Model Predictive Control Design for SISO LTI system subjected to bounded constraints

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Abstract

In this paper Model Predictive Control (MPC)for SISO LTI system is considered. MPC emanates from dynamicoptimisation where a dynamical system is controlled while minimising the associated performance index and satisfying all the constraints.Using current measurement of states (if all states are available for measurement elsestate observer is used) a predicted plant model is obtained on a prediction horizon (N) and using this predicted plant model we get the predicted performance index which is minimized to get a future control sequence. The first component of the control sequence is used while rest gets discarded and the at the next instance the whole process is repeated. In this paper a LTI SISO system is subjected to constraints on both input and output is considered. Simulation results verify the efficacy of the designed algorithm.

Key Words: Model Predictive Control, LTI,Control Variable,Dynamix matrix control

1. INTRODUCTION

Model Predictive Control (MPC) is a control strategy for a dynamical system based on repeated optimization. With the help of current information (current measurements) from the plant (to be controlled), future controllers and in turn the future plant responses are predicted using the model of the system and optimized at regular time instants with respect to a performance index which is to be minimized. The advantage of MPC lies in the fact that it is easy to design and implement and in the past emerging as the favourite control strategy for the control engineers because of its efficient ability to deal uncertain multi-parametric variable dynamical systems subjected to constraints on input and states. Such constraints are present in all control engineering applications and are bound to be there due to safety, economic and physical concerns. And these constraints are dealt explicitly by solving a dynamical constrained optimization problem in real time. MPC gives theoretical basis for the understanding of feasibility, stability, optimality and robustness properties.

The successful implementation of MPC which was first reported with the application of Identification and command (IDCOM) by Richalet et al. (1978) leads to the surge in the interest of MPC in which a Model Predictive Heuristic Control (MPHC) was applied on a Fluid Catalytic Cracking Unit (FCCU) main fractionator column in a PVC plant. But the idea of MPC started since 1960's (Garcia et al.1989) by: Propoi (1963) who gave the root idea of MPC algorithm while Lee and Markus (1967) anticipated MPC algorithm in their optimal control text book.

The application of MPC is not only confined to petrochemical and refining fields but also in automotive, food processing, pulp and paper, metallurgy: aerospaceand defence industries (Qin and Bagwell, 1997). Yamamoto and Hashimoto (1991) made a survey in 139 Japanese industries and found that 25.4% were making use of MPC while 21.1 % were looking out for the possibility. Similarly, Ohshima et al. in 1995 found that during 1991 to 1995 use of MPC made a steady move in Japanese industries. The advantage of MPC lies in handling multivariable control operations where care is taken over interaction between manipulated variables (MVs) and control variables (CVS) while in conventional methods pairing is possible between MVs and CVs only if there is a strong interaction between the two and is done with the help of a decoupler.

In addition, MPC has a very good constraints handling capability which is ignored in conventional methods. However, there are some disadvantages associated with MPC like complexity involved in deriving the control law. translucent analysis of stability and robustness, and most importantly lack of appropriate tuning procedure since effect of parameters variation on the closed loop behaviour solely depends upon the accuracy of the model (Camacho and Bordon,1998). Lundstormet al. (1998) introduced some limitations of Dynamix matrix control (DMC) including poor performance in handling multivariable plants.

Until recently, the other name for MPC is Linear model predictive control (LMPC). The DMC' from Aspen and HIECONTN from Adersa (Qin and Bagwell. 1997) are the software which make use of linear models. Most of the processes are nonlinear in nature but there are some reasons which pose problems in using nonlinear models like difficulties in generating nonlinear model through empirical data, complexity in computations which takes longer time and sometimes may become non-convex. On the other hand, the LMPC can be solved analytically through simple least square method (LS) where the problem is quadratic programming (QP) in nature. Many LP and QP problems can be found in Fletcher (1987). In industries LMPC is acceptable since various processes works with single set point and the work of controller is confined to rejection of disturbance only (Qin and Bagwell, 2000).

The closed loop stability of MPC is not easy to prove since MPC is a feedback mechanism resulting from RHC behaviour. Rawling and Muske (1993) gave the first generous results about MPC stability using infinite horizon problem and had successfully proved asymptotically stability in the presence of constraints using infinite prediction horizon (N). But since it is difficult to handle constraints another method, where a terminal constraint is imposed which brings states to the desired value at the end of horizon, is used. Terminal constraint can be equalityconstraint (Meadow al el., 1995) or it can inequality constraint (Polak and Yang, 1993). Robustness analysis of unconstrained MPC is given by Garcia and Morari, (1989) which later known as Internal Model Control (IMC). Zafiriou (1998) had used contraction properties of MPC to give robustness and stability conditions for input and output constraints. Polak and Yang (1993) also contributed for robustness properties by putting contraction constraint on the state. The conventional way to deal with plant model uncertainty is to detune the parameters. However, in the recent past the subject of research is to explicitly incorporate the plant model uncertainty in the MPC problem formulation.

