find the estimator that has the least possible Bayesian risk..

In (2009) the researcher (Al-Nasser)[1] came up with using the Bayes method and comparing it with the standard Bayes method by using a variety of loss functions to estimate the measurement parameter and the reliability function for the Whipple distribution with two parameters using simulation, and in (2010) the researcher (Al-Hadithi) [3] compared the standard Bayes estimations of the Pareto distribution parameter using different loss functions. In (2011), the researcher (Al-Issawi) [4] compared the standard Bayesian estimations for the parameter of the exponential distribution using different Ωloss functions.

2. Beta Distribution

The beta distribution is one of the important statistical distributions used in many fields. It is used in a continuous random variables model whose range is (0,1)[6] and the probability density function is given as follows:-

$$\mathbf{f\ (x;\ \alpha\ ,\beta) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}} \qquad (\mathbf{1})$$

where $0 \leq x \leq 1$, $a > 0$, and $b > 0$ are called shape parameters, and B(a,b) is the beta function defined as:

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

where Γ(a) is the gamma function defined as:

$$\Gamma(\alpha) = \int_{t=0}^{\infty} t^{\alpha-1}e^{-t}dt$$

3.loss function

The loss function is a function of two variables: the real value of the parameter , let it be θ, and its estimated value, let it be (θ)̂, and we will symbolize it with the symbol L ((θ)̂,θ) and as follows:-

$$L\left(\hat{\theta},\theta\right) > 0\ for\ all\ \theta$$

$$L\left(\hat{\theta},\theta\right)=0\ \boldsymbol{where}\ \hat{\theta}=\theta$$

Let X be a random variable dependent on the random parameter θ and **Ω** be the space of this parameter that includes all possible values for this parameter. Let us suppose that ∏θ represents the initial probability function of the parameter θ and that I(x/θ) represents the possibility function of the observations, as the subsequent probability density function, let it be h(x/θ) according to the inverse Bayes rule in probability is given by the following formula:

$$h(x/\theta)=\frac{I(x/\theta)\prod}{f(x)} \qquad (2)$$

$$\mathrm{f(x)}=\int_{\Omega}\mathrm{I}(\mathrm{x}/\theta)\prod\ \mathrm{d}\theta \qquad (3)$$

It is the Marginal probability density function of the observed random variable (X).

The conditional probability function of the beta distribution is:

$$I(\vec{\mathrm{x}}/\alpha,\beta)=\prod_{i=1}^{n}\frac{\Gamma_{\alpha+\beta}}{\Gamma\alpha\,\Gamma\beta}\mathrm{x}_i{}^{\alpha-1}(1-\mathrm{x}_i)^{\beta-1}$$

$$=\left(\frac{\Gamma_{\alpha+\beta}}{\Gamma\alpha\,\Gamma\beta}\right)^{n}\prod_{i=1}^{n}\mathrm{x}_i{}^{\alpha-1}(1-\mathrm{x}_i)^{\beta-1} \qquad (4)$$

Assuming that the parameter β is known and that α is a random variable that follows a gamma distribution with parameters (α,b) and as defined in the following formula:

$$\boldsymbol{\alpha}\sim\textbf{Gamma}\ (\boldsymbol{\alpha},\mathbf{b})$$

$$\prod(\alpha)=\frac{\mathrm{b^a}}{\Gamma\mathrm{a}}\alpha^{\mathrm{a}-1}\mathrm{e}^{-\mathrm{b}\,\alpha} \qquad 0<\alpha<\infty \qquad (5)$$

4. The Loss Functions Used In The Search

- **Squared Error Loss Function**

It is also called Squared Error Loss Function, a Symmetric Loss Function, and it gives equal importance to the losses caused by the exaggerated estimators, which are the overestimation and the underestimation of equal amouns. The squared loss in life-test experiments and reliability problems is inappropriate in some cases despite the estimation problems so they use other loss functions, the asymmetric loss functions[8].

The formula for the squared loss function is as follows:

$$L\left(\hat{\theta},\theta\right)=\mathrm{c}\,(\hat{\theta}-\theta)^2$$

Since c is a positive real constant.

Without losing generalization, we will take into account the squared loss function

$$L\left(\hat{\theta},\theta\right)=\ (\hat{\theta}-\theta)^2$$

because the constant C will be omitted when differentiating the piez estimator for this function.

