On Analytical Solutions of Some Types of Ordinary Differential Difference Equations

Hawazen Issa Razouki[a], Muayyad Mahmood Khalil[b], Ghassan Ezaldeen[c]

[a] College of Education for Pure Sciences, Tikrit University, wwhhh05@st.tu.edu.iq.
[b] Assistant Professor in College of Education for Pure Sciences, Tikrit University, medomath80@gmail.com.
[c] Assistant Professor in College of Education for Pure Sciences, Tikrit University, ghasanarif@tu.edu.iq.

Abstract: We studied in this work the applicability of Laplace decomposition method (LDM) and modified Laplace decomposition method (MLDM) of differential-difference equations of order (2,1), which means the differential equation is of order 2 and the difference equation is of order 1. Finally, we give two illustrative examples to show the applicability and the effectiveness of the methods above.

Keywords: Laplace, decomposition, modified, differential, difference, applicability.

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1. Introduction

The problems of differential difference equations of order (2,1) have relation with the problem of solving a singularly perturbed second order differential - difference equation where singular perturbation parameter and the delay parameter are selected as small as possible. These problems play an important role in variety of physical problems [1] such as microscale heat transfer, diffusion in polymers, control of chaotic systems and so on. In the present paper, we study the following differential - difference equation of order (2,1):

$$\boldsymbol{\epsilon y''(t) = y'(t) - \big[f(t) + F\big(y(t-\omega)\big)\big]\,, \qquad t > \omega}$$

So, we set $\boldsymbol{\epsilon = 1}$ and put an initial condition:

$$\boldsymbol{y(t) = a + bt \quad , \qquad t \in [0, \omega]}$$

remembering these facts, the current research looks into two analytical methods: the LDM and the MLDM. [2-14]. So, both approaches can solve any linear or nonlinear second order differential equations, with the fundamental notion being that they can solve linear or nonlinear differential - difference equations of order (2,1) with the right initial interval condition. More details can be found in [16-23]. The following simple types of differential difference equations are considered for the study:

1) Linear differential - difference equation of order (2,1)

$$\boldsymbol{y''(t) + c_1 y'(t) + c_2 y(t-\omega) = f(t) \quad , \quad t > \omega \quad , \qquad (1.1)}$$
$$\boldsymbol{y(t) = a + bt \quad , \qquad 0 \le t \le \omega \quad ,}$$

where $\boldsymbol{c_2 \neq 0}$, $\boldsymbol{c_1}$, $\boldsymbol{a}$ and $\boldsymbol{b}$ are real constants. $\boldsymbol{f(t)}$ is a given function of exponential order and $\boldsymbol{\omega}$ is appositive difference parameter.

2) Nonlinear differential difference equation of order (2.1):

$$\boldsymbol{y''(t) = f\big(y(t-\omega)\big) \quad , \quad t > \omega \quad , \qquad (1.2)}$$
$$\boldsymbol{y(t) = a + bt \quad , \qquad 0 \le t \le \omega}$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are real constants, $\boldsymbol{f(y)}$ is a given nonlinear function of the exponential order and $\boldsymbol{\omega}$ is a positive difference parameter.

2.Reviewing the method of solution

Assuming that we have the following kind of the linear or nonlinear differential - difference equation with 2[nd] order differential and 1[st] order difference:

$$y''(t) = cy'(t) + f\big(y(t-\omega)\big) \quad , \quad t > \omega \quad , \qquad (2.1)$$
$$y(t) = a + bt \quad , \qquad 0 \le t \le \omega$$

Where a, b, c are real constants, $f(t)$ is a given linear or nonlinear function of exponential order and ω in a positive difference parameter First we note that,

$$\int_{\omega}^{\infty} y'(t)e^{-pt}dt = L\{y'(t)\} - \int_{0}^{\omega} be^{-pt}dt = L\{y'(t)\} + \frac{b}{p}e^{-\omega p} - 1$$

and

$$\int_{0}^{\omega} y''(t)e^{-pt}dt = 0 \Rightarrow \int_{\omega}^{\infty} y''(t)e^{-pt}\,dt = L\{y''(t)\}$$

Hence, For the nonlinear differential - difference equation, we will discuss the Laplace decomposition method and the modified Laplace decomposition method (2.1).

2.1 Laplace Decomposition Method (LDM)

Now let us multiply both sides of (2.1) by e^{-pt} , $s > 1$ and integrate between ωand ∞,to get

$$\int_{\omega}^{\infty} y''(t)e^{-pt}dt = c\int_{\omega}^{\infty} y'(t)e^{-pt}dt + \int_{\omega}^{\infty} f(y(t-\omega))e^{-pt}dt$$

By applying suitable shifting of variables and use initial interval condition to obtain.

$$L\{y''(t)\} = cL\{y'(t)\} + \frac{bc}{p}(e^{-\omega p} - 1) + e^{-\omega p}L\{f(y(t))\}$$

After applying the formula of Laplace transform for second order derivative, finally, we have

$$L\{y(t)\} = \frac{a}{p} + \frac{b}{p^2} - \frac{ca}{p^2} - \frac{bc}{p^3} + \frac{bc}{p^3}e^{-\omega p} + \frac{c}{p}L\{y(t)\} + \frac{e^{-\omega p}}{p^2}L\{f(y(t))\} \quad , \qquad (2.2)$$