DAY-TO-DAY APPLICATION EXAMPLE OF PREDICTIVE CONTROL

Suppose a work is assigned to a group of people. The objective should be to complete the work as early as possible and in the known best way. So, completing the task will be a function of various factors like

1. How much effort to put in,

2. How well the group work as a team.

These are the manipulated variables in the planning problem.

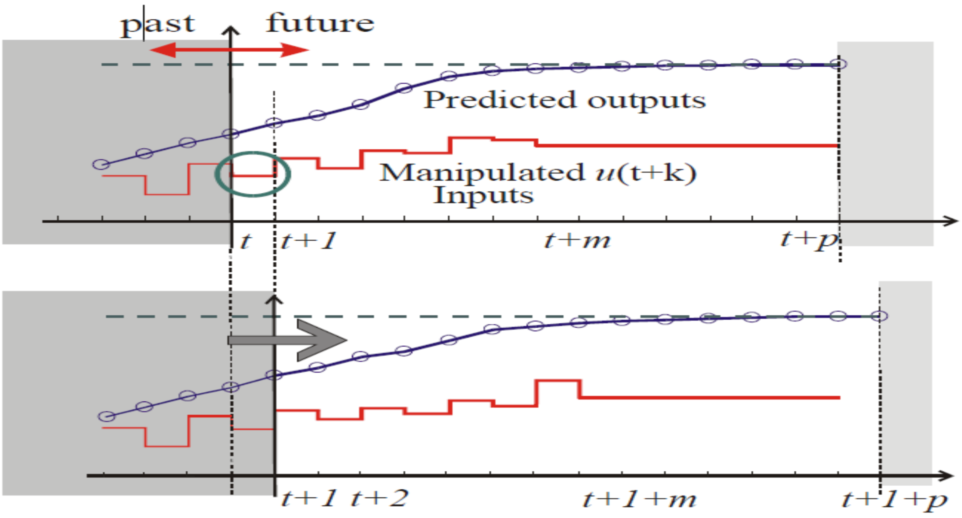


Fig. 1. Algorithm for MPC

Also, there mayhave limitations(constraints), like

1. Ability to understand the objective of the task, and
2. Whether people have good skills of computer hardware and softwareengineering.

These are the hard and soft constraints in the planning.

Now how they proceed. Firstly, backgroundinformation is acquired for this planning work.After everything is considered, there would be a plan for the next8 hours as functions of the manipulated variables. Then calculate hour-byhourwhat is needed to do in order to complete the tasks. In this calculation,based on the background information, limitations are considered, and find the best way to achieve the goal. Theend result of this planning gives the projected activities from 9 o'clock to5 o'clock. Then start working by implementing the activities for the firsthour of our plan.

At 10 o'clock, they check how much they have actually done for the firsthour. This information is used for the planning of next phase of activities.Maybethey have done less than they planned because one of the key members went for an emergencymeeting. Nevertheless, at 10 o'clock, they make an assessment on what they haveachieved, and use this updated information for planning our activities for thenext 8 hours. The objective may remain the same or may change. The lengthof time for the planning remains the same (8 hours).Theplanning and implementation process is repeated every hour until the originalobjective is achieved.

There are three key elements required in the planning. The key issuesin the planning exercise are:
1. The time window for the planning is fixed at 8 hours;
2. Need to know the current status before the planning;
3. Take the best approach for the 8 hours work by taking the constraintsinto consideration, and the optimization is performed in real-time with amoving horizon time window and with the latest information available.

The planning activity described here involves the principle of MPC. They are introduced as below.
1. Moving horizon window: the time-dependent window from an arbitrarytime $t_i$to $t_i+ T_p$. The length of the window $T_p$remains constant. In thisexample, the planning activity is performed within an 8-hour window,thus T_p= 8, with the measurement taken every hour. However, *ti*, whichdefines the beginning of the optimization window, increases on an hourlybasis, starting with t_i= 9.
2. Prediction horizon: dictates how ‘far’ we wish the future to be predictedfor. This parameter equals the length of the moving horizon window, T_p.
3. Receding horizon control: although the optimal trajectory of future controlsignal is completely described within the moving horizon window, theactual control input to the plant only takes the first sample of the controlsignal, while neglecting the rest of the trajectory.
4. In the planning process, we need the information at time *ti*in order topredict the future. This information is denoted as $x(t_i)$ which is a vectorcontaining many relevant factors, and is either directly measured orestimated.
5. A given model that will describe the dynamics of the system is paramountin predictive control. A good dynamic model will give a consistent andaccurate prediction of the future.
6. In order to make the best decision, a criterion is needed to reflect the objective. The objective is related to an error function based on the differencebetween the desired and the actual responses. This objective function isoften called the cost function *J*, and the optimal control action is foundby minimizing this cost function within the optimization window.

