Inclusion Properties of Certain Subclasses of Analytic Functions Involving Mittag-Leffler Function

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Abstract
The purpose of the present paper is to introduce certain new subclasses of analytic univalent functions defined by a linear operator involving Mittag-Leffler function and study their inclusion relationships. Some applications involving integral operators are also considered.
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Introduction

Let $\mathbf{A}$ denote the class of functions of the form:

$$\mathbf{f}(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in $\mathsf{U} = \{z \in \mathsf{C} : |z| < 1\}$.

For $0 \le \eta, \xi < 1$, let $\mathbf{S}(\eta)$, $\mathbf{K}(\eta)$, $\mathsf{C}(\eta,\xi)$ and $\mathsf{C}(\eta,\xi)$ be the subclasses of $\mathbf{A}$ consisting of all analytic functions which are, respectively, starlike of order η, convex of order η, close-to-convex functions of order ξ and type η and quasiconvex functions of order ξ and type η.

Let S be the class of analytic and univalent functions ϕ for which $\phi(\mathsf{U})$ is convex with $\phi(0) = 1$ and $\operatorname{Re}\{\phi(z)\} > 0$, $z \in \mathsf{U}$.

For $0 \le \eta, \xi < 1$ and $\phi, \psi \in S$, $z \in \mathsf{U}$ (cf. [3] and [11]), let

$$\mathbf{S}(\eta;\phi) = \left\{ \mathbf{f} \in \mathbf{A} : \frac{1}{1-\eta}\left(\frac{z\,\mathbf{f}'(z)}{\mathbf{f}(z)} - \eta \right) \prec \phi(z) \right\},$$

$$\mathbf{K}(\eta;\phi) = \left\{ \mathbf{f} \in \mathbf{A} : \frac{1}{1-\eta}\left(1 + \frac{z\,\mathbf{f}''(z)}{\mathbf{f}'(z)} - \eta \right) \prec \phi(z) \right\},$$

$$\mathsf{C}(\eta,\xi;\phi,\psi) = \left\{ \mathbf{f} \in \mathbf{A} \ :\exists\, g \in \mathbf{S}(\eta;\phi) \text{ s.t.} \frac{1}{1-\xi}\left(\frac{z\,\mathbf{f}'(z)}{g(z)} - \xi \right) \prec \psi(z) \right\}.$$

and

$$\mathsf{C}(\eta,\xi;\phi,\psi) = \left\{ \mathbf{f} \in \mathbf{A} \ :\exists\, g \in \mathbf{K}(\eta;\phi) \text{ s.t.} \frac{1}{1-\xi}\left(\frac{(z\,\mathbf{f}'(z))'}{g'(z)} - \xi \right) \prec \psi(z) \right\},$$

where $\prec$ denotes the subordination. For special choices for the functions ϕ and ψ, we can obtain well-known subclasses of $\mathbf{A}$. For examples:

$$\mathbf{S}\left(\eta;\frac{1+z}{1-z}\right) = \mathbf{S}(\eta),\ \mathbf{K}\left(\eta;\frac{1+z}{1-z}\right) = \mathbf{K}(\eta),$$

$$\mathsf{C}\left(\eta,\xi;\frac{1+z}{1-z},\frac{1+z}{1-z}\right) = \mathsf{C}(\eta,\xi) \text{ and } \mathsf{C}\left(\eta,\xi;\frac{1+z}{1-z},\frac{1+z}{1-z}\right) = \mathsf{C}(\eta,\xi).$$

The function

$$E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}, \qquad (z \in \mathsf{C},\ \operatorname{Re}(\alpha) > 0).$$

was introduced by Mittag-Leffler [15, 16] and is known as the Mittag-Leffler function. Srivastava and Tomovski [17] generalized this function by $E_{\alpha,\beta}^{\gamma,\delta}(z)$, where

$$E_{\alpha,\beta}^{\gamma,\delta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n\delta}\ z^n}{\Gamma(\alpha\, n+\beta)n!} \quad (z \in \mathsf{U}), \tag{1.2}$$

$$(\alpha,\beta,\gamma \in \mathsf{C};\ \operatorname{Re}(\alpha) > \max\{0,\ \operatorname{Re}(\delta)-1\};\ \operatorname{Re}(\delta) > 0),$$

and proved that it is an entire function in the complex z-plane, where

$$(\gamma)_{\sigma} = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)} = \begin{cases} 1, & \sigma = 0 \\ \gamma(\gamma+1)\ldots(\gamma+\sigma-1) & \sigma \in \mathsf{N} = \{1,2,\ldots\} \end{cases}.$$

Several properties of Mittag-Leffler function and its generalization can be found e.g. in ([1], [5]-[8], [12], [13], [15]-[17]).

