Research Article

FRACTIONAL CALCULUS APPROACH IN RLCCIRCUIT USING HYPERGEOMETRIC SERIES

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Abstract: In this paper, fractional calculus approach is applied in solving differential equation which is associated with an electrical circuit i.e. RLC circuit using hypergeometric series. The solution of the fractional differential equation of RLC circuit comes in the form ofMittag-Leffler function and the Ali's et.al.[8] results are special cases of our main result.
Mathematics Subject Classification: 26A33, 33E12.
Keywords and Phrases: RLC Electrical CircuitFractional Derivative, Mittag-Leffler function, Hypergeometric Series.

1.Introduction: Thefractional calculus is a generalization of ordinary differentiation to non-integer case. In other words, the fractional calculus operators deal with integrals and derivatives of arbitrary (i.e. real or complex) order. The name "fractional calculus" is actually a misnomer; the designation, "integration and differentiation of arbitrary order" is more appropriate. The classical calculus was independently discovered in seventeenth century by Isaac Newton and Gottfried Wilhelm Leibnitz[11]. The question raised by Leibnitz for the existence of fractional derivative of order half was an ongoing topic amongst mathematicians for more than three hundred years, consequently several aspects of fractional calculus were developed and studied. The first accurate use of a derivative of non-integer order is due to the French mathematician S. F. Lacroix [5] in 1819 who expressed the derivative of non-integer order ½ in terms of Legendre's factorial symbol Γ.

$$\Gamma(a) = \int_0^\infty t^{\alpha-1} e^{-t} dt$$

Starting, with a function $y = x^m$, Lacroix [5]expressed it as follows

$$\frac{d^n y}{dx^n} = \frac{m!}{(m-n)!} x^{m-n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n}$$

Replacing with $\frac{1}{2}$ and putting $m = 1$, he obtained the derivative of order $\frac{1}{2}$ of the function x.

$$\frac{d^{1/2} y}{dx^{1/2}} = \frac{\Gamma(2)}{\Gamma(3/2)} x^{1/2} = \frac{2}{\sqrt{\pi}} \sqrt{x}$$

The credit of first application of fractional calculus goes to Abel's [6] who employed it in the solution of an integral equation which emerged in the formulation of the tautochrone problem of finding the shape of a frictionless wire lying in a vertical plane such that the time of slide of a bead placed on the wire to the lowest point of the wire is the same regardless of position of the bead on the wire.

Abel's [6] solutions attracted a group of mathematicians and scientists to this branch of knowledge and first logical definition of fractional derivative was given by Riemann−Liouville[10]. Later on several attempts were made to define different forms of fractional integral and derivatives.On the other hand, several applications of the calculus of fractional order were sought by various mathematicians, engineers and scientists. The efforts were so rewarding that the subject of fractional calculus itself was categorized as applicable mathematics. The definitions of fractional derivative as under:

Definition 1. (Riemann Liouville operator [10]): Let f be a continuous function, $\alpha \in R^+, and\ t \in R. The$ fractional integral of order α is defined as: $J^\alpha f\ (t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} f(u)du$ ... (1.1)**Definition 2.** Pick some$\alpha \in R^+,$ let n be the nearest integer Greater than α. The Riemann – Liouville [10] fractional derivative of order α of a function f (t) is givenby:$D^\alpha f\ (t) = \frac{d^n}{dt^n} J^{n-\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} (\frac{d}{dt})^n \int_0^t (t-u)^{n-\alpha-1} f(u)du$…(1.2)

Definition 3. Pick some$\alpha \in R^+,$ let n be the nearest integer greater than α. The Caputo fractional derivative [7] of order α of a function f (t) is given by: $D_*^\alpha f\ (t) = J^{n-\alpha} \frac{d^n}{dt^n} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-u)^{n-\alpha-1} f^n(u)du$…(1.3)

The applications of the Mittag-Leffler function and its extension are discussed in [2] recently in a rapidly increasing numbers of papers, related to fractional calculus and fractional order differential and integral equations and systems modling in various phenomena.