3. PROBLEM STATEMENT

Model predictive control systems are designed based on a mathematical modelof the plant. The model to be used in the control system design is taken to bea state-space model. By using a state-space model, the current informationrequired for predicting ahead is represented by the state variable at the currenttime.
For simplicity, we begin our study by assuming that the underlying plant is a LTISISO system described as
$x_m(k+1) = A_m x_m(k)+B_m u(k)$ (1)
$y(k)=C_m x_m(k)$ (2)
where *u* is the manipulated variable or input variable; *y* is the process output;and $x_m$is the state variable vector.To incorporate the effect of rate of control input we use augmented model.Taking a difference operation on both sides of (1.1), we obtain that

$$x_m(k+1)-x_m(k)=A_m(x_m(k)-x_m(k-1))+B_m(u(k)-u(k-1)).$$

The difference of the state variable can be given as

$$\Delta x_m(k+1) = x_m(k+1) - x_m(k);\ \Delta x_m(k) = x_m(k) - x_m(k-1),$$

and the difference of the control variable by

$$\Delta u(k) = u(k) - u(k-1).$$

These are the increments of the variables $x_m(k)$ andu(k). With this transformation,the difference of the state-space equation is:

$$\Delta x_m(k+1) = A_m\Delta x_m(k)+B_m\Delta u(k) \quad (3)$$

Note that the input to the state-space model is Δu(k). The next step is toconnect $\Delta x_m(k)$ to the output y(k). To do so, a new state variable vector ischosen to be

$$x(k) = [\Delta x_m(k)T\ y(k)]^T$$

where superscript T indicates matrix transpose. Note that
$y(k+1)-y(k)=C_m(x_m(k+1)- x_m(k))=C_m\Delta x_m(k+1)= C_m A_m\Delta x_m(k) + C_m B_m\Delta u(k)$ (4)
Putting together (1.3) with (1.4) leads to the following state-space model:

$$\overbrace{\begin{bmatrix}\Delta x_m(k+1)\\ y(k+1)\end{bmatrix}}^{x(k+1)} = \overbrace{\begin{bmatrix}A_m & o_m^T\\ C_m A_m & 1\end{bmatrix}}^{A}\overbrace{\begin{bmatrix}\Delta x_m(k)\\ y(k)\end{bmatrix}}^{x(k)} + \overbrace{\begin{bmatrix}B_m\\ C_m B_m\end{bmatrix}}^{B}\Delta u(k)$$

$$y(k) = \overbrace{\begin{bmatrix}o_m & 1\end{bmatrix}}^{C}\begin{bmatrix}\Delta x_m(k)\\ y(k)\end{bmatrix},$$

The triplet (*A*,*B*,*C*) is called the augmented model, which will be used in the design of predictive control.

4. PREDICTIVE CONTROL WITHIN ONE OPTIMIZATION WINDOW

Upon formulation of the mathematical model, the next step in the design of apredictive control system is to calculate the predicted plant output with thefuture control signal as the adjustable variables we assume that the currenttime is *ki* and the length of the optimization window is $N_p$as the number ofsamples.

Prediction of State and Output Variables

The future control trajectory is denoted by$\Delta u(k_i)$,$\Delta u(k_i+1)$, . . .,$\Delta u(k_i+N_c-1)$, where $N_c$is called the control horizon dictating the number of parametersused to capture the future control trajectory. With given information $x(k_i)$,the future state variables are predicted for $N_p$number of samples, where $N_p$is called the prediction horizon. $N_p$is also the length of the optimizationwindow. We denote the future state variables as

$$x(k_i+1 \mid k_i),\ x(k_i+2 \mid k_i),\ \ldots,\ x(k_i+m \mid k_i),\ \ldots,\ x(k_i+N_p \mid k_i),$$

where $x(k_{i+m}| k_i)$ is the predicted state variable at $k_{i+m}$with given currentplant information $x(k_i)$. The control horizon $N_c$is chosen to be less than (orequal to) the prediction horizon N_p. Based on the state-space model (A,B,C), the future state variables arecalculated sequentially using the set of future control parameters.

$x(k_{i+1}| k_i) = Ax(k_i) + B\Delta u(k_i)$

$x(k_{i+2}| k_i) = Ax(k_{i+1} | k_i) + B\Delta u(k_{i+1})$

$= A^2x(k_i) + AB\Delta u(k_i) + B\Delta u(k_{i+1})$

.

$$x(k_{i+Np}| k_i) = A^{Np}x(k_i) + A^{Np-1}B\Delta u(k_i) + A^{Np-2}B\Delta u(k_i+1)+ \ldots + A^{Np-Nc}B\Delta u(k_i+N_{c-1}).$$

From the predicted state variables, the predicted output variables are, bysubstitution

$y(k_{i+1}|k_i)=CAx(k_i)+CB\Delta u(k_i)$ (5)

$y(k_{i+2}|k_i)=CA^2x(k_i)+ CAB\Delta u(k_i) + CB\Delta u(k_{i+1})$

.

$$y(k_i+N_p|k_i)=CA^{Np}x(k_i)+CA^{Np}-1B\Delta u(k_i)+CA^{Np-2}B\Delta u(k_{i+1})+...+CA^{Np-Nc}B\Delta u(k_i+N_c-1) \quad (6)$$

Note that all predicted variables are formulated in terms of current statevariable information $x(k_i)$ and the future control movement $\Delta u(k_{i+j})$, wherej = 0, 1, . . .N_{c-1}.