Now we seek the subsequent sort of decomposition for$L\{y(t)\}$

$$L\{y(t)\} = \sum_{n=0}^{\infty} e^{-n\omega p}L\{y_n(t)\} \quad , \qquad (2.3)$$

which may be considered as Laplace decomposition. Then $L\{f(y(t))\}$ is given by

$$\mathrm{L}\{f(y(t))\} = L\{f(y_0(t))\} + e^{-\omega p}L\left\{\left[\frac{\frac{d}{dy}f(y(t))|y=y_0}{1!}\right]y_1(t)\right\}$$

$$+e^{-2\omega p}L\{\left[\frac{\frac{d}{dy}f(y(t))|y=y_0}{1!}\right]y_2(t) + \left[\frac{\frac{d^2}{dy^2}f(y(t))|y=y_0}{2!}\right]y_1^2(t)\}$$

$$+e^{-3\omega p}L\{\left[\frac{\frac{d}{dy}f(y(t))|y=y_0}{1!}\right]y_3(t) + \left[\frac{\frac{d^2}{dy^2}f(y(t))|y=y_0}{2!}\right]2y_1(t)y_2(t) + \left[\frac{\frac{d^3}{dy^3}f(y(t))|y=y_0}{3!}\right]y_1^3(t)\} + \cdots$$

$$= \sum_{n=0}^{\infty} e^{-n\omega p}L\{A_n(t)\} \qquad (2.4)$$

By the equation (2.4), we have

$$A_0(t) = f(y_0(t))$$

$$A_1(t) = \left[\frac{\frac{d}{dy}f(y(t))|y=y_0}{1!}\right]y_1(t)$$

and in general, for $n \geq 2$, A_n is n^{th} degree Adomian Polynomial [15] of $f(y(t)$ in the powers of $y_1(t), y_2(t), \ldots, y_n(t)$ given by:

$$A_n(t) = \left[\frac{\frac{d}{dy}f(y(t))|y=y_0}{1!}\right]y_n(t) + \left[\sum_{k=2}^{n}\frac{\frac{d^k}{dy^k}f(y(t))|y=y_0}{k!}\right]\sum_{i1+i2+\cdots+ik=n} y_{i1}(t)\,y_{i2}(t)\ldots y_{ik}(t)$$

Now the main idea of Laplace decomposition is to set an iteration as follows:

$$(1-\frac{c}{p})\sum_{n=0}^{\infty} e^{-n\omega p}L\{y_n(t)\} = \left(\frac{a}{p} + \frac{b}{p^2} - \frac{ca}{p^2} - \frac{bc}{p^3}\right) + \frac{bc}{p^3}e^{-\omega p} + \frac{1}{p^2}\sum_{n=1}^{\infty} e^{-n\omega p}L\{A_{n-1}(t)\}$$

$$= \left(1-\frac{c}{p}\right)\left(\frac{a}{p}+\frac{b}{p^2}\right)+\left(\frac{bc}{p^3}+\frac{1}{p^2}L\{A_0(t)\}\right)e^{-\omega p} + \frac{1}{p^2}\sum_{n=2}^{\infty} e^{-n\omega p}\, L\{A_{n-1}(t)\} \qquad (2.5)$$

One may compute $L\{y_n(t)\}$iteratively us follows:

$$L\{y_0(t)\} = \frac{a}{p}+\frac{b}{p^2}$$

$$L\{y_1(t)\} = \left(1-\frac{c}{p}\right)^{-1}\left(\frac{bc}{p^3}+\frac{1}{p^2}L\{A_0(t)\}\right)$$

$$L\{y_n(t)\} = \left(1-\frac{c}{p}\right)^{-1}\left(\frac{1}{p^2}L\{A_{n-1}(t)\}\right) \quad , \quad n = 2,3,4,\ldots$$

By applying inverse Laplace transform for the Laplace decomposition series, we get

$$y(t) = \sum_{n=0}^{\infty} y_n(t-n\omega)e^{(t-n\omega)} \quad , \qquad (2.6)$$

where$e^{(t-n\omega)}$is a unit step function given by

$$e^{(t-n\omega)} = \begin{cases} 0, & t < n\omega, \\ 1, & t > n\omega, \end{cases}$$

Hence the approximate solution for each interval is given by

$$y(t) = \sum_{n=0}^{N} y_n(t-n\omega) \quad , \quad N\omega \le t \le (N+1)\omega \quad , \qquad (2.7)$$

$$N = 0,1,2,\ldots$$

2.2 Modified Laplace Decomposition Method (MLDM)

In the LDM, $L\{y_0(t)\} = \frac{a}{p}+\frac{b}{p^2}$. The term $\frac{b}{s^2}$ will lead to unnecessary simplifications in each iteration. So, in order to avoid this, we modify the LDM only at the first and second term. Let us apply the following modified Laplace decomposition.

$$L\{y(t)\} = L\{\tilde{y}(t)\} = \sum_{n=0}^{\infty} e^{-n\omega p} L\{\tilde{y}_n(t)\} \quad , \qquad (2.8)$$

