Building a probabilistic model (Maxwell-Rayleigh) using the (T-X family) method

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Abstract: In this study, a new distribution was created, which is the Maxwell-Rayleigh distribution with two parameters (θ, λ), using the (TX Family method), and some different characteristics of the Maxwell-Rayleigh distribution were studied, and the shape of the distribution was skewed towards the left, and its parameters were estimated using The weighted least squares method, and for the purpose of comparing the estimation methods for parameters, the Monte-Carlo simulation method was employed using the Wolfram Mathematica 12.2 program to conduct several experiments with different sample sizes (small "25,50", medium "100"). and large "150" and through the use of the statistical scale Mean Squares Error (MSE), and the results showed the preference of the weighted least squares method in estimating parameters, at medium and small sample sizes, and the preference of the method of greatest possibility at large sample sizes

Keywords: Maxwell-Rayleigh, Estimation, moments, T-X family

1. Introduction

Statistical distributions are widely used in the analysis of real phenomena of nature, and although many distributions have been defined and studied over the years, they have a limited range of capabilities and therefore cannot be applied in all situations, because the characteristics of the phenomenon do not remain the same with the passage of time. Time, which prompted the researchers to develop and study new distributions to be more flexible and accurate in the data to study the real phenomenon, by expanding the already existing distributions in several ways, through which a new form of an already existing probability distribution can be produced, and accordingly it is possible to produce a new set of probability distributions. Ibeh, et. (2021) distribution in the family of generalized exponential distributions generated using the transformed-transformer. introduces Ekum, et. (2020) using the T-X framework T-Dagum a way of generalizing Dagum distribution using Lomax quantile. Introduce Hamed. et. (2018) some new families of generalized Pareto distributions using the T-X framework. Osatohanmwen, et. (2019) a new Member from the (T− X Family) of: distribution called the Gumbel-Burr and account of some mathematical properties of the new distribution, and including the maximum likelihood estimation of its parameters is presented. Aldeni, et. (2017). the quantile of generalized lambda distribution has been proposed using the T-X framework to be much more broad and flexible. So our aim is to in This study introduce a new distribution in the family of generalized maxwell. distribution generated using, the (T-X family) or ,transformed-transformer introduced by Alzaatreh et al. (2013). . Rest of the paper is organized as follows. In Section 2 we define distribution the Maxwell. In Section 3,, we define distribution the Rayleigh. In section 4, we define distribution the Maxwell-Rayleigh, by using the (T-X family) framework. we define some of its basic properties in section 5, then we Parameters estimation and simulationin the last section of this paper.

2. Maxwell distribution

The Maxwell or (Maxwell–Boltzmann) distribution has many applications in physics and chemistry and other allied sciences. that was first introduced by (James Clerk Maxwell, 1860), and described it again by (Ludwig Boltzmann, 1970), with a few assumptions. and since then, the study and application of Maxwell distribution have been received. and since then, the study and application of Maxwell distribution have been received great attention, The probability density function (p.d.f) of Maxwell distribution is given by:[1] [6]

$$f(t;\alpha)=\sqrt{\frac{2}{\pi}}\frac{t^2 e^{-\frac{t^2}{2\alpha^2}}}{\boldsymbol{\alpha^3}}\qquad ,\alpha>0\,,t\geq 0 \qquad (1)$$

The cumulative distribution function (c.d.f) is given by:

$$F(t)=\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{t^2}{2\alpha^2}\right) \qquad (2)$$

Wher

$\alpha > 0$ is scale parameter.

and γ (-,-) is the lower incomplete gamma function $\gamma(\alpha,t) = \int_0^t t^{\alpha-1} e^{-u} du$

The Maxwell distribution is a special case of a generalized Whipple distribution.

3. Rayleigh Distribution

Apart Rayleigh distribution was Discovered by Scientist English physicist Lord Riley, the random variable x is said to have the Rayleigh distribution, if its probability density function (p.d.f) is given by: [9]

$$f(x,\theta) = \frac{x}{\theta^2} e^{-\frac{x^2}{2\theta^2}} \quad , \qquad x\,,\theta > 0 \qquad (3)$$

The cumulative distribution function (c.d.f) is given by:

$$F(x) = 1 - e^{-\frac{x^2}{2\theta^2}} \qquad (4)$$

Wher:

$\theta > 0$ is scale parameter.