Define vectors

$$Y = [y(k_{i+1}| k_i)\ y(k_{i+2}| k_i)\ y(k_{i+3}| k_i) \ldots y(k_{i+Np|ki})]$$
$$\Delta U=[\Delta u(k_i)\ \Delta u(k_{i+1})\ \Delta u(k_{i+2}) \ldots \Delta u(k_{i+Nc-1})\ ]$$

where in the single-input and single-output case, the dimension of Y is $N_p$and the dimension of ΔU is N_c. We collect (1.5) and (1.6) together in acompact matrix form as

$$Y=Fx(k_i)+\Phi\Delta U \quad (7)$$

where

$$F = \begin{bmatrix} CA \\ CA^2 \\ CA^3 \\ \vdots \\ CA^{N_p} \end{bmatrix} ; \Phi = \begin{bmatrix} CB & 0 & 0 & \ldots & 0 \\ CAB & CB & 0 & \ldots & 0 \\ CA^2B & CAB & CB & \ldots & 0 \\ \vdots & & & & \\ CA^{N_p-1}B & CA^{N_p-2}B & CA^{N_p-3}B & \ldots & CA^{N_p-N_c}B \end{bmatrix}$$

5. OPTIMIZATION

For a given reference signal r(ki) at sample time ki, within a prediction horizon the objective of the predictive control system is to bring the predicted output as close as possible to the reference signal, where we assume it remains constant in the optimization window. This objective is then translated into a design to find the 'best' control parameter vector ΔU such that an error function between the reference and the predicted output is minimized.

Assuming that the data vector that contains the reference, information is

$$R_s^T = \overbrace{[1\ 1\ \ldots\ 1]}^{N_p} r(k_i),$$

we define the cost function J that reflects the control objective as

$$J = (Rs - Y)^T(Rs - Y) + \Delta U^T R \Delta U \quad (8)$$

where the first term is linked to the objective of minimizing the errors between the predicted output and the reference while the second term reflects the consideration given to the size of ΔU when the objective function J is made to be as small as possible. R is a diagonal matrix called control weighted matrix in the form that

$$R= r_w I_{Nc\times Nc}\ (r_w \geq 0)$$

where r_w is used as a tuning parameter forthedesired closed-loop performance.For the case that $r_w = 0$, the cost functionis interpreted as the situation where we would not want to pay anyattention to how large the ΔU might be and our goal would be solely tomake the error $(R_s - Y)^T(R_s - Y)$ as small as possible. For the case of larger r_w,the cost function is interpreted as the situation where we wouldcarefully consider how large the ΔU might be and cautiously reduce the error$(R_s - Y)^T(R_s - Y)$.

To find the optimal ΔU that will minimize J, by using (1.7), J is expressedas

$$J = (R_s - Fx(k_i))^T(R_s - Fx(k_i)) - 2\Delta U^T\Phi^T(R_s - Fx(ki)) + \Delta U^T(\Phi^T\Phi + R)\Delta U \quad (9)$$

From the first derivative of the cost function J:

$$\partial J/\partial \Delta U = -2\Phi^T(R_s - Fx(ki)) + 2(\Phi^T\Phi + R)\Delta U \quad (10)$$

the necessary condition of the minimum J is obtained as

$$\partial J/\partial \Delta U = 0,$$

from which we find the optimal solution for the control signal as

$$\Delta U = (\Phi^T\Phi + R)^{-1}\Phi^T(R_s - Fx(ki)) \quad (10)$$

6. RECEDING HORIZON CONTROL

Although the optimal parameter vector ΔU contains the controls $\Delta u(k_i)$,$\Delta u(k_i+1)$, $\Delta u(k_i+2)$, . . ., $\Delta u(k_i+N_{c-1})$, with the receding horizon control principle, we only implement the first sample of this sequence, i.e., $\Delta u(k_i)$, while ignoring the rest of the sequence. When the next sample period arrives, the more recent measurement is taken to form the state vector $x(k_i+1)$ for calculation of the new sequence of control signal. This procedure is repeated in real time to give the receding horizon control law.