Using (3.2.5), we obtain

$$= \frac{a}{p}\left(1-\frac{c}{p}\right)+\left(\frac{bc}{p^3}+\frac{1}{p^2}L\{\tilde{A}_0(t)\}\right)e^{-\omega p}$$

$$+\frac{1}{p^2}\sum_{n=2}^{\infty} e^{-n\omega p} L\{\tilde{A}_{n-1}(t)\} \quad , \qquad (2.9)$$

where

$$L\{\tilde{y}_0(t)\} = \frac{a}{p}$$

$$L\{\tilde{y}_1(t) - bt\} = \left(1-\frac{c}{p}\right)^{-1}\left(\frac{bc}{p^3}+\frac{1}{p^2}L\{\tilde{A}_0(t)\}\right)$$

$$L\{\tilde{y}_n(t)\} = \left(1-\frac{c}{p}\right)^{-1}\left(\frac{1}{p^2}L\{\tilde{A}_{n-1}(t)\}\right) \quad , \quad n = 2,3,4,\ldots$$

And $\tilde{A}\ \ _n$ is n^{th}degree Adomian Polynomial [15] of $f(y(t))$ in the powers of $\tilde{y}_1(t)$, $\tilde{y}_2(t), \ldots , \tilde{y}_n(t)$ and it is similar to $A_n\acute{s}$ of LDM. Again, by applying inverse Laplace transform for the MLD series (3.2.8), we obtain

$$y(t) = \tilde{y}(t) = \sum_{n=0}^{\infty} \tilde{y}_n(t-n\omega)e^{(t-n\omega)} \quad , \qquad (2.10)$$

$e^{(t-n\omega)}$ is a unit step function given by $e^{(t-n\omega)} = \begin{cases} 0, & t < n\omega, \\ 1, & t > n\omega, \end{cases}$

Hence the approximate solution for each interval $N = 0,1,2,\ldots$ is given by

$$\tilde{y}(t) = \sum_{n=0}^{N} \tilde{y}_n(t-n\omega) \quad , \quad N\omega \le t \le (N+1)\omega \quad , \qquad (2.11)$$

3.Examples

Example 1.

Let us consider the following Linear differential - difference equation with differential order two and difference of order one:

$$y''(t) - 2y'(t) + y(t-\omega) = 1 \quad , \quad t > \omega \quad , \qquad (3.1)$$

with the initial interval condition

$$y(t) = 2 + t \quad , \quad 0 \le t \le \omega \quad , \qquad (3.2)$$

First, we note that as $\omega \to 0$, the equation (3.1) becomes linear second order differential equation with exact solution $y(t) = 1 + e^t$ Hence we have selected the initial interval condition (3.2).

Laplace transform method for the differential equation

$$y''(t) - 2y'(t) + y(t) = 1 \quad , \quad y(0) = 2 \quad , \quad y'(0) = 1$$

By applying Laplace transform we obtain,

$$(p^2 L\{y(t)\} - 2p - 1) - 2(pL\{y(t)\} - 2) + L\{y(t)\} = \frac{1}{p}$$

$$(p^2 - 2p + 1)L\{y(t)\} = 2p - 3 + \frac{1}{p}$$

$$L\{y(t)\} = \frac{2p-1}{p(p-1)}$$

$$\Rightarrow y(t) = L^{-1}\left\{\frac{1}{p} + \frac{1}{p-1}\right\} = 1 + e^t$$

LDM for the equations (3.1) – (3.2)

First, we note that,

$$\int_\omega^\infty y'(t)e^{-pt}dt = L\{y'(t)\} - \int_0^\omega 1e^{-pt}dt$$

$$= L\{y'(t)\} + \frac{1}{p}(e^{-\omega p} - 1)$$

And

$$\int_0^\omega y''(t)e^{-pt}dt = 0 \Rightarrow \int_\omega^\infty y''(t)e^{-pt}dt = L\{y''(t)\}$$

Following initial steps of the LDM for (3.1) - (3.2), we obtain

$$L\{y''(t)\} - 2L\{y'(t)\} + \frac{2}{p}(e^{-\omega p} - 1) + e^{-\omega p}L\{y(t)\} = \frac{e^{-\omega p}}{p}$$

After applying formula of Laplace transform for first and second order derivative, finally, we arrive at

$$p(p-2)L\{y(t)\} = \left(2p - 3 - \frac{2}{p}\right) + \frac{3}{p}e^{-\omega p} - e^{-\omega p}L\{y(t)\}$$

$$\left(1 - \frac{2}{p}\right)L\{y(t)\} = \left(1 - \frac{2}{p}\right)\left(\frac{2}{p} + \frac{1}{p^2}\right) + \frac{3}{p^3}e^{-\omega p} - \frac{e^{-\omega p}}{p^2}L\{y(t)\} \qquad (3.3)$$

Now we seek the following type of decomposition for $L\{y(t)\}$:

$$L\{y(t)\} = \sum_{n=0}^{\infty} e^{-n\omega p} L\{y_n(t)\}$$

which may be regarded as Laplace decomposition. Now the basic idea of Laplace decomposition is to set an iteration as follows:

$$(1-\frac{2}{p})\sum_{n=0}^{\infty} e^{-n\omega p} L\{y_n(t)\} = \left(1-\frac{2}{p}\right)\left(\frac{2}{p}+\frac{1}{p^2}\right)+\left(\frac{3}{p^3}-\frac{L\{y_n(t)\}}{p^2}\right)e^{-\omega p}-\frac{1}{p^2}\sum_{n=2}^{\infty} e^{-n\omega p} L\{y_{n-1}(t)\} \quad (3.4)$$