Attiya [1] under the same conditions above defined the function $Q_{\alpha,\beta}^{\gamma,\delta}(z)$ by

$$Q_{\alpha,\beta}^{\gamma,\delta}(z) = \frac{\Gamma(\alpha\ +\ \beta)}{(\gamma)_{\delta}}\left(E_{\alpha,\beta}^{\gamma,\delta}(z) - \frac{1}{\Gamma(\beta)} \right)$$

$$= z + \sum_{n=2}^{\infty} \frac{\Gamma(\delta n + \gamma)\Gamma(\alpha + \beta)}{\Gamma(\delta + \gamma)\Gamma(\alpha n + \beta) n!} z^n. \qquad (1.3)$$

Corresponding to a function $Q^{\gamma,\delta}_{\alpha,\beta}(z)$ defined by (1.3), we introduce a function $Q^{\gamma,\delta}_{\lambda,\alpha,\beta}(z)$ given by

$$Q^{\gamma,\delta}_{\lambda,\alpha,\beta}(z) * Q^{\gamma,\delta}_{\alpha,\beta}(z) = \frac{z}{(1-z)^{\lambda+1}} \quad (\lambda > -1) \qquad (1.4)$$

and define the linear operator $\mathbf{H}^{\gamma,\delta}_{\lambda,\alpha,\beta}(\mathbf{f})(z) : \mathbf{A} \to \mathbf{A}$ as follows:

$$\mathbf{H}^{\gamma,\delta}_{\lambda,\alpha,\beta}\mathbf{f}(z) = Q^{\gamma,\delta}_{\lambda,\alpha,\beta}(z) * \mathbf{f}(z)$$

$$= z + \sum_{n=2}^{\infty} \frac{n\,\Gamma(\lambda+n)\Gamma(\delta+\gamma)\Gamma(\alpha n+\beta)}{\Gamma(\lambda+1)\Gamma(\delta n+\gamma)\Gamma(\alpha+\beta)} a_n z^n \qquad (1.5)$$

$(\lambda > -1;\ \beta,\gamma \in \mathbb{C};\ \mathrm{Re}(\alpha) > \max\{0,\ \mathrm{Re}(\delta)-1\};\ \mathrm{Re}(\delta) > 0$ and $\mathrm{Re}(\alpha) = 0$ when $\mathrm{Re}(\delta) = 1$ with $\beta \neq 0)$.

We note that $\mathbf{H}^{1,1}_{0,0,\beta}\mathbf{f}(z) = \mathbf{f}(z)$.

It is easily verified from (1.5) that

$$\delta z (\mathbf{H}^{\gamma+1,\delta}_{\lambda,\alpha,\beta}\mathbf{f}(z))' = (\gamma+\delta)\mathbf{H}^{\gamma,\delta}_{\lambda,\alpha,\beta}\mathbf{f}(z) - \gamma\,\mathbf{H}^{\gamma+1,\delta}_{\lambda,\alpha,\beta}\mathbf{f}(z), \qquad (1.6)$$

$$z (\mathbf{H}^{\gamma,\delta}_{\lambda,\alpha,\beta}\mathbf{f}(z))' = (\lambda+1)\mathbf{H}^{\gamma,\delta}_{\lambda+1,\alpha,\beta}\mathbf{f}(z) - \lambda\mathbf{H}^{\gamma,\delta}_{\lambda,\alpha,\beta}\mathbf{f}(z), \qquad (1.7)$$

and

$$\alpha z (\mathbf{H}^{\gamma,\delta}_{\lambda,\alpha,\beta}\mathbf{f}(z))' = (\alpha+\beta)\mathbf{H}^{\gamma,\delta}_{\lambda,\alpha,\beta+1}\mathbf{f}(z) - \beta\,\mathbf{H}^{\gamma,\delta}_{\lambda,\alpha,\beta}\mathbf{f}(z). \qquad (1.8)$$