The hazard function of the distribution Rayleigh is obtained by;-

$$[-\log(1 - F(x))] = \frac{x^2}{2\boldsymbol{\theta}^2} \,(5)$$

$$h(x) = \frac{f(x)}{1 - F(x)} = \frac{x}{\theta^2} \,(6)$$

4. Maxwell-Rayleigh Distribution

In this section, we present the probability density function (p.d.f) and the cumulative distribution function (c.d.f) of the new Maxwell-Rayleigh distribution by adopt a new method called the transformed-transformer or (T-X family) developed by Alzaatreh et al. (2013).. Here they used a random variable (X), with the probability density function (p.d.f), and the cumulative distribution function (c.d.f), to transform another random variable (T) with support [0, ∞] having probability density function (p.d.f) and cumulative distribution function (c.d.f), z(t) and Z(t) respectively to obtain the distribution (c.d.f) function of a new class of distributions the is can has defined by:-[4] [14]

$$F_{MR}(x) = \int_0^{-\log(1-F(x))} z(t)dt = Z[-\log(1 - F(x))] \qquad (7)$$

The probability density function (p.d.f) corresponding to (7) is given by:-

$$f_{MR}(x) = \left[\frac{\partial}{\partial x}\{-\log(1 - F(x))\}\right] z[-\log(1 - F(x))]$$

$$f_{MR}(x) = h(x)\, z[-\log(1 - F(x))] \qquad (8)$$

Where, h(x)=f(x)/1-F(x), is the hazerd function of the random variable (x). Then [-log(1-F (x))] is the hazerd cumulative function H(x) of the random variable (x). Assume that (t) follows the Maxwell distribution with scale parameter (α). Assume that (x) follows the Rayleigh distribution with scale parameter (θ). The distributive function (c.d.f) for the Maxwell-Rayleigh distribution will be obtained as follows:[5]

$$F_{MR}(x) = \int_0^t z(t)dt$$

$$t = (WF(x)) = -\log(1 - F(x)) = \frac{x^2}{2\theta^2}$$

$$F_{MR}(x) = \int_0^{\frac{x^2}{2\theta^2}} z\left(\frac{x^2}{2\theta^2}\right) d\,\frac{x^2}{2\theta^2}$$

$$= \int_0^{\frac{x^2}{2\theta^2}} z\left(\frac{x^2}{2\theta^2}\right) d\frac{x^2}{2\theta^2}$$

$$= \sqrt{\frac{2}{\pi}}\frac{1}{\alpha^3}\int_0^{\frac{x^2}{2\theta^2}} \left(\frac{x^2}{2\theta^2}\right)^2 e^{-\frac{\left(\frac{x^2}{2\theta^2}\right)^2}{2\lambda^2}} d\frac{x^2}{2\theta^2}$$

$$= \sqrt{\frac{2}{\pi}}\frac{1}{\alpha^3\theta^6}\int_0^{\frac{x^2}{2\theta^2}} x^5 e^{-\frac{x^4}{8\lambda^2\theta^4}} dx$$

$$let\ y = \frac{x^4}{8\boldsymbol{\lambda}^2\theta^4},\Leftrightarrow\ x^4 = 8y\boldsymbol{\lambda}^2\theta^4 \Leftrightarrow x = (8y\alpha^2\theta^4)^{\frac{1}{4}} \Leftrightarrow dx = \frac{1}{4}y^{\frac{-3}{4}}(8\alpha^2\theta^4)^{\frac{1}{4}}dy$$

$$F_{MR}(x) = \sqrt{\frac{2}{\pi}}\frac{1}{\alpha^3\theta^6}\int_0^{\frac{x^2}{2\theta^2}} (8y\alpha^2\theta^4)^{\frac{5}{4}}\, e^{-y}\,\frac{1}{4}y^{\frac{-3}{4}}(8\alpha^2\theta^4)^{\frac{1}{4}}dy$$

$$= \sqrt{\frac{2}{\pi}}\frac{8\ 8^{\frac{1}{2}}}{\mathbf{16}\ \alpha^3\theta^6}\alpha^3\theta^6\int_0^{\frac{x^2}{2\theta^2}} y^{\frac{1}{2}}\, e^{-y}dy$$

$$= \sqrt{\frac{2}{\pi}}\frac{2\sqrt{2}}{2}\gamma\left(\frac{3}{2},y\right)$$

$$F_{MR}(x) = \frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{x^4}{8\alpha^2\theta^4}\right) \qquad (9)$$

where is $\gamma(\alpha,y) = \int_0^x t^{\alpha-1}e^{-y}\,dt$the incomplete lower gamma function.