- **Modified Squared Error Loss Function**[3]

$$L\left(\hat{\theta},\theta\right)=\theta^{r}\ (\hat{\theta}-\theta)^2 \quad ,r\neq 0\ \ ,r\in Z \qquad (6)$$

Since (Z) is the set of integers.

5. The Standard Bayes Estimator Using Squared Error Loss Function[5][7]

Referring to equation (2) and substituting for the conditional probability function of the beta distribution in equation (1) and for the initial probability function in equation (4) and assuming that:

Let $I(\mathrm{X}/\alpha,\beta)\prod(\alpha)=\psi$

$$\psi=\frac{(\Gamma\alpha+\beta)^{n}b^{a}\alpha^{a-1}}{(\Gamma\alpha)^{n}(\Gamma\beta)^{n}\Gamma a}e^{-b\alpha}\prod_{i=1}^{n}x_i^{\alpha-1}\prod_{i=1}^{n}(1-x_i)^{\beta-1}$$

$$=\frac{(\Gamma\alpha+\beta)^{n}b^{a}\alpha^{a-1}}{(\Gamma\alpha)^{n}(\Gamma\beta)^{n}\Gamma a}e^{-b\alpha+(\alpha-1)\sum_{i=1}^{n}\log x_i+(\beta-1)\sum_{i=1}^{n}\log(1-x_i)} \quad (7)$$

$$h(\alpha|\vec{x})=\frac{\psi}{\int_0^\infty \psi\, d\alpha}$$

$$h(\alpha|\vec{x})=\frac{\frac{(\Gamma\alpha+\beta)^{n}b^{a}\alpha^{a-1}}{(\Gamma\alpha)^{n}(\Gamma\beta)^{n}\Gamma a}e^{-b\alpha+(\alpha-1)\sum_{i=1}^{n}\log x_i+(\beta-1)\sum_{i=1}^{n}\log(1-x_i)}}{\frac{b^{a}}{(\Gamma\beta)^{n}\Gamma a}e^{(\beta-1)\sum_{i=1}^{n}\log(1-x_i)}e^{-\sum_{i=1}^{n}\log x_i}\int_0^\infty\frac{(\Gamma\alpha+\beta)^{n}\alpha^{a-1}}{(\Gamma\alpha)^{n}}e^{-\alpha\left(b-\sum_{i=1}^{n}\log x_i\right)}\,d\alpha}$$

Let $\beta=1$

$$h(\alpha|\vec{x})=\frac{\frac{(\Gamma\alpha+1)^{n}}{(\Gamma\alpha)^{n}}\alpha^{a-1}e^{-b\alpha+\alpha\sum_{i=1}^{n}\log x_i}}{\int_0^\infty\frac{(\Gamma\alpha+1)^{n}}{(\Gamma\alpha)^{n}}\alpha^{a-1}e^{-\alpha\left(b-\sum_{i=1}^{n}\log x_i\right)}\,d\alpha} \quad (8)$$

To simplify the numerator in equation (8)

Let $\alpha\left(b-\sum_{i=1}^{n}\log x_i\right)=u\rightarrow\alpha=\frac{u}{b-\sum_{i=1}^{n}\log x_i}$, $d\alpha=\frac{1}{b-\sum_{i=1}^{n}\log x_i}\,du$

Substituting into equation (8), we get

$$\int_0^\infty\left(\frac{u}{b-\sum_{i=1}^{n}\log x_i}\right)^{a+n-1}e^{-u}\frac{1}{b-\sum_{i=1}^{n}\log x_i}\,du$$

$$\frac{1}{\left(b-\sum_{i=1}^{n}\log x_i\right)^{a+n}}\int_0^\infty(u)^{a+n-1}e^{-u}\,du$$

$$\boldsymbol{\Gamma(a+n)=\int_0^\infty(u)^{a+n-1}e^{-u}\,du}$$

$$\frac{\Gamma a+n}{\left(b-\sum_{i=1}^{n}\log x_i\right)^{a+n}} \quad (9)$$

$$h(\alpha|\vec{x})=\frac{\alpha^{a+n-1}e^{-\alpha\left(b-\sum_{i=1}^{n}\log x_i\right)}}{\frac{\Gamma a+n}{\left(b-\sum_{i=1}^{n}\log x_i\right)^{a+n}}}$$

$$h(\alpha|\vec{x})=\frac{\alpha^{a+n-1}e^{-\alpha\left(b-\sum_{i=1}^{n}\log x_i\right)}\left(b-\sum_{i=1}^{n}\log x_i\right)^{a+n}}{\Gamma a+n} \quad (10)$$