Definition4.Mittag – Leffler function[2]: The definition of Mittag-Leffler function is given by Mittag – Leffler[2] in 1903 which is defined as:

$E_\alpha(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k+1)}$ … (1.4)

Where $\alpha \epsilon c, Re(\alpha) > 0$

Its Laplace transform is given as:

$\sum_{k=0}^{\infty} \frac{\Gamma(k+1)}{\Gamma(\alpha k+1)} \frac{1}{S^{k+1}}$… (1.5)

The generalization of Mittag – Lefflerfunction introduced by Wiman [4] in 1905 which is given as:

$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k+\beta)},\ Re(\alpha) > 0, Re(\beta) > 0$… (1.6)

Its Laplace transform is as follows:

$\sum_{k=0}^{\infty} \frac{\Gamma(k+1)}{\Gamma(\alpha k+\beta)} \frac{1}{S^{k+1}}$… (1.7)

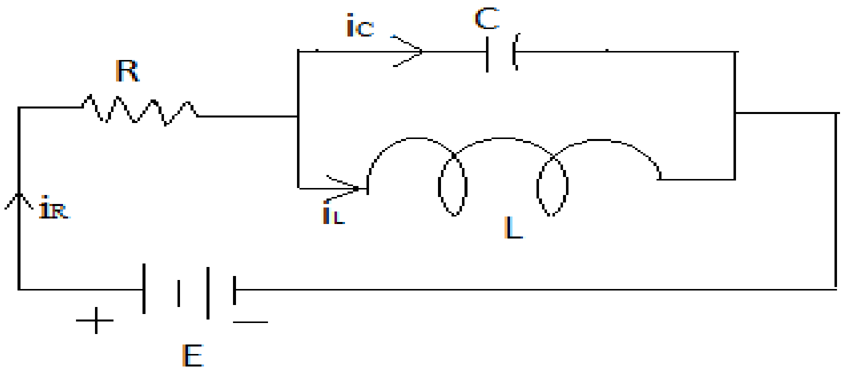
After that Prabhakar [3] introduced the definition of generalization of generalized Mittag-Leffler function in 1971 in the following form

$E_{\alpha,\beta}^{\gamma}(t) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k+\beta)} \frac{t^k}{k!}$… (1.8)

Where $\alpha, \beta, \gamma \in C, Re(\alpha) > 0, Re(\beta) > 0$
Its Laplace transform is as under:
$\sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k+\beta)} \frac{1}{S^{k+1}}$ … (1.9)
A general hypergeometric series[9] with p upper or numerator parameters and q lower or denominator parameters is denoted and defined as follows: ${}_pF_q \left[\begin{matrix} a_1, \dots, a_p; \\ b_1, \dots, b_q; \end{matrix} z\right] = {}_pF_q\left((a_p); (b_q); z\right) = {}_pF_q(z) = \sum_{r=0}^{\infty} \frac{(a_1)_r \dots (a_p)_r}{(b_1)_r \dots (b_q)_r} \frac{z^r}{r!}$…(1.10)
where $(a_j)_r$ and $(b_j)_r$ are the Pochhammer symbolsof the series in is defined when none of the $(b_j)_r$'s, j = 1, 2, ….,q, is a negative integer or zero. If a b_j is a negative integer or zero then $(b_j)_r$ will be zero for some r. A b_j can be zero provided there is a numerator parameter a_k such that $(a_k)_r$ become zero first before $(b_j)_r$ become zero.If any numerator parameter a_j is a negative integer or zero then terminates and becomes a polynomial in z. From the ratio test it is evident that the series is convergent for all z if $q \geq p$,it is convergent for $|z| < 1$ if p = q +1 and divergent if p > q + 1. When p = q+1 and $|z| = 1$ the series can converge in some cases. Let $\beta = \sum_{j=1}^{p} a_j - \sum_{j=1}^{q} b_j$.It can be shown that when p = q + 1 the series is absolutely convergent for $|z| = 1$ if R(β)< 0, conditionally convergent for z = -1 if 0 ≤R(β)< 1 and divergent for $|z| = 1$ if 1 ≤ R(β). Some special cases of a ${}_pF_q$ are the following when there is no upper or lower parameters we have,
${}_0F_0(;;\pm z) = \sum_{r=0}^{\infty} \frac{(\pm z)^r}{r!} = e^{\pm z}$….(1.11)
Thus ${}_0F_0$(.)is an exponential series. ${}_1F_0(\alpha;;z) = \sum_{r=0}^{\infty} \alpha_r \frac{z^r}{r!} = (1-z)^{-\alpha}$ for $|z| < 1$. …(1.12)This is the binomial series. ${}_1F_1(.)$is known as confluent hypergeometric series and ${}_2F_1(.)$ is known as Gauss' hypergeometric series[9].