For $n = 0,1,2, \dots$, equate the co - efficient of $e^{-n\omega s}$ on both sides of (3.4), we get

$$L\{y_0(t)\} = \frac{2}{p}+\frac{1}{p^2}$$

$$L\{y_1(t)\} = \left(1-\frac{2}{p}\right)^{-1}\left(\frac{3}{p^3}-\frac{L\{y_0(t)\}}{p^2}\right)$$

$$L\{y_n(t)\} = \left(1-\frac{2}{p}\right)^{-1}\left(-\frac{L\{y_{n-1}(t)\}}{p^2}\right) \quad , \qquad n \geq 2$$

$n = 1$, we have *For*

$$L\{y_1(t)\} = \left(1-\frac{2}{p}\right)^{-1}\left(\frac{3}{p^3}-\frac{L\{y_0(t)\}}{p^2}\right)$$

$$= \left(\frac{1}{p^3}-\frac{1}{p^4}\right)\left(1+\frac{2}{p}+\frac{4}{p^2}+\cdots+\frac{2^n}{p^n}+\cdots\right)$$

$$= \left(\frac{1}{p^3}-\frac{1}{p^4}\right)+\frac{1}{8}\left(\frac{2^4}{p^5}+\frac{2^5}{p^6}+\cdots+\frac{2^{n+4}}{p^{n+5}}+\cdots\right)$$

$$= \left(\frac{1}{p^3}-\frac{1}{p^4}\right)+\frac{1}{8}\left(\frac{1}{p-2}-\frac{1}{p}-\frac{2}{p^2}-\frac{4}{p^3}-\frac{8}{p^4}\right)$$

$n = 2$, we have Again for

$$L\{y_2(t)\} = \left(1-\frac{2}{p}\right)^{-1}\left(-\frac{L\{y_1(t)\}}{p^2}\right)$$

$$= -\left(\frac{1}{p^5}+\frac{1}{p^6}+\frac{2}{p^7}+\frac{4}{p^8}+\cdots+\frac{2^n}{p^{n+6}}+\cdots\right)\times\left(1+\frac{2}{p}+\frac{4}{p^2}+\cdots+\frac{2^n}{p^n}+\cdots\right)$$

$$= \left(\frac{1}{p^5}+\frac{1}{p^6}\right)-\frac{2}{p^5}-\frac{4}{p^6}-\frac{8}{p^7}-\frac{20}{p^8}-\frac{48}{p^9}-\frac{112}{p^{10}}-\cdots$$

$$= \left(\frac{1}{p^5}+\frac{1}{p^6}\right)-\frac{1}{2^3}\left(\frac{2^4}{p^5}+\frac{2^5}{p^6}+\frac{2^7}{p^7}+\cdots\right)-\frac{1}{2^5}\sum_{n=2}^{\infty}\left(\frac{2^{n+5}}{p^{n+6}}+\frac{2^{n+6}}{p^{n+7}}+\frac{2^{n+7}}{p^{n+8}}+\cdots\right)$$

$$= \left(\frac{1}{p^5}+\frac{1}{p^6}\right)-\frac{1}{2^3}\left(\frac{1}{p-2}-\frac{1}{p}-\frac{2}{p^2}-\frac{4}{p^3}-\frac{8}{p^4}\right)-\frac{1}{2^5}\sum_{n=2}^{8}\omega\left(\frac{1}{-2}-\frac{1}{p}-\frac{2}{p^2}-\frac{4}{p^3}-\cdots-\frac{2^{n+4}}{p^{n+4}}\right)$$

By using (2.6), we obtain an approximate solution of $y(t)$. for $t > 0$. For $2\omega \leq t \leq 3\omega$, the exact solution is

$$y(t) = \sum_{n=0}^{2} y_n(t-n\omega)$$

$$= 2+t+\left[\frac{(t-\omega)^2}{2!}+\frac{(t-\omega)^3}{3!}+\frac{1}{2^3}\left(e^{2(t-\omega)}-1-2(t-\omega)-\frac{4(t-\omega)^2}{2!}-\frac{8(t-\omega)^3}{3!}\right)\right]$$

$$+\left[\frac{(t-2\omega)^4}{4!}+\frac{(t-2\omega)^5}{5!}-\frac{1}{2^3}\left(e^{2(t-2\omega)}-1-2(t-2\omega)-\frac{4(t-2\omega)^2}{2!}-\frac{8(t-2\omega)^3}{3!}\right)\right.$$

$$\left.-\frac{1}{2^5}\sum_{n=2}^{\infty}\left(e^{2(t-2\omega)}-1-2(t-2\omega)-\cdots-2^{n+4}\frac{(t-2\omega)^{n+3}}{(n+3)!}\right)\right] \quad (3.5)$$

When we truncate the solution at third term, we obtain

$$y(t) \approx \sum_{n=0}^{2} y_n(t-n\omega)e^{(t-n\omega)}$$

$$y(t) \sim 2+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\frac{t^5}{5!} \sim 1+e^t \text{when } \omega \to 0 \text{ و} t \to 0$$

Example 2.