The equation (10) represents the probability density function of the random parameter α, then we can

Get the standard Bayes estimator for the random parameter by following these steps:

$$=\int_0^\infty(\hat{\alpha}-\alpha)^2\,h(\alpha|\vec{x})\,d\alpha$$

$$= \int_0^\infty (\hat{\alpha}^2 - 2\hat{\alpha}\alpha + \alpha^2)\, \mathrm{h}\,(\alpha|\vec{\mathrm{x}})\, d\alpha$$

$$= \hat{\alpha}^2 \int_0^\infty \mathrm{h}\,(\alpha|\vec{\mathrm{x}})\, d\alpha - 2\hat{\alpha} \int_0^\infty \alpha\, \mathrm{h}\,(\alpha|\vec{\mathrm{x}})\, d\alpha + \int_0^\infty \alpha^2\, \mathrm{h}\,(\alpha|\vec{\mathrm{x}})\, d\alpha$$

$$= \hat{\alpha}^2 - 2\hat{\alpha}E(\alpha) + K \quad (11)$$

By partial differentiation of equation (11) for the parameter α ̂, we get:

$$\frac{\partial}{\partial\hat{\alpha}} = 2\hat{\alpha} - 2E(\alpha) = 0$$

$$2\hat{\alpha} = 2E(\alpha)$$

$$\hat{\alpha}_{bayes\ 1} = E(\alpha) = \frac{\mathrm{a}+\mathrm{n}}{b - \sum_{i=1}^{n} \log \mathrm{x}_i} \qquad (12)$$

We can show the following:

1- The standard Bayes estimator is nothing but the maximum possible estimator for the parameter α, assuming that a, b takes very small values, according to the researcher (press s .j)

2- The standard Bayes estimator in the case of a quadratic loss function is the expectation of the subsequent distribution (Posterior Mean)

The subsequent probability density function for the random parameter (Posterior p.d.f) using the inverse Bayes Formula, as shown below:

$$\alpha \sim \text{Gamma}\left(\mathrm{a}+\mathrm{n}\ ,\frac{1}{b - \sum_{i=1}^{n} \log \mathrm{x}_i}\right)$$

$$\hat{\alpha} = E(\alpha) = \frac{\mathrm{a}+\mathrm{n}}{b - \sum_{i=1}^{n} \log \mathrm{x}_i} \quad (13)$$

In this research, small values for (a, b) will be assumed according to the opinion of the researcher Press[10], as shown below

a=b=0.1 e^{-11}

And to find the estimator of the reliability function for a beta distribution by the Bayesian method using a squared loss function:

Reliability function of the beta distribution (α bayes1 by offsetting in the parameter)

$$\hat{\mathrm{R}}_{bayes\ 1}(\mathrm{t}) = 1 - \frac{\mathrm{B_t}(\alpha_{\mathrm{bayes}\ 1}, \beta)}{\mathrm{B}(\alpha_{\mathrm{bayes}\ 1}, \beta)} \quad \ldots \quad (14)$$

6.The Standard Bayes Estimator Using Modified Squared Error Loss Function

Relying on the integrative equation (10-1), it is possible to obtain the standard bayes estimator for the random parameter using a modified quadratic loss function, let it be by following the following steps:

Standard bayes estimator under modified squared loss function:

$$= \int_0^\infty \alpha^r (\hat{\alpha} - \alpha)^2\, \mathrm{h}\,(\alpha|\vec{\mathrm{x}})\, d\alpha$$

$$= \int_0^\infty \alpha^r (\hat{\alpha}^2 - 2\hat{\alpha}\alpha + \alpha^2)\, \mathrm{h}\,(\alpha|\vec{\mathrm{x}})\, d\alpha$$

$$= \int_0^\infty (\hat{\alpha}^2\alpha^r - 2\hat{\alpha}\alpha^{r+1} + \alpha^{r+2})\, \mathrm{h}\,(\alpha|\vec{\mathrm{x}})\, d\alpha$$

$$= \hat{\alpha}^2 \int_0^{\infty} \alpha^r \, h(\alpha|\vec{x}) \, d\alpha - 2\hat{\alpha} \int_0^{\infty} \alpha^{r+1} \, h(\alpha|\vec{x}) \, d\alpha + \int_0^{\infty} \alpha^{r+2} \, h(\alpha|\vec{x}) \, d\alpha$$

$$= \hat{\alpha}^2 E(\alpha^r) - 2\hat{\alpha} E(\alpha^{r+1}) + K \qquad (15-1)$$

By partial differentiation of equation (15) with respect to the variable, we get

$$\frac{\partial}{\partial \hat{\alpha}} = 2\hat{\alpha} E(\alpha^r) - 2E(\alpha^{r+1}) = 0$$

$$2\hat{\alpha} E(\alpha^r) = 2E(\alpha^{r+1})$$

$$\hat{\alpha}_{bayes\ 1} = \frac{E(\alpha^{r+1})}{E(\alpha^r)} = \frac{a + n + r}{b - \sum_{i=1}^{n} \log x_i} \qquad , r \neq 0 \qquad (16)$$

And to find the estimator of the reliability function for a beta distribution by the Bayesian method using a modified squared loss function:

$$\hat{R}_{bayes\ 2}(t) = 1 - \frac{B_t(\alpha_{bayes\ 2}, \beta)}{B(\alpha_{bayes\ 2}, \beta)} \quad \ldots \qquad (17)$$

Describe the stages of the simulation experiment

7. Simulation

The simulation experiments included the following stages for applying the reliability function estimation methods in this section.

First stage:

It is the stage of choosing the default values, as it is one of the important stages on which the other stages depend. The default values were chosen as follows:

1- As for the different parameters and experiments, they were as in the following table:

Table.1. Shows default values for parameter (β, α) and for different experiments

Experiment	α	β
1	0.5	1
2	1.5	1
3	0.5	2
4	1.5	2
5	0.5	3
6	1.5	3

2- As for the assumed sample sizes, they were as follows:

n = 10, 20, 25, 40,75,100

The frequency of these experiments was equal to (1000) for each experiment.

Second stage: the generation of random numbers:

The mechanism of the Monte-Carlo method is carried out according to the following steps:

1- Generating a random setting that follows a continuous Uniform distribution defined over the period [0,1] by using the aggregate distribution function (c.d.f) that describes the model.

2- Transforming the regular random number to get a random variable that describes the model under the experiment and as shown below using the concept of the inverse of the function. If we have the following function F:

$u = F(x)$

$x = F^{-1}(u)$

where B=1,2,3

Third stage:

At this stage, the reliability function of the beta distribution is estimated

Fourth stage:

It is the stage of comparison between the efficiency of the estimation methods for the parameters for the purpose of reaching the most efficient estimator, and based on the statistical scale Mean Squared Error (MSE) as in the following formula

$$\boldsymbol{MSE}[\boldsymbol{\alpha}] = \frac{1}{R}\sum_{i=1}^{R}(\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha})^2 \qquad \textbf{(18)}$$

whereas :

α: represents the default values of the parameters

$\hat{\alpha}$: represents the estimated values of the parameters.

R: represents the number of repetitions of the experiment.

8. Discuss simulation experiments:

In this section, a simulation experiment was conducted with different sample sizes and based on (MSE), and these results were obtained by adopting a program written in (Mathematica 12.2) by the researcher, and the results are shown in the following tables

Table.2. It shows the MSE mean values for the assumed parameters and for all sample sizes