For $\phi,\psi \in S,\ \lambda > -1,\ 0 \le \eta,\gamma < 1$ and $\mathbf{H}^{\gamma,\delta}_{\lambda,\alpha,\beta}\mathbf{f}(z)$, let

$$\mathbf{S}^{\gamma,\delta}_{\lambda,\alpha,\beta}(\eta;\phi) = \left\{\mathbf{f} \in \mathbf{A} :\ \mathbf{H}^{\gamma,\delta}_{\lambda,\alpha,\beta}\mathbf{f}(z) \in \mathbf{S}(\eta;\phi)\right\};$$

$$\mathbf{K}^{\gamma,\delta}_{\lambda,\alpha,\beta}(\eta;\phi) = \left\{\mathbf{f} \in \mathbf{A} :\ \mathbf{H}^{\gamma,\delta}_{\lambda,\alpha,\beta}\mathbf{f}(z) \in \mathbf{K}(\eta;\phi)\right\};$$

$$\mathbf{C}_{\lambda,\alpha,\beta}^{\gamma,\delta}(\eta,\xi;\phi,\psi)=\left\{\mathbf{f}\in\mathbf{A}:\ \mathbf{H}_{\lambda,\alpha,\beta}^{\gamma,\delta}\mathbf{f}(z)\in\ \mathbf{C}(\eta,\xi;\phi,\psi)\right\};$$

and

$$\mathrm{C}_{\lambda,\alpha,\beta}^{\gamma,\delta}(\eta,\xi;\phi,\psi)=\left\{\mathbf{f}\in\mathbf{A}:\ \mathbf{H}_{\lambda,\alpha,\beta}^{\gamma,\delta}\mathbf{f}(z)\in\ \mathrm{C}(\eta,\xi;\phi,\psi)\right\}.$$

Note that

$$\mathbf{f}(z)\in\mathbf{K}_{\lambda,\alpha,\beta}^{\gamma,\delta}(\eta;\phi)\Leftrightarrow z\,\mathbf{f}'(z)\in\mathbf{S}_{\lambda,\alpha,\beta}^{\gamma,\delta}(\eta;\phi) \tag{1.9}$$

and

$$\mathbf{f}(z)\in\mathrm{C}_{\lambda,\alpha,\beta}^{\gamma,\delta}(\eta;\phi)\Leftrightarrow z\,\mathbf{f}'(z)\in\mathbf{C}_{\lambda,\alpha,\beta}^{\gamma,\delta}(\eta;\phi). \tag{1.10}$$

In particular, set

$$\mathbf{S}_{\lambda,\alpha,\beta}^{\gamma,\delta}(\eta;\frac{1+Az}{1+Bz})=\mathbf{S}_{\lambda,\alpha,\beta}^{\gamma,\delta}(\eta;A,B)\quad(-1\le B<A\le 1)$$

and

$$\mathbf{K}_{\lambda,\alpha,\beta}^{\gamma,\delta}(\eta;\frac{1+Az}{1+Bz})=\mathbf{K}_{\lambda,\alpha,\beta}^{\gamma,\delta}(\eta;A,B)\quad(-1\le B<A\le 1).$$

Inclusion properties involving the operator $\mathbf{H}_{\lambda,\alpha,\beta}^{\gamma,\delta}\mathbf{f}(z)$

The following lemmas will be required in our investigation.

Lemma 1 [4]**.** Let φ be convex univalent in U with $\varphi(0)=1$ and $\mathrm{Re}\{\kappa\varphi(z)+\nu\}>0\ \ (\kappa,\nu\in\mathrm{C})$. If p is analytic in U with $p(0)=1,$ then

$$p(z)+\frac{z\,p'(z)}{\kappa\,p(z)+\nu}\prec\varphi(z)\Rightarrow p(z)\prec\varphi(z)$$

Lemma 2 [14]. Let φ be convex univalent in U and ω be analytic in U with $\mathrm{Re}\{\omega(z)\}\ge 0.$ If p is analytic in U and $p(0)=\varphi(0)$, then

$$p(z)+\omega(z)zp'(z)\prec\varphi(z)\Rightarrow p(z)\prec\varphi(z)$$

Theorem 1. Let $\mathrm{Re}\{\frac{\gamma}{\delta}\},\ \lambda>0$ and $\phi\in S.$ Then

$$\mathbf{S}_{\lambda+1,\alpha,\beta}^{\gamma,\delta}(\eta;\phi)\subset\mathbf{S}_{\lambda,\alpha,\beta}^{\gamma,\delta}(\eta;\phi)\subset\mathbf{S}_{\lambda,\alpha,\beta}^{\gamma+1,\delta}(\eta;\phi).$$