Let us consider the following nonlinear differential - difference equation with both differential and difference of order one:

$$y''(t)+\sin\big(y(t-\omega)\big)=0 \quad , \quad t>\omega \quad , \qquad (3.6)$$

with the initial interval condition

$$y(t)=1+t \quad , \quad 0 \le t \le \omega \quad , \qquad (3.7)$$

MLDM

The following starting steps of the MLDM for (3.6) – (3.7),we directly obtain

$$L\{y(t)\}=\frac{1}{p}+\frac{1}{p^2}-\frac{e^{-\omega p}}{p^2}L\{\sin\ (y(t))\} \quad , \qquad (3.8)$$

Now we seek the following type of MLD for $L\{y(t)\}$:

$$L\{y(t)\}=\sum_{n=0}^{\infty} e^{-n\omega p}L\{\tilde{y}_n(t)\} \quad , \qquad (3.9)$$

Now let us expanding $L\{\sin(y(t))\}$ at $y=\tilde{y}_0$. by using Laplace decomposition as follows

$$L\{\sin\ (y(t))\}=L\{\sin(\tilde{y}_0(t))\}+e^{-\omega p}L\{\tilde{y}_1(t)\cos(\tilde{y}_0(t))\}+e^{-2\omega p}L\big\{\tilde{y}_2(t)\cos(\tilde{y}_0\ (t)\big)-\frac{1}{2}\tilde{y}_1^2(t)\sin(\tilde{y}_0(t))\}$$

$$+e^{-3\omega p}L\big\{\tilde{y}_3(t)\cos(\ \tilde{y}_0(t)\big)-\tilde{y}_1(t)\tilde{y}_2(t)\sin\big(\tilde{y}_0(t)\big)-\frac{1}{6}\tilde{y}_1^3(t)\cos\big(\tilde{y}_0(t)\big)\}+\cdots=\sum_{n=0}^{\infty} e^{-n\omega p}\ L\big\{\tilde{B}_n(t)\big\} \ (3.10)$$

where $\tilde{B}_i \acute{s}$ are Adomian Polynomials [15] given below,

$$\tilde{B}_0(t)=\sin\big(\tilde{y}_0(t)\big)$$

$$\tilde{B}_1(t)=\tilde{y}_1(t)\cos\big(\tilde{y}_0(t)\big)$$

$$\tilde{B}_2(t)=\tilde{y}_2(t)\cos\big(\tilde{y}_0(t)\big)-\frac{1}{2}\tilde{y}_1^2(t)\sin\big(\tilde{y}_0(t)\big)$$

$$\tilde{B}_3(t)=\tilde{y}_3(t)\cos\big(\tilde{y}_0(t)\big)-\tilde{y}_1\ (t)\tilde{y}_2(t)\sin\big(\tilde{y}_0(t)\big)-\frac{1}{6}\tilde{y}_1^3(t)\cos(\tilde{y}_0(t))$$

$$\tilde{B}_4(t)=\tilde{y}_4(t)\cos\big(\tilde{y}_0(t)\big)-\frac{1}{2}y_2^2(t)\sin\big(\tilde{y}_0(t)\big)-\tilde{y}_1(t)\tilde{y}_3(t)\sin(\tilde{y}_0(t))$$
$$-\frac{1}{2}\tilde{y}_1^2(t)\tilde{y}_2(t)\cos\big(\tilde{y}_0(t)\big)+\frac{1}{24}\tilde{y}_1^4(t)\sin\big(\tilde{y}_0(t)\big)$$

And so on.

Now by using (3.3.9) and (3.3.10), we get.

$$\sum_{n=0}^{\infty} e^{-n\omega p}\ L\{\tilde{y}_n(t)\}=\frac{1}{p}+\frac{1}{p^2}-\frac{e^{-\omega p}}{p^2}\sum_{n=0}^{\infty} e^{-n\omega p}\ L\{\tilde{B}_n(t)\}$$

$$L\{\tilde{y}_0(t)\}+L\{\tilde{y}_1(t)-t\}e^{-n\omega p}+\sum_{n=2}^{\infty} e^{-n\omega p}\ L\{\tilde{y}_n(t)\}$$

$$=\frac{1}{p}-\frac{1}{p^2}\sum_{n=1}^{\infty} e^{-n\omega p}\, L\{\tilde{B}_{n-1}(t)\} \quad , \qquad (3.11)$$

For $n = 0,1,2, \dots$, equate the co - efficient of $e^{-n\omega s}$ on both sides of (3.11) to get $L\{\tilde{y}_n(t)\}$ and apply inverse Laplace transform to obtain $\tilde{y}_n(t)$.

$$forn = 0, \qquad L\{\tilde{y}_0(t)\} = \frac{1}{p} \Rightarrow \tilde{y}_0(t) = 1$$

$$forn = 1, \qquad L\{\tilde{y}_1(t) - t\} = -\frac{1}{p^2} L\{B_0(t)\}$$

$$= -\frac{1}{p^2} L\{\sin(\tilde{y}_0(t))\} = -\frac{\sin(1)}{p^3}$$

$$\Rightarrow \tilde{y}_1(t) = t - \sin(1)\frac{t^2}{2!}$$

$$forn = 2, \qquad L\{\tilde{y}_2(\mathrm{t})\} = -\frac{1}{p^2} L\{\tilde{B}_1(t)\}$$

$$= -\frac{1}{p^2} L\{\tilde{y}_1(t)\cos(\tilde{y}_0(t)\}$$

$$= \frac{-\cos(1)}{p^4} + \frac{\sin(1)\cos(1)}{p^5}$$

$$\Rightarrow \tilde{y}_2(t) = -\cos(1)\frac{t^3}{3!} + \sin(1)\cos(1)\frac{t^4}{4!}$$