2.Electrical Circuit[8]:

In this section, we present the three elements of RLC electrical circuit where C is a capacitance, L is an inductance, R is a resistance and we consider here only positive value of all these constants.
The constitutive equations associated with three elements of RLC electrical circuit are defined as under:
The voltage drop across resistance R=$U_R(t) = RI(t)$,
Where I (t) is current.
The voltage drop across inductor L= $U_L(t) = L\frac{dI}{dt}$
And the voltage drop across capacitance C= $U_c(t) = \frac{1}{c}\int_0^t I(v)dv$
Kirchhoff law:The algebraic sum of the voltage drop around any closed circuit is equal to resultant EMF in the circuit.
By applying the Kirchhoff law in the non-homogeneous second order ordinary differential equations. We get
RC$\frac{d^2U_c(t)}{dt^2} + \frac{dU_c(t)}{dt} + \frac{R}{L}U_c(t) = \frac{d}{dt}\theta(t)$… (2.1)
Where $U_c(t)$ is the voltage on the capacitor, it is the similar on the inductor as we can see in figure, because these are connected in parallel [8].



Three element LCR electrical circuit

Fig-1

Again, consider another non-homogeneous second order ordinary differential equation associated with current on the inductor as follows:

RLC$\frac{d^2 I_L(t)}{dt^2} + L\frac{dI_L(t)}{dt} + RI_L(t) = \theta(t)$… (2.2)

Using the constitutive equation for the inductor, these two non-homogeneous second order ordinary differential equations can be led to correspondent integro-differential equations, then we get

$R\frac{di_c(t)}{dt} + \frac{1}{c}i_c(t) + \frac{R}{LC}\int_0^t i_c(v)dv = \frac{d}{dt}\theta(t)$… (2.3)

$RC\frac{dU_L(t)}{dt} + U_L(t) + \frac{R}{L}\int_0^t U_L(v)dv = \theta(t)$… (2.4)

We consider the initial condition $I_c(t) = 0$ at $t = 0$ i.e. the initial current on the capacitor is zero and we get the solution in terms of an exponential function [8]

3. Fractional integro-differential equation:

The fractional integro-differential equation with current on the capacitor is as :

R$\frac{d^\alpha}{dt^\alpha}i_c(t) + \frac{1}{c}i_c(t) + \frac{R}{LC}\frac{1}{\Gamma(\alpha)}\int_0^t (t-v)^{\alpha-1} i_c(v)dv = \frac{d\theta(t)}{dt}$… (3.1)

The classical integro-differential equation associated with the RLC electrical circuit because for α=1 we improve the result get in equation (3.1). Its replacement is very important in discussing the corresponding numerical problem for a particular value of the parameter because the solution is obtained in terms of a closed expression [8]

The Laplace integral transform

$L[i_c(t)] = F(s) = \int_0^\infty e^{-st} i_c(t)dt\,,\ Re(s) > 0$… (3.2)

Let $\theta(t)$= hypergeometricfunction in equation (3.2)

$$R\frac{d^\alpha}{dt^\alpha}i_c(t) + \frac{1}{c}i_c(t) + \frac{R}{LC}\frac{1}{\Gamma(\alpha)}\int_0^t (t-v)^{\alpha-1} i_c(v)dv = \frac{d\theta(t)}{dt}$$