It is possible to simplify the cumulative distribution function in the formula (9) as follows:-

$$\gamma(\alpha,y) + \Gamma($$

Where is $\Gamma(.,x)$ و $\gamma(.,x)$These are incomplete gamma functions, and the formula for the lower gamma function is:

$$\gamma(\alpha,y) = \Gamma($$

$$\gamma\left(\frac{3}{2},\frac{x^4}{8\alpha^2\theta^4}\right)$$

By substituting the above equation into equation (9), we get the cumulative distribution function (c.d.f) as follows-:

$$F_{MR}(x) = \frac{2}{\sqrt{\pi}}$$

$$F_{MR}(x) = 1 - \frac{2}{\sqrt{\pi}}\,\Gamma\left(\frac{3}{2},\frac{x^4}{8\alpha^2\theta^4}\right) \qquad (11)$$

To get the probability density function (p.d.f) for the proposed Maxwell-Rayleigh probability distribution, we follow:

$$f_{MR}(x) = \frac{d}{dx} F_{MR}$$

$$f_{MR}(x) = \frac{d}{dx}\Big[1 -$$

Using the program (Wolfram Mathematica 12.2), the above equation was derived to obtain a function (p.d.f) as follows-:

$$f_{MR}(x) = \sqrt{\frac{2}{\pi}} \frac{1}{4\alpha^3\theta^6} x^5 e^{-\frac{x^4}{8\alpha^2\theta^4}} \quad ; \quad x \geq 0 \ , \ \alpha, \theta > 0 \qquad (12)$$

Since (θ,α) is the scale parameter.

In order for the above function to be a probabilistic function, it must fulfill the following two conditions:

1- $f_{MR}(x) \geq 0 \ ; \quad for\ All\ x$

2- $\int_0^\infty f_{MR}(x) = 1$

To achieve the above conditions, we follow the following:-

$$f_{MR}(x) = \sqrt{\frac{2}{\pi}}$$

Let

$$y = \frac{x^4}{8\alpha^2\theta^4}$$

$$dx = \frac{1}{4} y^{-\frac{3}{4}} [$$

$$f_{MR}(x) = \sqrt{\frac{2}{\pi}}$$

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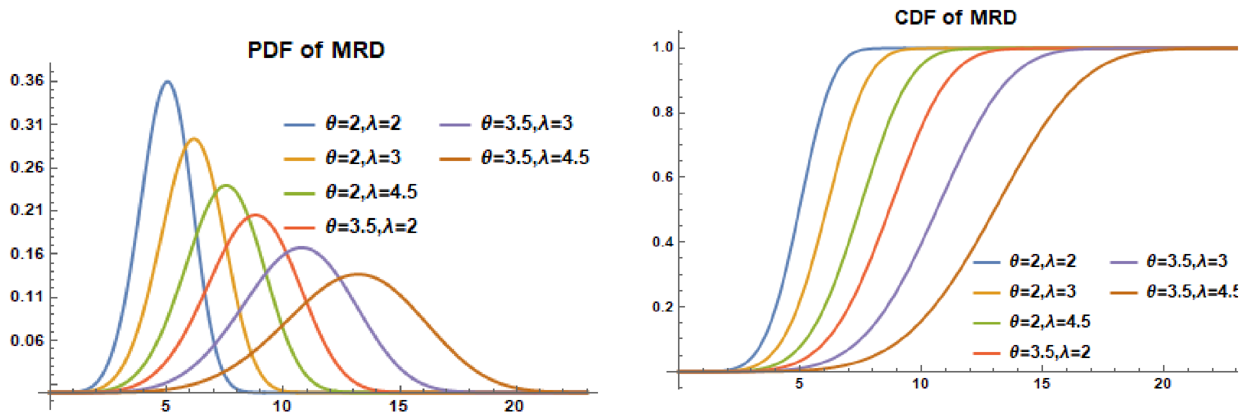
$$f_{MR}(x) = \frac{2}{\sqrt{\pi}}$$

$$But\ \Gamma\left(\frac{3}{2}\right) =$$

$$\therefore f_{MR}(x) = 1$$

The condition that the function is a positive probability is satisfied for all values of the random variable X



(a) (b)

Figure 1. Plots of (a) PDF and (b) CDF of MR distribution.