SIMULATION RESULTS

The plant state-space model is given by:

$$x_m(k+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_m(k) + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u(k) \quad (11)$$

$$y(k) = [1 \quad 0]x_m(k)$$

and R=10, Np=10, Nc=4 & Rs=1

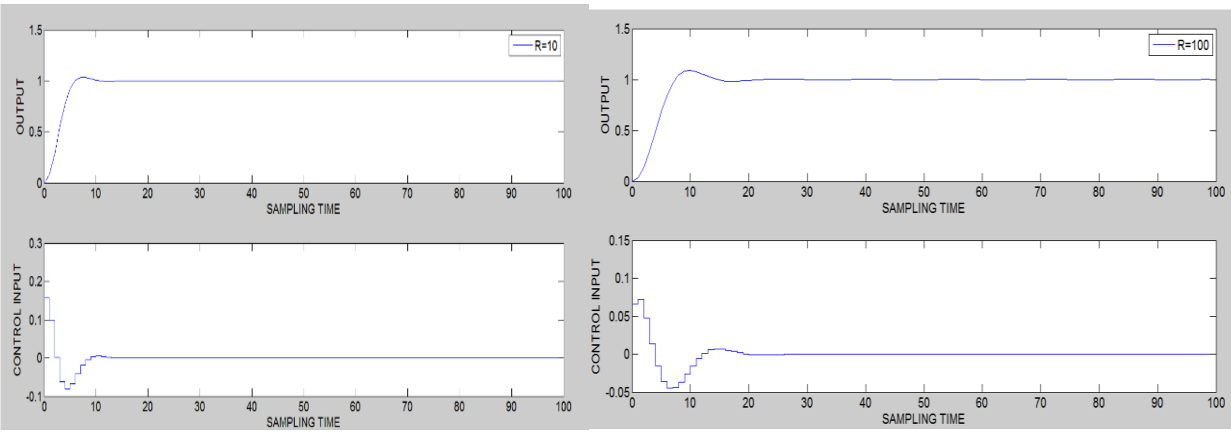


FIGURE.2(a,b)

INFERENCE

1. In Fig.2(a) when R=10 output tracks the reference at 10 sampling time but value of control input is ranging between 0.15 to -0.1
2. In Fig.2(a) when R=100 output tracks the reference at 15 sampling time but value of control input is ranging between 0.07 to -0.05 (smaller than when R=10)

CONSTRAINED MODEL PREDICTIVE CONTROL

A motivational example to illustrate how the performance of a control system can deteriorate significantly when the control signals from the original design meet with operational constraints.

FREQUENTLY USED OPERATIONAL CONSTRAINTS

There are **three major types of constraints** frequently encountered in applications. The first two types deal with constraints imposed on the control variables u(k), and the third type of constraint deals with output y(k) or state variable x(k) constraints.

(a) Constraints on the Control Variable Incremental Variation

These are hard constraints on the size of the control signal movements, i.e.,on the rate of change of the control variables (Δu(k)). The constraints are specified in the form

$$\Delta u^{min} \le \Delta u(k) \le \Delta u^{max}$$

The rate of change constraints can be used to impose directional movement constraints on the control variables; for instance, if u(k) can only increase, not decrease, then we select $0 \le \Delta u(k) \le \Delta u^{max}$. The constraint on Δu(k) can be used to cope with the cases where the rate of change of the control amplitude is restricted or limited in value. For example, in a control system implementation, assuming that the control variable u(k) is only permitted to increase or decrease in a magnitude less 0.1 unit, then the operational constraint is. These are not the upper or lower limits of the control input.

$$-0.1 \le \Delta u(k) \le 0.1.$$

(b). Constraints on the Amplitude of the Control Variable

These are the most commonly encountered constraints among all constrainttypes. For instance, we cannot expect a valve to open more than 100 percentnor a voltage to go beyond a given range. These are the physical hardconstraints on the system. Simply, we demand that

$$u^{min} \le u(k) \le u^{max}.$$

Here, we need to pay particular attention to the fact that u(k) is an incrementalvariable, not the actual physical variable. The actual physical controlvariable equals the incremental variable u plus its steady-state value u^{ss}. For instance, if a valve is allowed to openin the range between 15% and 80% and the valve's normal operating value isat 30%, then u^{min}= 15% − 30% = −15% and u^{max}= 80% − 30% = 50%. These are the upper or lower limits of the control input.

(c). Output Constraints

We can also specify the operating range for the plant output. For instance, supposing that the output y(k) has an upper limit $y^{max}$and a lower limit y^{min}, then the output constraints are specified as

$$y^{min} \le y(k) \le y^{max}.$$

Output constraints are often implemented as 'soft' constraints in the way that a slack variable s_v>0 is added to the constraints, forming

$$y^{min} - s_v \le y(k) \le y^{max} + s_v$$

There is a primary reason why we use a slack variable to form 'soft' constraints for output. Output constraints often cause large changes in both the control and incremental control variables when they are enforced (i.e when they become active). When that happens, the control orincremental control variables can violate their own constraints and the problemof constraint conflict occurs. In the situations where the constraints onthe control variables are more essential to plant operation, the output constraintsare often relaxed by selecting a larger slack variable $s_v$to resolve theconflict problem.Similarly, we can impose constraints on the state variables if they aremeasurable or impose the constraints on observer state variables. They alsoneed to be in the form of 'soft' constraints for the same reasons as the output.