$$forn = 3, \qquad L\{\tilde{y}_3(t)\} = -\frac{1}{p^2} L\{\tilde{B}_2(t)\}$$

$$= -\frac{1}{p^2} L\{\tilde{y}_2(t)\cos\big(\tilde{y}_0(t)\big) - \frac{1}{2}\tilde{y}_1^2(t)\sin(\tilde{y}_0(t))\}$$

$$= \frac{\sin(1)}{p^5} + \frac{\cos^2(1) - 3\sin^2(1)}{p^6}$$

$$+ \frac{3\sin^3(1) - \sin(1)\cos^2(1)}{p^7}$$

$$\Rightarrow \tilde{y}_3(t) = \sin(1)\frac{t^4}{4!} + [\cos^2(1) - 3\sin^2(1)]\frac{t^5}{5!} + [3\sin^3(1) - \sin(1)\cos^2(1)]\frac{t^4}{4!}$$

Note that, by applying inverseLaplace transform for the Laplace decomposition series (3.9), we get

$$y(t) = \sum_{n=0}^{\infty} \tilde{y}_n(t - n\omega)e^{(t-n\omega)}$$

$$\approx \sum_{n=0}^{3} \tilde{y}_n(t - n\omega)e^{(t-n\omega)}$$

$$= 1 + [(t-\omega) - \sin(1)\frac{(t-\omega)^2}{2!}]e^{(t-\omega)}$$

$$+[-\cos(1)\frac{(t-2\omega)^3}{3!} + \sin(1)\cos(1)\frac{(t-2\omega)^4}{4!}]e^{(t-2\omega)}$$

$$+[\sin(1)\frac{(t-3\omega)^4}{4!} + [\cos^2(1) - 3\sin^2(1)]\frac{(t-3\omega)^5}{5!}$$

$$+[\,3\sin^3(1) - \sin(1)\cos^2(1)]\frac{(t-3\omega)^6}{6!}]e^{(t-3\omega)}$$

$\omega \to 0$, we get Hence as

$$y(t) \sim\ 1 + t - \sin(1)\frac{t^2}{2!} - \cos(1)\frac{t^3}{3!} + [\sin(1)\cos(1) + \sin(1)]\frac{t^4}{4!}$$

$$+[\cos^2(1) - 3\sin^2(1)]\frac{t^5}{5!} + [3\sin^3(1) - \sin(1)\cos^2(1)]\frac{t^4}{4!}$$

$$\sim \sum_{n=0}^{6} \frac{y^{(n)}(0)}{n!} t^n \quad , \quad \text{at} \omega \to 0 \quad \text{and} t \to 0$$

Example 3.

Let us consider the following nonlinear differential - difference equation with differential order two and difference of order one:

$$y''(t) = y^2(t - \omega) \quad , \quad t > \omega \quad , \qquad (3.12)$$

with the initial interval condition:

$$y(t) = \frac{2}{3} - \frac{4}{9}t \quad , \qquad 0 \le t \le \omega \quad , \qquad (3.13)$$

First, we note that as $\omega \to 0$, the equation (3.12) becomes nonlinear second order differential equation

$$y''(t) = y^2(t) \quad , \quad y(0) = \frac{2}{3} \quad , \quad y'(0) = \frac{-4}{9} \quad , \qquad (3.14)$$

Exact Solution of The Equation (3.14)

$$y''(t) = y^2(t) \quad , \quad y(0) = \frac{2}{3}, y'(0) = \frac{-4}{9}$$

$$\Rightarrow \quad y'(t)y''(t) = y^2(t)y'(t)$$

$$\Rightarrow \frac{\left(y'(t)\right)^2}{2} - \frac{\left(-\frac{4}{9}\right)^2}{2} = \frac{y^3(t)}{3} - \frac{\left(\frac{2}{3}\right)^3}{3}$$

$$\Rightarrow \quad y'(t) = -\frac{\sqrt{2}}{\sqrt{3}}\left(y(t)\right)^{\frac{3}{2}} \quad , \quad y(0) = \frac{2}{3}$$

$$\Rightarrow \left(y(t)\right)^{-\frac{1}{2}} = \frac{\sqrt{3}}{\sqrt{2}} - \frac{1}{2}\left[-\frac{\sqrt{2}}{\sqrt{3}}t\right]$$

$$\left(y(t)\right)^{-\frac{1}{2}} = \frac{1}{\sqrt{6}}(t + 3)$$

so, the exact $u(t)$ is given by, $y(t) = \frac{6}{(t+3)^2}$.

Hence, we have selected the initial interval condition (3.13)

LDM

Following initial steps of the LDM for (3.12) - (3.13). we directly have

$$L\{y(t)\} = \frac{2}{3}\frac{1}{p} - \frac{4}{9}\frac{1}{p^2} + \frac{e^{-\omega p}}{p^2} L\{y^2(t)\} \quad , \qquad (3.15)$$

where the nonlinear term $y^2(t)$ is decomposed in terms of the Adomian Polynomial by using Laplace decomposition as follows:

$$L\{y^2(t)\} = L\{y_0^2(t)\}$$
$$+e^{-p}L\{2y_0(t)y_1(t)\}$$
$$+e^{-2p}L\{2y_0(t)y_2(t) + y_1^2(t)\}$$
$$+e^{-3p}L\{2y_0(t)y_3(t) + 2y_1(t)y_2(t)\}$$