5. Distribution Properties of Maxwell-Rayleigh

Here we list some important distributional properties of Maxwell-Rayleigh distribution:

5.1 Non-central r^th moment[12]

$$\mu'_r = E(x^r) = \int_0^\infty x^r\, g(x)\, dx \quad (14)$$

Substituting equation (13) into equation (14), we get:

$$\mu'_r = E(x^r) = \sqrt{\frac{2}{\pi}} \frac{1}{4\alpha^3\theta^6} \int_0^\infty [y\, 8\, \alpha^2\theta^4]^{\frac{5+r}{4}} e^{-y} \frac{1}{4} y^{-\frac{3}{4}} [\,8\, \alpha^2\theta^4]^{\frac{1}{4}}\, dy$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{16\, \alpha^3\theta^6} 8^{\frac{5+r}{4}} \alpha^{\frac{5+r}{2}} \theta^{5+r} 8^{\frac{1}{4}} \alpha^{\frac{1}{2}}\, \theta \int_0^\infty y^{\frac{5+r}{4}} y^{-\frac{3}{4}} e^{-y}\, dy$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{16\, \alpha^3\theta^6} 8^{\frac{3}{2}}\, 8^{\frac{r}{4}} \alpha^3 \alpha^{\frac{r}{2}} \theta^6 \theta^r \int_0^\infty y^{\frac{2+r}{4}} e^{-y}\, dy$$

$$\mu'_r = E(x^r) = \frac{2}{\sqrt{\pi}} 8^{\frac{r}{4}} \alpha^{\frac{r}{2}} \theta^r\, \Gamma\left(\frac{r+6}{4}\right); r > -6 \quad (15)$$

So that :

(a) Mean $= \mu'_1 = E(x) = \frac{3}{2}\sqrt{\frac{\alpha}{\pi}}\, 8^{\frac{1}{4}}\, \theta\, \Gamma\left(\frac{3}{4}\right)$ (16)

(b) Mean square error = $\boldsymbol{\mu'_2} = E(x^2) = \frac{4\sqrt{2}}{\sqrt{\pi}}\theta^2\alpha$ (**17**)

5.2 Central r^th moment[12]

$$\mu_r = E(x-\mu'_1)^r = \int_0^\infty (x-\mu'_1)^r\, g(x)\, dx \quad (\mathbf{18})$$

$$(x-\mu'_1)^r = \sum_{j=0}^{r} C_j^r\, (x)^j (-\mu'_1)^{r-j}$$

$$\mu_r = E(x-\mu'_1)^r = \sqrt{\frac{2}{\pi}} \frac{1}{4\alpha^3\theta^6} \left\{ \sum_{j=0}^{r} C_j^r\, (-\mu'_1)^{r-j} \right\} \int_0^\infty [y\, 8\, \alpha^2\theta^4]^{\frac{5+j}{4}} e^{-y} \frac{1}{4} y^{-\frac{3}{4}} [\,8\, \alpha^2\theta^4]^{\frac{1}{4}}\, dy$$

$$= \sqrt{\frac{2}{\pi}}\frac{1}{16\,\alpha^{3}\theta^{6}}\left\{\sum_{j=0}^{r} C_j^r\ (-\mu_1')^{r-j}\right\}8^{\frac{5+j}{4}}\alpha^{\frac{5+j}{2}}\theta^{5+j}8^{\frac{1}{4}}\alpha^{\frac{1}{2}}\,\theta\int_0^\infty y^{\frac{5+j}{4}}y^{-\frac{3}{4}}e^{-y}dy$$

$$= \sqrt{\frac{2}{\pi}}\frac{1}{16\,\alpha^{3}\theta^{6}}\left\{\sum_{j=0}^{r} C_j^r\ (-\mu_1')^{r-j}\right\}8^{\frac{3}{2}}\,8^{\frac{j}{4}}\alpha^{3}\,\alpha^{\frac{j}{2}}\theta^{6}\theta^{j}\int_0^\infty y^{\frac{2+j}{4}}\,e^{-y}\,dy$$

$$= \sqrt{\frac{2}{\pi}}\sqrt{2}\sum_{j=0}^{r} C_j^r\ (-\mu_1')^{r-j}\,8^{\frac{j}{4}}\alpha^{\frac{j}{2}}\,\theta^{j}\,\Gamma(\frac{6+j}{4})$$

$$\therefore \mu_r = E(x-\mu_1')^r = \frac{2}{\sqrt{\pi}}\left\{\sum_{j=0}^{r} C_j^r\ (-\mu_1')^{r-j}\,8^{\frac{j}{4}}\alpha^{\frac{j}{2}}\theta^{j}\,\Gamma(\frac{6+j}{4})\right\} \tag{19}$$

So that :

(a) **Variance** $= \mu_2 = \mu_1^2 - \frac{4}{\sqrt{\pi}}\left\{\mu\,8^{\frac{1}{4}}\alpha^{\frac{1}{2}}\theta\,\Gamma\left(\frac{7}{4}\right) + \frac{4\sqrt{2}}{\sqrt{\pi}}\alpha\ \ \theta^2\right\}$