Proof: Let $\mathbf{f}\ \in\mathbf{S}_{\lambda+1,\alpha,\beta}^{\gamma,\delta}(\eta;\phi)$ and

$$p(z)=\frac{1}{1-\eta}\left(\frac{z(\mathbf{H}^{\gamma,\delta}_{\lambda,\alpha,\beta}\mathbf{f}(z))'}{\mathbf{H}^{\gamma,\delta}_{\lambda,\alpha,\beta}\mathbf{f}(z)}-\eta\right). \tag{2.1}$$

Applying (1.7) in (2.1), we obtain

$$\frac{1}{1-\eta}\left(\frac{z(\mathbf{H}^{\gamma,\delta}_{\lambda+1,\alpha,\beta}\mathbf{f}(z))'}{\mathbf{H}^{\gamma,\delta}_{\lambda+1,\alpha,\beta}\mathbf{f}(z)}-\eta\right)=p(z)+\frac{zp'(z)}{(1-\eta)\,p(z)+\lambda+\eta}. \tag{2.2}$$

Since $\lambda>0$ and $\phi\in S,$ we see that

$$\operatorname{Re}\{(1-\eta)\phi(z)+\lambda+\eta\}>0.$$

Applying Lemma 1 to (2.2), it follows that $p\prec\phi$, that is, $\mathbf{f}\in\mathbf{S}^{\gamma,\delta}_{\lambda,\alpha,\beta}(\eta;\phi).$

Using arguments similar to those detailed above with (1.6), we can prove the second part.

Theorem 2. Let $\operatorname{Re}\{\frac{\gamma}{\delta}\},\ \lambda>0$ and $\phi\in S.$ Then

$$\mathbf{K}^{\gamma,\delta}_{\lambda+1,\alpha,\beta}(\eta;\phi)\subset\mathbf{K}^{\gamma,\delta}_{\lambda,\alpha,\beta}(\eta;\phi)\subset\mathbf{K}^{\gamma+1,\delta}_{\lambda,\alpha,\beta}(\eta;\phi).$$

Proof: Applying (1.9) and Theorem 1, we observe that

$$\mathbf{f}(z)\in\mathbf{K}^{\gamma,\delta}_{\lambda+1,\alpha,\beta}(\eta;\phi)\Leftrightarrow z\,\mathbf{f}'(z)\in\mathbf{S}^{\gamma,\delta}_{\lambda+1,\alpha,\beta}(\eta;\phi)$$
$$\Rightarrow z\,\mathbf{f}'(z)\in\mathbf{S}^{\gamma,\delta}_{\lambda,\alpha,\beta}(\eta;\phi)\Leftrightarrow\mathbf{f}(z)\in\mathbf{K}^{\gamma,\delta}_{\lambda,\alpha,\beta}(\eta;\phi),$$
$$\mathbf{f}(z)\in\mathbf{K}^{\gamma,\delta}_{\lambda,\alpha,\beta}(\eta;\phi)\Leftrightarrow z\,\mathbf{f}'(z)\in\mathbf{S}^{\gamma,\delta}_{\lambda,\alpha,\beta}(\eta;\phi)$$
$$\Rightarrow z\,\mathbf{f}'(z)\in\mathbf{S}^{\gamma+1,\delta}_{\lambda,\alpha,\beta}(\eta;\phi)\Leftrightarrow\mathbf{f}(z)\in\mathbf{K}^{\gamma+1,\delta}_{\lambda,\alpha,\beta}(\eta;\phi),$$

which evidently proves Theorem 2.