$$\vdots$$

$$+e^{-np}L\{y_0(t)y_n(t)+y_1(t)y_{n-1}(t)+\cdots+y_n(t)y_0(t)\}$$

$$\vdots$$

$$=\sum_{n=0}^{\infty}e^{-np}L\{C_n(t)\} \qquad , \qquad (3.16)$$

where $C_i\acute{s}$ are Adomian Polynomials [15],

$$C_0(t)=y_0^2(t)$$

$$C_n(t)=y_0(t)y_n(t)+y_1(t)y_{n-1}(t)+\cdots+y_n(t)y_0(t) \ \ , \ \ n\geq 1$$

Using (3.16), the equation (3.15) becomes

$$\sum_{n=0}^{\infty}e^{-n\omega p}L\{y_n(t)\}=\left(\frac{2}{3}\frac{1}{p}-\frac{4}{9}\frac{1}{p^2}\right)+\frac{1}{p^2}\sum_{n=1}^{\infty}e^{-n\omega p}\,L\{C_{n-1}(t)\} \ ,(3.17)$$

Equating the terms with co- efficient of $e^{-n\omega s}$ on both sides of (3.17), we get $L\{y_n(t)\}$. An application of inverse Laplace transform will yield $y_n(t)$.

$$y_0(t)=\frac{2}{3}-\frac{4}{9}t \qquad , \qquad (3.18)$$

$$y_n(t)=L^{-1}\left\{\frac{1}{p^2}L\{C_{n-1}(t)\}\right\}, \qquad n\geq 1 \qquad (3.19)$$

The four-term truncated approximate solution is

$$y(t)\ \approx\ \sum_{n=0}^{3}y_n(t-n\omega)e^{(t-n\omega)}$$

$$=\left(\frac{2}{3}-\frac{2\times 2}{3^2}t\right)e^t+\left(\frac{2\times 3}{3^3}(t-\omega)^2-\frac{2\times 4}{3^4}(t-\omega)^3+\frac{2\times 2}{3^5}(t-\omega)^4\right)e^{(t-\omega)}$$

$$+(\frac{2\times 3}{3^5}(t-2\omega)^4-\frac{2\times 6}{3^6}(t-2\omega)^5+\frac{2\times 4}{3^7}(t-2\omega)^6-\frac{2\times 8}{3^8\times 7}(t-2\omega)^7)e^{(t-2\omega)}$$

$$+(\frac{2\times 3}{3^7}(t-3\omega)^6-\frac{2\times 48}{3^8\times 7}(t-3\omega)^7+\frac{2\times 45}{3^9\times 7}(t-3\omega)^8$$

$$-\frac{2\times 20}{3^{10}\times 7}(t-3\omega)^9+\frac{2\times 4}{3^{11}\times 7}(t-3\omega)^{10})e^{(t-3\omega)} \qquad , \qquad (3.20)$$

We note that, by applying inverse Laplace transform for the Laplace decomposition series (3.15), we get

$$y(t)=\sum_{n=0}^{\infty}y_n(t-n\omega)e^{(t-n\omega)}\approx\sum_{n=0}^{3}y_n(t-n\omega)e^{(t-n\omega)}$$

$\omega\rightarrow 0$, (3.20) becomes *As*

$$y(t)\approx\ y_A(t)=\frac{2}{3}-\frac{2\times 2}{3^2}t+\frac{2\times 3}{3^3}t^2-\frac{2\times 4}{3^4}t^3+\frac{2\times 5}{3^5}t^4$$

$$-\frac{2\times 6}{3^6}t^5+\frac{2\times 7}{3^7}t^6-\frac{2\times 8}{3^8}t^7+\frac{2\times 45}{3^9\times 7}t^8$$

$$-\frac{2\times 20}{3^{10}\times 7}t^9+\frac{2\times 4}{3^{11}\times 7}t^{10} \qquad , \qquad (3.21)$$

MLDM

Now, let us apply the following MLD

$$L\{y(t)\} = L\{\tilde{y}(t)\} = \sum_{n=0}^{\infty} e^{-n\omega p} L\{\tilde{y}_n(t)\} \quad , \quad (3.22)$$

Using (3.15), we get

$$L\{\tilde{y}_0(t)\} + L\left\{\tilde{y}_1(t) + \frac{4}{9}t\right\} e^{-\omega p} + \sum_{n=2}^{\infty} e^{-n\omega p} L\{\tilde{y}_n(t)\}$$

$$= \frac{2}{3}\frac{1}{p} + \frac{1}{p^2}\sum_{n=1}^{\infty} e^{-n\omega p} L\{\tilde{C}_{n-1}(t)\} \quad , \quad (3.23)$$

$\tilde{C}_i$ are Adomian Polynomials [15],were

$\tilde{C}_0(t) = \tilde{y}_0^2(t)$

$\tilde{C}_n(t) = \tilde{y}_0(t)\tilde{y}_n(t) + \tilde{y}_1(t)\tilde{y}_{n-1}(t) + \cdots + \tilde{y}_n(t)\tilde{y}_0(t)$, for all$n \geq 1$

For $n = 0,1,2, \ldots$ equate the coefficients of $e^{-n\omega s}$ on both sides of (3.23), we have

$$L\{\tilde{y}_0(t) = \frac{2}{3}\frac{1}{p} \quad , \quad (3.24)$$

$$L\left\{\tilde{y}_1(t) + \frac{4}{9}t\right\} = \frac{1}{p^2} L\{\tilde{C}_0(t)\}$$

$$L\{\tilde{y}_n(t) = \frac{1}{p^2} L\{\tilde{C}_{n-1}(t)\} \quad , \quad n \geq 1$$