(b) **Coefficients of Variation :**

$$C.V = \frac{\sqrt{(\mu_1')^2 - \frac{4}{\sqrt{\pi}}\left((\mu_1')8^{\frac{1}{4}}\alpha^{\frac{1}{2}}\theta\,\Gamma\left(\frac{7}{4}\right) + \frac{4\sqrt{2}}{\sqrt{\pi}}\alpha\ \ \theta^2\right)}}{\frac{3}{2}\sqrt{\frac{\alpha}{\pi}}8^{\frac{1}{4}}\,\theta\,\Gamma(\frac{3}{4})} * 100$$

(c) The coefficient of skewness is :

$$C.S = \frac{-\mu^3 + \frac{2}{\sqrt{\pi}}\left[\frac{9}{4}\mu 8^{\frac{1}{4}}\alpha^{\frac{1}{2}}\theta\,\gamma\left(\frac{3}{4}\right) + \frac{12\sqrt{2}}{\sqrt{\pi}}\alpha\ \ \theta^2 + \frac{10\ 8^{\frac{3}{4}}}{4\sqrt{\pi}}\alpha^{\frac{3}{2}}\theta^3\,\gamma\left(\frac{5}{4}\right)\right]}{\left[\mu^2 - \frac{4}{\sqrt{\pi}}\ [\mu 8^{\frac{1}{4}}\alpha^{\frac{1}{2}}\theta\,\gamma\left(\frac{7}{4}\right) + \frac{4\sqrt{2}}{\sqrt{\pi}}\alpha\theta^2\ \ \right]^{\frac{3}{2}}}$$

(d) Also the measure of kurtosis becomes :

$$= \frac{\left[\mu^4 - \frac{6}{\sqrt{\pi}}\mu^3 8^{\frac{1}{4}}\alpha^{\frac{1}{2}}\theta\,\Gamma\left(\frac{3}{4}\right) + \frac{12}{\sqrt{\pi}}\mu^2 8^{\frac{1}{2}}\theta^2 - \frac{10}{\sqrt{\pi}}\mu\ 8^{\frac{3}{4}}\alpha^{\frac{3}{2}}\theta^3\,\Gamma\left(\frac{5}{4}\right) +\ 12\alpha^{2}\,\theta^4\right]}{\left[\mu^2 - \frac{4}{\sqrt{\pi}}\left(\mu 8^{\frac{1}{4}}\alpha^{\frac{1}{2}}\theta\,\Gamma\left(\frac{7}{4}\right) + \frac{4\sqrt{2}}{\sqrt{\pi}}\alpha\ \ \theta^2\right)\right]^2}$$

6. Parameter Estimation

6.1. Maximum Likelihood Estimation:

In this section, we use the Maximum likelihood Method (MLE) to address the parameters estimation, and conduct a simulation to examine of this method.

Let x1,x2,…, xn be a random sample of size n from a Maxwell-Rayleigh distribution defined in equation (9), then the log-likelihood function is given by:[12] [3]

$$\mathrm{Ln}\ L(\alpha,\theta) = \frac{n}{2}\ \mathrm{Ln}\left(\frac{2}{\pi}\right) - n\mathrm{Ln}(4) - 3n\ \mathrm{Ln}(\alpha) - \mathbf{6nLn(\theta)} + \mathbf{Ln}\prod_{i=1}^{n} xi^5 - \frac{\sum_{i=1}^{n} xi^4}{8\ \alpha^{2}\theta^4} \tag{20}$$

Differentiating equation (10) with respect to θ, we get:

$$\frac{\partial \mathrm{Ln}\ L(\alpha,\theta)}{\partial\theta} = -\frac{6n}{\hat{\theta}} + \frac{32\ \alpha^2\hat{\theta}^3\sum_{i=1}^{n} xi^4}{64\ \alpha^4\hat{\theta}^8} = \text{zero}$$

$$\hat{\theta}^4 = \frac{\sum_{i=1}^{n} xi^4}{12\ n\ \alpha^2}$$

$$\hat{\theta} = \left(\frac{\sum_{i=1}^{n} xi^4}{12\ n\ \alpha^2}\right)^{\frac{1}{4}} \qquad (21)$$