$R\frac{d^\alpha}{dt^\alpha}i_c(t) + \frac{1}{c}i_c(t) + \frac{R}{LC}\frac{1}{\Gamma(\alpha)}\int_0^t (t-v)^{\alpha-1} i_c(v)dv = \frac{d}{dt}[{}_pF_q(\text{t})]$ … (3.3)

$$R\frac{d^{\alpha}}{dt^{\alpha}}i_c(t)+\frac{1}{c}i_c(t)+\frac{R}{LC}\frac{1}{\Gamma(\alpha)}\int_0^t(t-v)^{\alpha-1}\,i_c(v)dv=$$
$\frac{d}{dt}\left[\sum_{k=0}^{\infty}\frac{(a_1)_k\ldots(a_p)_k}{(b_1)_k\ldots(b_q)_k}\frac{t^k}{k!}\right]\ldots$ (3.4)

$R\frac{d^{\alpha}}{dt^{\alpha}}i_c(t)+\frac{1}{c}i_c(t)+\frac{R}{LC}\frac{1}{\Gamma(\alpha)}\int_0^t(t-v)^{\alpha-1}\,i_c(v)dv=\left[\sum_{k=0}^{\infty}\frac{(a_1)_k\ldots(a_p)_k}{(b_1)_k\ldots(b_q)_k}\frac{kt^{k-1}}{\Gamma(k+1)}\right]\ldots$ (3.5)

Applying the Laplace transform of the both sides, we get

$Rs^{\alpha}F(s)+\frac{F(s)}{C}+\frac{R}{LC}\frac{F(s)}{S^{\alpha}}=\sum_{k=0}^{\infty}\frac{(a_1)_k\ldots(a_p)_k}{(b_1)_k\ldots(b_q)_k}\frac{k\Gamma(k)}{\Gamma(k+1)}\frac{1}{s^k}\ldots$ (3.6)

$F(s)[Rs^{\alpha}+\frac{1}{C}+\frac{R}{LC\,S^{\alpha}}]=\sum_{k=0}^{\infty}\frac{(a_1)_k\ldots(a_p)_k}{(b_1)_k\ldots(b_q)_k}\frac{1}{s^k}\ldots$ (3.7)

$RF(s)=\sum_{k=0}^{\infty}\frac{(a_1)_k\ldots(a_p)_k}{(b_1)_k\ldots(b_q)_k}\frac{1}{s^k}\frac{1}{[S^{\alpha}+\frac{1}{RC}+\frac{1}{LC\,S^{\alpha}}]}\ldots$ (3.8)

Let $a=\frac{1}{RC}$, $b=\frac{1}{LC}$

$RF(s)=\sum_{k=0}^{\infty}\frac{(a_1)_k\ldots(a_p)_k}{(b_1)_k\ldots(b_q)_k}\frac{1}{s^k[S^{\alpha}+a+\frac{b}{S^{\alpha}}]}\ldots$ (3.9)

$RF(s)=\sum_{k=0}^{\infty}\frac{(a_1)_k\ldots(a_p)_k}{(b_1)_k\ldots(b_q)_k}\frac{S^{\alpha-k}}{[S^{2\alpha}+aS^{\alpha}+b]}\ldots$ (3.10)

$F(s)=\frac{1}{R}\sum_{k=0}^{\infty}\frac{(a_1)_k\ldots(a_p)_k}{(b_1)_k\ldots(b_q)_k}\frac{S^{\alpha-k}}{[S^{2\alpha}+aS^{\alpha}+b]}\ldots$ (3.11)

Taking the inverse Laplace transform of the both sides, then we have

$i_c(t)=\frac{1}{R}\sum_{k=0}^{\infty}\frac{(a_1)_k\ldots(a_p)_k}{(b_1)_k\ldots(b_q)_k}L^{-1}\left\{\frac{S^{\alpha-k+1-1}}{S^{2\alpha}+AS^{\alpha}+B}\right\}\ldots$ (3.12)

We know the following relation by [12]

$L^{-1}\left\{\frac{S^{\gamma-1}}{S^{\alpha}+AS^{\beta}+B}\right\}=t^{\alpha-\gamma}\sum_{r=0}^{\infty}(-A)^r\,t^{(\alpha-\beta)r}E_{\alpha,\alpha+1-\gamma+(\alpha-\beta)r}^{r+1}(-Bt^{\alpha})\ldots$ (3.13)