Constraints as Part of the Optimal Solution

Having formulated the constraints as part of the design requirements, the next step is to translate them into linear inequalities, and relate them to the original model predictive control problem. The key here is to parameterize the constrained variables using the same parameter vector ΔUas the ones used in the design of predictive control. Therefore, the constraints are expressed in a set of linear equations based on the parameter vector ΔU. Since the predictive control problem is formulated and solved in the framework of receding horizon control, the constraints are taken into consideration for each moving horizon window. This allows us to vary the constraints at the beginningof each optimization window. Based on this idea, if we want to impose the constraints on the rate of change of the control signal Δu(k) at time ki, the constraints at sample time ki are expressed as

$$\Delta u^{min} \le \Delta u(k_i) \le \Delta u^{max}.$$

From the time instance ki, the predictive control scheme looks into the future. The constraints at future samples, for example on the first three samples,$\Delta u(k_i)$,$\Delta u(k_{i+1})$, $\Delta u(k_{i+2})$ are imposed as

$$\Delta u^{min} \le \Delta u(k_i) \le \Delta u^{max}$$
$$\Delta u^{min} \le \Delta u(k_{i+1}) \le \Delta u^{max}$$
$$\Delta u^{min} \le \Delta u(k_{i+2}) \le \Delta u^{max}.$$

In principle, all the constraints are defined within the prediction horizon.However, in order to reduce the computational load, we sometimes choose asmaller set of sampling instants at which to impose the constraints, insteadof all the future samples. The following example shows how to express theconstraints from the design specification in terms of a function of ΔU.

Example . In the motor control system, suppose that the input voltagevariation is limited to 2 V and 6 V. The steady state of the control signal is at4 V. Assuming that the control horizon is selected to be Nc = 4, express theconstraint on $\Delta u(k_i)$ and $\Delta u(k_{i+1})$ in terms of ΔU for the first two sampletimes.

Solution:

The parameter vector to be optimized in the predictive control system at time ki is

$$\Delta U = [\Delta u(k_i)\ \Delta u(k_{i+1})\ \Delta u(k_{i+2})\ \Delta u(k_{i+3})]$$

$$u(k_i) = u(k_{i-1}) + \Delta u(k_i) = u(k_{i-1}) + [1\ 0\ 0\ 0]\Delta U$$

$$u(k_{i+1}) = u(k_i) + \Delta u(k_{i+1}) = u(k_{i-1}) + \Delta u(k_i) + \Delta u(k_{i+1}) = u(k_{i-1}) + [1\ 1\ 0\ 0]\Delta U \qquad (12)$$

With the limits on the control variables, by subtracting the steady-state value of the control, as $u^{min} = 2 - 4 = -2$ and $u^{max} = 6 - 4 = 2$, the constraints are expressed as

$$\begin{bmatrix} -2 \\ -2 \end{bmatrix} \le \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k_i - 1) + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \Delta U \le \begin{bmatrix} 2 \\ 2 \end{bmatrix} \qquad (13)$$

the constraints needto be decomposed into two parts to reflect the lower limit, and the upper limitwith opposite sign. Namely, for instance, the constraints

$$\Delta U^{min} \le \Delta U \le \Delta U^{max}$$

will be expressed by two inequalities:

$$-\Delta U \le -\Delta U^{min}$$

$$\Delta U \le \Delta U^{max}$$

In the matrix form

$$\begin{bmatrix} -I \\ I \end{bmatrix} \Delta U \le \begin{bmatrix} -\Delta U^{min} \\ \Delta U^{max} \end{bmatrix}_x$$

In the case of a manipulated variable constraint, we write:

$$\begin{bmatrix} u(k_i) \\ u(k_i+1) \\ u(k_i+2) \\ \vdots \\ u(k_i+N_c-1) \end{bmatrix} = \begin{bmatrix} I \\ I \\ I \\ \vdots \\ I \end{bmatrix} u(k_i-1) + \begin{bmatrix} I & 0 & 0 & \dots & 0 \\ I & I & 0 & \dots & 0 \\ I & I & I & \dots & 0 \\ \vdots \\ I & I & \dots & I & I \end{bmatrix} \begin{bmatrix} \Delta u(k_i) \\ \Delta u(k_i+1) \\ \Delta u(k_i+2) \\ \vdots \\ \Delta u(k_i+N_c-1) \end{bmatrix} \qquad (14)$$