Differentiating equation (10) with respect to α, we get:

$$\frac{\partial \text{Ln L}(\alpha,\theta)}{\partial\alpha} = -\frac{3n}{\hat{\alpha}} + \frac{16\ \hat{\alpha}\theta^4 \sum_{i=1}^{n} xi^4}{64\ \hat{\alpha}^4\theta^8} = \text{zero}$$

$$\hat{\alpha}^2 = \frac{\sum_{i=1}^{n} xi^4}{12n\ \theta^4}$$

$$\hat{\alpha} = \sqrt{\frac{\sum_{i=1}^{n} xi^4}{12n\ \theta^4}} \qquad (22)$$

6.2. Weighted Least Square Method

This method aims to make the sum of squares of deviations as small as possible by reducing the amount of K and using the cumulative function of the complex distribution (MR) as follows:-using the moment estimator for θ:-[8] [16]

$$k = \sum_{i=1}^{n} Wi \left\{G(xi) - \frac{i}{n+1}\right\}^2$$

$$Wi = \frac{1}{V(G(xi))} = \frac{(n+1)^2\ (n+2)}{i(n-i+1)}$$

$$K = \sum_{i=1}^{n} Wi \left\{1 - \frac{2}{\sqrt{\pi}}\ \Gamma\left(\frac{3}{2},\frac{x^4}{8\alpha^2\theta^4}\right) - \frac{i}{n+1}\right\}^2$$

By deriving equation (2.62) for the parameter (θ) and dividing by (2) and setting it equal to zero, we get:

$$\frac{dK}{d\theta} = \sum_{i=0}^{n} Wi\left\{1 - \frac{2}{\sqrt{\pi}}\ \Gamma\left(\frac{3}{2},\frac{x^4}{8\alpha^2\theta^4}\right) - \frac{i}{n+1}\right\}\left(-\frac{x^6}{2\sqrt{2\pi}\,\alpha^3\theta^7}\ e^{-\frac{x^4}{8\alpha^2\theta^4}}\right) = 0 \qquad (23)$$

The derivative inside the bracket was obtained for the parameter θ in the above equation using the program (Wolfram Mathematica 12.2) and as follows:

$$0 = -\sum_{i=0}^{n} Wi\ \frac{x^6}{2\sqrt{2\pi}\,\alpha^3\theta^7}\ e^{-\frac{x^4}{8\lambda^2\theta^4}} + \sum_{i=0}^{n} Wi\ \frac{2}{\sqrt{\pi}}\ \Gamma\left(\frac{3}{2},\frac{x^4}{8\alpha^2\theta^4}\right)\left(\frac{x^6}{2\sqrt{2\pi}\,\alpha^3\theta^7}\ e^{-\frac{x^4}{8\alpha^2\theta^4}}\right)$$
$$+ \sum_{i=0}^{2} Wi\left(\frac{x^6}{2\sqrt{2\pi}\,\alpha^3\theta^7}\ e^{-\frac{x^4}{8\alpha^2\theta^4}}\right)\frac{i}{n+1}$$

$$\sum_{i=0}^{2} Wi\ \frac{2}{\sqrt{\pi}}\ \Gamma\left(\frac{3}{2},\frac{x^4}{8\alpha^2\theta^4}\right)\left(\frac{x^6}{2\sqrt{2\pi}\,\alpha^3\theta^7}\ e^{-\frac{x^4}{8\alpha^2\theta^4}}\right) - \sum_{i=0}^{n} Wi\ \frac{x^6}{2\sqrt{2\pi}\,\alpha^3\theta^7}\ e^{-\frac{x^4}{8\alpha^2\theta^4}}$$
$$+ \sum_{i=0}^{n} Wi\left(\frac{x^6}{2\sqrt{2\pi}\,\alpha^3\theta^7}\ e^{-\frac{x^4}{8\alpha^2\theta^4}}\right)\frac{i}{n+1} = 0 \qquad (24)$$

By deriving equation (2.62) for the parameter (λ) and dividing by (2) and setting it equal to zero, we get:

$$\frac{dK}{d\lambda} = \sum_{i=0}^{n} Wi\left[1 - \frac{2}{\sqrt{\pi}}\ \Gamma\left(\frac{3}{2},\frac{x^4}{8\alpha^2\theta^4}\right) - \frac{i}{n+1}\right]\left(-\frac{x^6}{4\sqrt{2\pi}\alpha^4\theta^6}\ e^{-\frac{x^4}{8\alpha^2\theta^4}}\right) = 0 \qquad (25)$$