Valid for $\left|\frac{AS^{\beta}}{S^{\alpha}+B}\right|<1,\alpha\geq\beta$

Using the relation (3.13), we get,

$L^{-1}\left\{\frac{S^{\alpha-k+1-1}}{S^{2\alpha}+aS^{\alpha}+b}\right\}=t^{\alpha+k-1}\sum_{r=0}^{\infty}(-a)^r\,t^{\alpha r}E_{2\alpha,\alpha+k+\alpha r}^{r+1}(-bt^{2\alpha})\ldots$ (3.14)

Comparing the above these equations (3.13) and (3.14), then we get

$i_c(t)=\frac{1}{R}\sum_{k=0}^{\infty}\frac{(a_1)_k\ldots(a_p)_k}{(b_1)_k\ldots(b_q)_k}L^{-1}\left\{\frac{S^{\alpha-k+1-1}}{S^{2\alpha}+aS^{\alpha}+b}\right\}\ldots$ (3.15)

$$i_c(t)=\frac{1}{R}\sum_{k=0}^{\infty}\frac{(a_1)_k\ldots(a_p)_k}{(b_1)_k\ldots(b_q)_k}\left\{t^{\alpha+k-1}\sum_{r=0}^{\infty}(-a)^r\,t^{\alpha r}E_{2\alpha,\alpha+k+\alpha r}^{r+1}(-bt^{2\alpha})\right\}\qquad\ldots(3.16)$$

Here $E_{\mu,w}^{\rho}(t)$ is the Mittag-Leffler function of three parameters

Special Cases: 1. When $\theta(t)={}_2F_1(a_1,a_2;\,b_1;t)=\sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k}{(b_1)_k}\frac{t^k}{k!}$ is a Gauss's Hypergeometric function [9] then equation (3.16) reduces to

$i_c(t)=\frac{1}{R}\sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k}{(b_1)_k}\left\{t^{\alpha+k-1}\sum_{r=0}^{\infty}(-a)^r\,t^{\alpha r}E_{2\alpha,\alpha+k+\alpha r}^{r+1}(-bt^{2\alpha})\right\}\ldots$(3.17)

2. When $\theta(t)={}_1F_1(a_1;\,b_1;t)=\sum_{k=0}^{\infty}\frac{(a_1)_k}{(b_1)_k}\frac{t^k}{k!}$ is a confluenthypergeometric function[9]then equation (3.17) reduces to

$$i_c(t) = \frac{1}{R}\sum_{k=0}^{\infty}\frac{(a_1)_k}{(b_1)_k}\left\{t^{\alpha+k-1}\sum_{r=0}^{\infty}(-a)^r\, t^{\alpha r}E_{2\alpha,\alpha+k+\alpha r}^{r+1}(-bt^{2\alpha})\right\} \quad \ldots(3.18)$$

3. When we put $(a_1)_{k,}\ldots,(a_p)_k = 1$ and $(b_1)_{k,}\ldots,(b_q)_k = 1$ and k=1 in equation (3.17) then we get the Ali's [8] result

$i_c(t) = \frac{1}{R}\left\{t^{\alpha}\sum_{r=0}^{\infty}(-a)^r\, t^{\alpha r}E_{2\alpha,\alpha+1+\alpha r}^{r+1}(-bt^{2\alpha})\right\}$…(3.19)

4. When we put $(a_1)_{k,}\ldots,(a_p)_k = 1$ and $(b_1)_{k,}\ldots,(b_q)_k = 1$ and k=2 in equation (3.17) then we get the Ali's [8] result

$i_c(t) = \frac{1}{R}\left\{t^{\alpha+1}\sum_{r=0}^{\infty}(-a)^r\, t^{\alpha r}E_{2\alpha,\alpha+2+\alpha r}^{r+1}(-bt^{2\alpha})\right\}$…(3.20)

This completes the analysis.

4. Conclusion: The applications of fractional calculus can be seen in many areas.It has been played an important role in electrical engineering. In this paper we have obtained the closed form solution of fractional integro-differential equation associated with RLC circuitusing the hypergeometric functions in terms of Mittag-Leffler function and Ali's [8] results are special cases of our result.

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