Decomposed form

$$-(C_1 u(k_{i-1}) + C_2 \Delta U) \le -U^{min}$$

$$(C_1 u(k_{i-1}) + C_2 \Delta U) \le U^{max},$$

where U^{min} and U^{max} are column vectors with N_c elements of u^{min} and u^{max}, respectively. Similarly, for the increment of the control signal, we have the constraints:

$$-\Delta U \le -\Delta U^{min}$$

$$\Delta U \le \Delta U^{max},$$

where ΔU^{min} and ΔU^{max} are column vectors with Nc elements of Δu^{min} and Δu^{max}, respectively. The output constraints are expressed in terms of ΔU:

$$Y^{min} \le Fx(k_i) + \Phi \Delta U \le Y^{max} \qquad (15)$$

Finally, the model predictive control in the presence of hard constraints is proposed as finding the parameter vector ΔU that minimizes

$$J=(Rs-Fx(ki))^T(Rs-Fx(ki))-2\Delta U^T\Phi^T(Rs-Fx(ki))+\Delta U^T(\Phi^T\Phi+R)\Delta U,$$

subject to the inequality constraints

$$\begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix} \Delta U \le \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix},$$

Where the data matrices are

$$M_1 = \begin{bmatrix} -C_2 \\ C_2 \end{bmatrix};\ N_1 = \begin{bmatrix} -U^{min} + C_1 u(k_i - 1) \\ U^{max} - C_1 u(k_i - 1) \end{bmatrix};\ M_2 = \begin{bmatrix} -I \\ I \end{bmatrix};$$

$$N_2 = \begin{bmatrix} -\Delta U^{min} \\ \Delta U^{max} \end{bmatrix};\ M_3 = \begin{bmatrix} -\Phi \\ \Phi \end{bmatrix}; N_3 = \begin{bmatrix} -Y^{min} + Fx(k_i) \\ Y^{max} - Fx(k_i) \end{bmatrix}.$$

The matrix $\Phi^T\Phi + R$ is the Hessian matrix and is assumed to be positive definite. For compactness of expression, we denote the data matrices by

$$M\Delta U \leq \gamma$$

where M is a matrix representing the constraints.

7. NUMERICAL SOLUTIONS USING QUADRATIC PROGRAMMING

The standard quadratic programming problem has been extensively studied in the literature. The required numerical solution for MPC is often viewed as an obstacle in the application of MPC. So to understand the essence of quadratic programming is important to produce the essential computational programs required. To be consistent with the literatures of quadratic programming, the decision variable is denoted by x. The objective function J and the constraints are expressed as

$$J = 0.5x^TEx + x^TF$$
$$Mx \leq \gamma,$$

here E, F, M and γ are compatible matrices and vectors in the quadratic programming problem. Without loss of generality, E is assumed to be symmetric and positive definite.

QUADRATIC PROGRAMMING FOR EQUALITY CONSTRAINTS

The simplest problem of quadratic programming is to find the constrained minimum of a positive definite quadratic function with linear equality constraints. Each linear equality constraint defines a hyperplane. Positive definite quadratic functions have their level surfaces as hyperellipsoids. Intuitively**,** the constrained minimum is located at the point of tangency between the boundaryof the feasible set and the minimizing hyperellipsoid.

LAGRANGE MULTIPLIERS

ENABLES TO INCORPORATE EQUALITY CONSTRAINTS DIRECTLY,

Lagrange expression

$$L = J + \lambda^T (Mx - \gamma)$$

where J is the objective function

$$L = \frac{1}{2}x^TEx + x^T F + \lambda^T (Mx - \gamma)$$
$$\text{s.t } Mx = \gamma$$

Minimization gives :

$$\frac{\partial L}{\partial x} = Ex + F + M^T\lambda = 0$$

$$\frac{\partial L}{\partial x} = Mx - \gamma$$

The optimal x &λ are obtained as

$$\lambda = -(ME^{-1}M^T)^{-1}\gamma + ME^{-1}F$$
$$x = -E^{-1}(M^T\lambda + F)$$

Or

$$x = -E^{-1}F - E^{-1}M^T\lambda = x^0 - E^{-1}M^T\lambda$$

where $x^0 = -E^{-1}F$ is the global optimal (in absence of constraint), and the second term is a correction term due to the equality constraints.

8. MINIMIZATION WITH INEQUALITY CONSTRAINTS

In the minimization with inequality constraints, the number of constraints could be larger than the number of decision variables. The inequality constraints $Mx \leq \gamma$ as in may comprise active constraints and inactive constraints. An inequality $Mix \leq \gamma i$is said to be active if $M_ix = \gamma_i$and inactive if $M_ix < \gamma_i$, where $M_i$together with $\gamma_i$form the ithinequality constraint and are the ithrow of M matrix and the ithelement of γ vector, respectively. We introduce the Kuhn-Tucker conditions, which define the active and inactive constraints in terms of the Lagrange multipliers.