The derivative inside the bracket was obtained for the parameter θ in the above equation using the program (Wolfram Mathematica (12.2) and as follows:-

$$= -\sum_{i=0}^{m} \mathrm{Wi} \, \frac{x^6}{4\sqrt{2\pi}\alpha^4\theta^6} \, e^{-\frac{x^4}{8\alpha^2\theta^4}} + \sum_{i=0}^{n} \mathrm{Wi} \frac{2}{\sqrt{\pi}} \, \Gamma\left(\frac{3}{2}, \frac{x^4}{8\alpha^2\theta^4}\right) \frac{x^6}{4\sqrt{2\pi}\alpha^4\theta^6} \, e^{-\frac{x^4}{8\alpha^2\theta^4}}$$

$$+ \sum_{i=0}^{n} \mathrm{Wi} \, \frac{x^6}{4\sqrt{2\pi}\alpha^4\theta^6} \, e^{-\frac{x^4}{8\alpha^2\theta^4}} \frac{\mathrm{i}}{\mathrm{n}+1}$$

$$\sum_{i=0}^{n} \mathrm{Wi} \frac{2}{\sqrt{\pi}} \, \Gamma\left(\frac{3}{2}, \frac{x^4}{8\alpha^2\theta^4}\right) \frac{x^6}{4\sqrt{2\pi}\alpha^4\theta^6} \, e^{-\frac{x^4}{8\alpha^2\theta^4}} - \sum_{i=0}^{m} \mathrm{Wi} \, \frac{x^6}{4\sqrt{2\pi}\alpha^4\theta^6} \, e^{-\frac{x^4}{8\alpha^2\theta^4}} + \sum_{i=0}^{n} \mathrm{Wi} \, \frac{x^6}{4\sqrt{2\pi}\alpha^4\theta^6} \, e^{-\frac{x^4}{8\alpha^2\theta^4}} \frac{\mathrm{i}}{\mathrm{n}+1} = 0 \qquad (26)$$

7. Simulation.

In this section, we made an estimation by using 2 methods of the Maxwell-Rayleigh distribution parameters, and compared between them to assess the performance of each method by a simulation study. We consider different sample sizes n =25, 50, 100, 150 for the two parameters combinations, Set-1: θ=2, α=3, θ=2, α=4.5, , Set-II: θ=3.5, α=3, θ=3.5, α=4.5. For each estimate, we obtain the average values of the estimates (mean) and their corresponding mean squares error (MSE). The performance of different estimators is evaluated in terms of MSE, i.e., the most efficient method of estimating will be those whose (MSE) values are closer to zero. Simulation results are obtained via the (Wolfram Mathematica 12.2) program, in Table 1. We noted that the (MSE) decrease as the sample size increases. Thus, the (MLE) performs well in estimating the parameters of Maxwell-Rayleigh distribution.

Table 1. Simulation results showing and the (MSE) of the Maxwell-Rayleigh distribution

habits.

		Set 1				Set 2.			
N	Model	θ=2	α =3	θ =2	α =4.5	θ=3.5	α =3	θ=3.5	α =4.5
25	parameters	1.97763	3.07747	1.95933	4.62334	3.45011	3.07721	3.46262	4.61636
	MSE_ML	0.00762	0.01023	0.00991	0.02555	0.02598	0.01012	0.02382	0.02254
	parameters	1.98562	3.07905	1.9673	4.61718	3.4576	3.07626	3.4735	4.6161
	MSE_WLS	0.00773	0.01073	0.0102	0.0258	0.02532	0.00998	0.02297	0.02304
50	parameters	1.98261	3.05035	1.98506	4.58732	3.46468	3.0592	3.47203	4.58159
	MSE_ML	0.00357	0.00434	0.00421	0.01253	0.01495	0.00541	0.01206	0.01107
	parameters	1.98818	3.04666	1.98834	4.58162	3.47269	3.05542	3.47903	4.57813
	MSE_WLS	0.0033	0.00349	0.00407	0.01151	0.0151	0.00568	0.01196	0.01005
100	parameters	1.98758	3.03792	1.98698	4.55516	3.4783	3.03435	3.47774	4.55438
	MSE_ML	0.00219	0.00261	0.00211	0.0057	0.00538	0.00202	0.00586	0.00485
	parameters	1.98964	3.03493	1.98799	4.55294	3.48426	3.03194	3.47911	4.55212
	MSE_WLS	0.00215	0.00209	0.00215	0.00459	0.00506	0.00168	0.00616	0.00432
150	parameters	1.99321	3.02787	1.98867	4.55255	3.48209	3.02519	3.48453	4.54119
	MSE_ML	0.00103	0.00122	0.00159	0.00412	0.00315	0.00108	0.00336	0.00246
	parameters	1.9967	3.02651	1.99122	4.54682	3.48434	3.02545	3.48645	4.53934
	MSE_WLS	0.00107	0.00126	0.00148	0.0035	0.00351	0.00106	0.00336	0.0025

Interpretation of table-1.