Notice that by applying inverse Laplace transform for the Laplace decomposition series (3.22), we get

$$\tilde{y}(t) = \sum_{n=0}^{\infty} \tilde{y}_n(t - n\omega) e^{(t-n\omega)} \approx \sum_{n=0}^{5} \tilde{y}_n(t - n\omega) e^{(t-n\omega)}$$

Because $\tilde{y}_0(t)$ is a constant term when compared to $y_0(t)$ which is a first-degree polynomial we compute the first 6 terms truncated approximate solution for $\tilde{y}_0(t)$ so that it matches with four term truncated approximate solution for $y(t)$.

$$\tilde{y}(t) \approx \sum_{n=0}^{5} \tilde{y}_n(t - n\omega) e^{(t-n\omega)} = \frac{2}{3} e^{(t)} + \left(-\frac{2\times 2}{3^2}(t-\omega) + \frac{2\times 3}{3^3}(t-\omega)^2\right) e^{(t-\omega)}$$
$$+ \left(-\frac{2\times 4}{3^4}(t-2\omega)^3 + \frac{2\times 3}{3^5}(t-2\omega)^4\right) e^{(t-2\omega)}$$
$$+ \left(\frac{2\times 2}{3^5}(t-3\omega)^4 - \frac{2\times 6}{3^6}(t-3\omega)^5 + \frac{2\times 3}{3^7}(t-3\omega)^6\right) e^{(t-3\omega)}$$
$$+ \left(\frac{2\times 4}{3^7}(t-4\omega)^6 - \frac{2\times 48}{3^8\times 7}(t-4\omega)^7 + \frac{2\times 18}{3^9\times 7}(t-4\omega)^8\right) e^{(t-4\omega)} + (-\frac{2\times 8}{3^8\times 7}(t-5\omega)^7$$
$$+ \frac{2\times 45}{3^9\times 7}(t-5\omega)^8 - \frac{2\times 50}{3^{10}\times 7}(t-5\omega)^9$$
$$+ \frac{2\times 15}{3^{11}\times 7}(t-5\omega)^{10}) e^{(t-5\omega)} \quad , \quad (3.25)$$

$\omega \to 0$, (3.25)As

$$y(t) \approx \tilde{y}_A(t) = \frac{2}{3} - \frac{2\times 2}{3^2}t + \frac{2\times 3}{3^3}t^2 - \frac{2\times 4}{3^4}t^3 + \frac{2\times 5}{3^5}t^4 - \frac{2\times 6}{3^6}t^5 + \frac{2\times 7}{3^7}t^6 - \frac{2\times 8}{3^8}t^7 + \frac{2\times 9}{3^9}t^8$$
$$+ \frac{2\times 50}{3^{10}\times 7}t^9 - \frac{2\times 15}{3^{11}\times 7}t^{10} \quad , \quad (3.26)$$

When $\omega \to 0$, equation (3.12) become nonlinear second order differential equation

$y''(t) = y^2(t), y(0) = \frac{2}{3}, y'(0) = \frac{-4}{9}$and the exact solution is given by$\frac{6}{(t+3)^2}$.

The following table gives a comparative study of two approximate solutions $y_A(t)$ given by (3.21) and $\tilde{y}_A(t)$ given by (3.26) with the exact solution given by $\frac{6}{(t+3)^2}$:

t	Exact solution $y(t)$	Solution by LDM $y_A(t)$	Solution by MLDM $\tilde{y}_A(t)$
0	0.66666667	0.6666667	0.6666667
0.1	0.6243497	0.6243497	0.6243497
0.2	0.5859375	0.5859375	0.5859375
0.3	0.5509642	0.5509642	0.5509642
0.4	0.5190312	0.5190310	0.5190312
0.5	0.4897959	0.4897953	0.4897960
0.6	0.4629630	0.4629604	0.4629635
0.7	0.4382761	0.4382682	0.4382779
0.8	0.4155125	0.4154915	0.4155177
0.9	0.3944773	0.3944282	0.3944906
1.0	0.3750000	0.3748961	0.3750300

4.Conclusions

The above three examples explain the fact that LDM and MLDM are applicable and quite suitable for solving linear or nonlinear differential difference equations of order (2.1). The first example clearly pointing to the computation of exact solution in each interval is not simple but the asymptotic relations guide in each step to get the exact solution of the differential equation when time and difference parameter approach to 0. At the second example the usual LDM leads to integration of successive iterative sine functions like $\sin(\sin(y_0))$ which is not desirable. As a substitute for that the MLDM is quite suitable in this situation. The third example indicates that the LDM shows flexibility in rearranging the terms of the decomposition series which will lead to modified decomposition method suitable for the computation. The Numerical results of the third example give us the following facts:

(i) $y_A(t)$ compares with $\tilde{y}_A(t)$better near $t = 0$ than $t = 0.5$ and $t = 1.0$.

(ii) $y_A(t)$ shows better rate of convergence near $t = 0$ than $t = 0.5$ and $t = 1.0$.

(iii) $y_A(t)$ shows better rate of convergence than $\tilde{y}_A(t)$.

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