KUHN-TUCKER CONDITIONS(TO FIND ACTIVE CONSTRAINTS)

1. $\nabla J - \sum_{i=1}^{m} \lambda i \nabla \Phi_i - \sum_{r=1}^{k} u_r \nabla \varphi_r$

2. $\Phi_i = 0$ for all i

3. $\varphi_r = 0$ for all r

4. $\varphi_r u_r = 0$ for all r(complementary slackness)

5. $u_r \geq 0$ for all r

Where J is the objective function,

$\Phi_1(x_1,x_2 \ldots\ldots\ldots x_n) = 0$
.
$\Phi_m(x_1,x_2 \ldots\ldots\ldots x_n) = 0$

Equality constraints &
$\varphi_1(x_1,x_2 \ldots\ldots\ldots x_n) \geq 0$
.
$\varphi_m(x_1,x_2 \ldots\ldots\ldots x_n) \geq 0$

Inequality constraints
Treat inequality constraint as equality constraint and if u (sensitivity coefficient)comes out to be positive means assumption is correct else omit the inequality constraint since it is inactive.

Let us consider an example of an undamped oscillator is given by mathematical model as:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

Suppose that the design objective is to design the predictive control system such that the output of the plant has to track a unit step reference signal as fast as possible. To this end, we select the prediction horizon $N_p= 10$ and the control horizon $N_c= 3$. There is no weight on the control signal, i.e., R = 0. Examine what happens if the control amplitude is limited between -5 and 5 by saturation.

Case A. Without control saturation

The closed-loop response is illustrated in Figure 2(a). It is seen that the output converges to the reference signal after 25 samples. Indeed, the design objective has been achieved. If the control amplitude is of concern, then we note that this optimal control has a large amplitude that is close to 10 at its maximum.

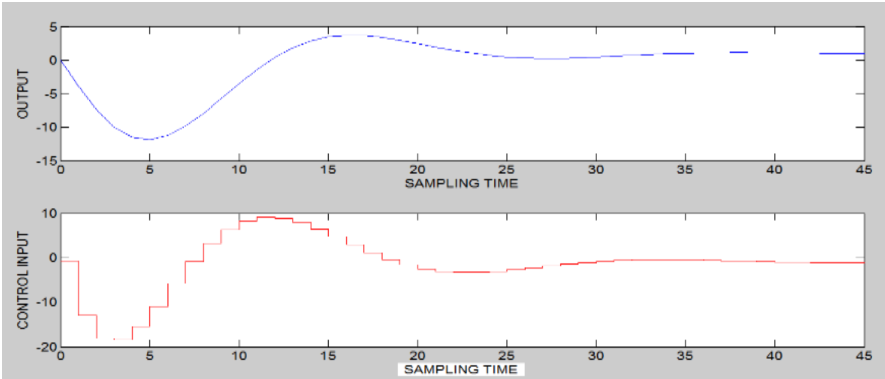
Case B. With control saturation

Assume that the control amplitude has limits at ±5 due to operational constraint,

$$-5 \leq u(k) \leq 5$$

Then, this limit prevents the control signal from being implemented to the plant when its amplitude exceeds this limit.

Thus, u(k) = 5, if u(k) >5; and u(k) = -5 if u(k) <−5. When this is done, the closed-loop performance significantly deteriorates, as shown in Figure 2(b) system becomes oscillatory.



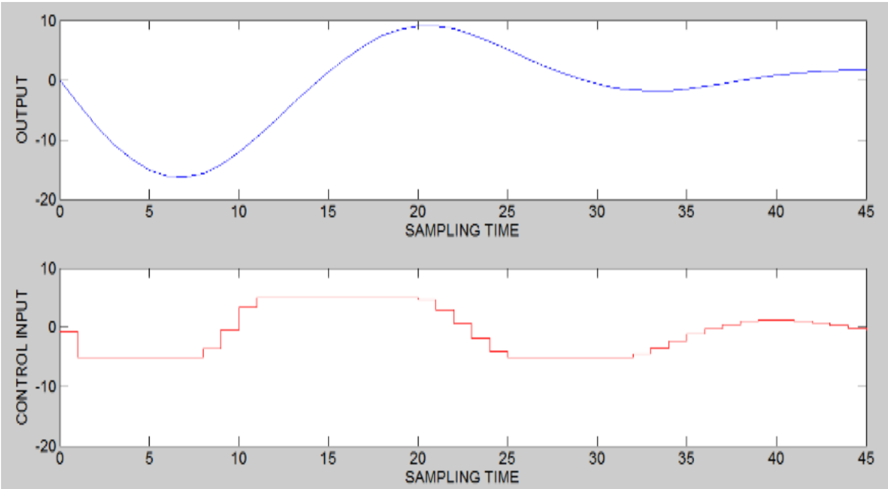


Fig.3(a)Without saturation
Fig.3(b)With saturation

9. CONCLUSION

It is clear that an iterative procedure is required to solve the optimization problem with inequality constraints, because we did not know which constraints would become active constraints. If the active set could be identified in advance, then the iterative procedure would be shortened.

The conditions for the inequality constraints are more relaxed than the case of imposing equality constraints. For instance, the number of constraints is permitted to be greater than the number of decision variables, and the set of inequality constraints is permitted to be linearly dependent. However, these relaxations are only permitted to the point that the active constraints need to be linearly independent and the number of active constraints needs to be less than or equal to the number of decision variables.

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