It is clear from the above table and by comparing the mean squares error (MSE) and for the different sample sizes that the weighted least squares estimation method has the least (MSE) compared to the greatest possibility method.

8. Applications

The MR distribution was applied to a set of data representing the machine breakdowns of the textile lab, consisting of (91) observations taken in months, as shown in the following table:

Table 2. It represents the operating time of the machine until a breakdown, measured in months

2.06	4.3	4.8	5.43	5.93	6.26	6.76	7.36	8.23
2.1	4.39	4.8	5.46	6.02	6.26	6.8	7.36	8.23
2.26	4.39	4.82	5.46	6.06	6.3	6.8	7.5	8.3
3.1	4.4	4.88	5.56	6.06	6.3	6.8	7.56	8.36
3.2	4.42	4.9	5.6	6.1	6.3	6.83	7.6	8.36
3.56	4.57	4.94	5.6	6.1	6.33	6.83	7.63	8.46
3.66	4.64	5.23	5.63	6.13	6.53	7.02	7.63	8.83
3.7	4.71	5.36	5.7	6.13	6.6	7.1	7.73	8.93
4.2	4.78	5.4	5.83	6.13	6.7	7.33	7.73	8.93
4.26	4.79	5.43	5.83	6.16	6.76	7.36	8.1	9.1
								9.36

Table.3 The following table shows the most important statistical indicators for statistical data.

Index	Value
Mean	6.04648
Variance	2.57022
Skewness	-0.195492
Kurtosis	2.79538
Median	6.1
Standard Deviation	1.60319

These data were tested using chi-squared statistic and it was found that they follow the suggested distribution Maxwell-Rayleigh. The results of the test are shown in the following table**.**

Table.4 Represents aGoodness of fit

distribution	Parameter		Pearson Chi-Square	
	θ	α	statistic	P-Value
Maxwell-Rayleigh	2.22741	2.45911	13.4	0.339

The test for the best representation of this data was also conducted by comparing the Maxwell-Rayleigh distribution with the Maxwell-Rayleigh distribution, and the Rayleigh distribution, using the AIC and AICc criterion. The results showed that the Maxwell-Rayleigh distribution is the most appropriate and best way to represent the data.

Table.5 The test standard represents the best distribution.

distribution	Parameter estimation		AIC	AICc
	θ	α		
Maxwell-Rayleigh	2.22741	2.45911	353.74	535.877

Maxwell	-	3.75131	378.096	378.141
Rayleigh	4.7443	-	406.975	407.02

The following figure shows the appropriateness of the (Maxwell-Rayleigh) distribution compared to the (Maxwell-Rayleigh) distributions:

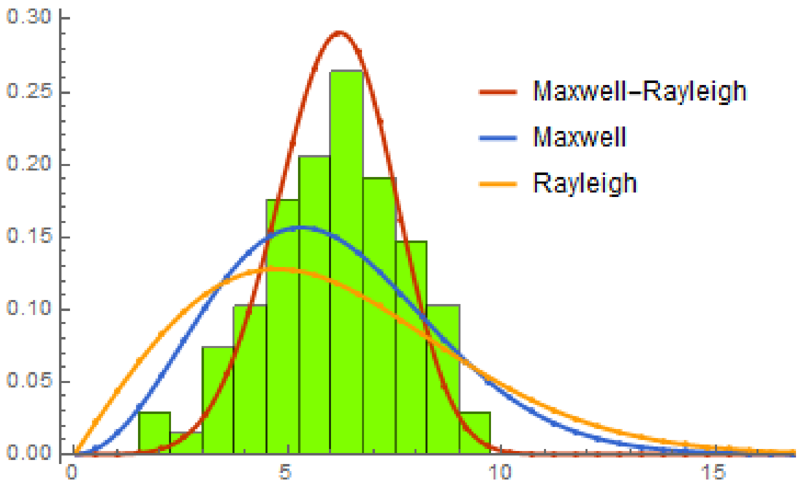


Figure 2shows the fit of the proposed (MR) probabilistic model to the real data compared to the two distributions (Maxwell) and (Rayleigh)

9. Conclusion

The experimental side showed, based on the statistical criterion (MSE), that the best way to estimate the parameters of the proposed model that have the most number of times of advantage is the weighted least squares (WLS) method. The larger the sample size, the lower the MSE values for each experiment, which is consistent with the statistical theory. As through the test results, it was proved that the proposed distribution is the best in representing the data compared with (Maxwell distribution) and (Rayleigh) distribution based on the two criteria (AIC, AICc).

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