Model	α=0.5				
	n	SLF		MSLF	
		α	MSE	α	MSE
β=1	10	0.764716	0.070074	0.841187	0.116409
	20	0.632846	0.017648	0.664489	0.027057
	25	0.554747	0.002997	0.576937	0.005919
	40	0.548006	0.002305	0.561707	0.003808
	75	0.542020	0.001766	0.548795	0.002381
	100	0.471601	0.000806	0.474745	0.000638
	α=1.5				
	10	2.294150	0.630668	2.523560	1.047680
	20	1.898540	0.158833	1.993470	0.243508
	25	1.664240	0.026975	1.730810	0.053273
	40	1.644020	0.020742	1.685120	0.034269
	75	1.626060	0.015891	1.646380	0.021428
	100	1.414800	0.007258	1.424240	0.005740
Mode2	α=0.5				

	n	SLF		MSLF	
		α	MSE	α	MSE
β=2	10	0.519306	0.000373	0.571237	0.005075
	20	0.451466	0.002356	0.474039	0.006040
	25	0.405753	0.008883	0.421983	0.006087
	40	0.396336	0.010746	0.406245	0.008790
	75	0.399993	0.010001	0.404993	0.009026
	100	0.358278	0.020085	0.360666	0.019414
	α=1.5				
	10	1.466620	0.054467	1.393280	0.011389
	20	1.315750	0.147645	1.331540	0.107885
	25	1.656450	0.243596	1.646700	0.205478
	40	1.976378	0.274180	1.660790	0.241213
	75	1.995161	0.294862	1.707600	0.242457
	100	1.900354	0.359576	1.906356	0.352413
Mode3	α=0.5				
	n	SLF		MSLF	
		α	MSE	α	MSE
β=3	10	0.433040	0.004484	0.476344	0.000560
	20	0.383803	0.013502	0.402993	0.009410
	25	0.349205	0.022739	0.363173	0.018722
	40	0.340775	0.025353	0.349294	0.022712
	75	0.345266	0.026943	0.349581	0.022826
	100	0.313198	0.034895	0.315286	0.034119
	α=1.5				
	10	1.648131	0.304559	1.642940	0.208900
	20	1.655315	0.415619	1.698081	0.362307
	25	1.683307	0.513648	1.714640	0.469719
	40	1.758582	0.549700	1.777547	0.521939
	75	1.777523	0.551973	1.787242	0.538024
	100	1.715151	0.615988	1.719919	0.608527

Interpretation of table-2. The best estimator for parameter is (α)when the value of parameter is (β=1,2,3).

It can be seen from Table 2 that the best value of the parameter alpha and the least mean square error is when β = 1

Conclusion

The use of the standard Bayes method in estimating the α parameter has been discussed in this research, assuming that the β parameter is known and takes different values. The statistician is the mean of the squares of error, and we would also like to point out that the more the β parameter increases, the α parameters begin to move away from the real values. Therefore, we recommend that in the case of assuming that the beta parameter is known in subsequent studies to be equal or close to the one, we also recommend experimenting and studying the **opposite** of the known of the parameters

references

1. Al Nasser, &MakkiAkram Mohammed Saleh. (2009). The use of a generalized loss function of the Bayes method to estimate the measurement parameter and the reliability function of the Whipple distribution. IRAOI JOURNAL OF STATISTICAL SCIENCES, 9(16).
2. Al-Bayati, HussamNajmAbboud, (2002), "Comparing the methods of estimating the Whipple failure model using simulation", PhD thesis, College of Administration and Economics - University of Baghdad. Modified squared loss function.
3. Al-Hadithi, Ikhlas Ali Hammoudi, “Comparing the standard Bayes estimations of the Pareto distribution parameterUsing different loss functions", Master's Thesis, College of Administration and Economics - University of (Baghdad (2010 .).
4. Al-Issawi (2011)Comparison of standard Bayesian estimations for the exponential distribution parameter using different loss functions.
5. Ghosh , J.K , Delampady , M &Samanta , J (2006): An Introduction to Bayesian Analysis , Theory & Methods , Springer , First Edision.
6. Hormuz, Amir Hanna, (1990), "Mathematical Statistics", College of Administration and Economics, Mosul University Press. Beta distribution.
7. Koch, K. R. (2007). Introduction to Bayesian statistics. Springer Science & Business Media.
8. Podder, C. K. (2004). Comparison of two risk functions using the Pareto distribution. Pakistan Journal of Statistics, 20(3)
9. Press, S. J. (2001), The Subjectivity of Scientists and the Bayesian Approach, Wiley, New York.
10. Press, S. J., &Tanur, J. M. (2012). The subjectivity of scientists and the Bayesian approach (Vol. 775). John Wiley & Sons.