Taking $\phi(z)=\dfrac{1+Az}{1+Bz}\quad(-1\le B<A\le 1;\ z\in\mathrm{U}),$ in Theorems 1 and 2, we have

Corollary 1. Let $\operatorname{Re}\{\frac{\gamma}{\delta}\},\ \lambda>0.$ Then

$$\mathbf{S}^{\gamma,\delta}_{\lambda+1,\alpha,\beta}(\eta;A,B)\subset\mathbf{S}^{\gamma,\delta}_{\lambda,\alpha,\beta}(\eta;A,B)\subset\mathbf{S}^{\gamma+1,\delta}_{\lambda,\alpha,\beta}(\eta;A,B),$$

and

$$\mathbf{K}^{\gamma,\delta}_{\lambda+1,\alpha,\beta}(\eta;A,B)\subset\mathbf{K}^{\gamma,\delta}_{\lambda,\alpha,\beta}(\eta;A,B)\subset\mathbf{K}^{\gamma+1,\delta}_{\lambda,\alpha,\beta}(\eta;A,B).$$

Theorem 3. Let $\mathrm{Re}\{\frac{\gamma}{\delta}\}$, $\lambda > 0$ and $\phi, \psi \in S$. Then

$$\mathbf{C}^{\gamma,\delta}_{\lambda+1,\alpha,\beta}(\eta,\xi;\phi,\psi) \subset \mathbf{C}^{\gamma,\delta}_{\lambda,\alpha,\beta}(\eta,\xi;\phi,\psi) \subset \mathbf{C}^{\gamma+1,\delta}_{\lambda,\alpha,\beta}(\eta,\xi;\phi,\psi).$$

Proof: To prove the first inclusion, let $\mathbf{f} \in \mathbf{C}^{\gamma,\delta}_{\lambda+1,\alpha,\beta}(\eta,\xi;\phi,\psi)$. Then, from the definition of $\mathbf{C}^{\gamma,\delta}_{\lambda+1,\alpha,\beta}(\eta,\xi;\phi,\psi)$, $\exists\ g \in \mathbf{S}^{\gamma,\delta}_{\lambda+1,\alpha,\beta}(\eta;\phi)$ such that

$$\frac{1}{1-\xi}\left(\frac{z(\mathbf{H}^{\gamma,\delta}_{\lambda+1,\alpha,\beta}\mathbf{f}(z))'}{\mathbf{H}^{\gamma,\delta}_{\lambda+1,\alpha,\beta}g(z)} - \xi\right) \prec \psi(z).$$

Now, let

$$p(z) = \frac{1}{1-\xi}\left(\frac{z(\mathbf{H}^{\gamma,\delta}_{\lambda,\alpha,\beta}\mathbf{f}(z))'}{\mathbf{H}^{\gamma,\delta}_{\lambda,\alpha,\beta}g(z)} - \xi\right), \tag{2.3}$$

using (1.7), we obtain

$$(1-\xi)zp'(z)\mathbf{H}^{\gamma,\delta}_{\lambda,\alpha,\beta}g(z) + [(1-\xi)p(z)+\xi]z(\mathbf{H}^{\gamma,\delta}_{\lambda,\alpha,\beta}g(z))'$$
$$= (\lambda+1)z(\mathbf{H}^{\gamma,\delta}_{\lambda+1,\alpha,\beta}\mathbf{f}(z))' - \lambda z(\mathbf{H}^{\gamma,\delta}_{\lambda,\alpha,\beta}\mathbf{f}(z))'. \tag{2.4}$$

Since $g \in \mathbf{S}^{\gamma,\delta}_{\lambda+1,\alpha,\beta}(\eta;\phi) \subset \mathbf{S}^{\gamma,\delta}_{\lambda,\alpha,\beta}(\eta;\phi)$, by Theorem 1, we set

$$q(z) = \frac{1}{1-\eta}\left(\frac{z(\mathbf{H}^{\gamma,\delta}_{\lambda,\alpha,\beta}g(z))'}{\mathbf{H}^{\gamma,\delta}_{\lambda,\alpha,\beta}g(z)} - \eta\right) \prec \phi(z). \tag{2.5}$$

By using (1.7) again, we obtain

$$(\lambda+1)\frac{\mathbf{H}^{\gamma,\delta}_{\lambda+1,\alpha,\beta}g(z)}{\mathbf{H}^{\gamma,\delta}_{\lambda,\alpha,\beta}g(z)} = (1-\eta)q(z)+\lambda+\eta. \tag{2.6}$$

From (2.4) and (2.6), we get

$$\frac{1}{1-\xi}\left(\frac{z(\mathbf{H}^{\gamma,\delta}_{\lambda+1,\alpha,\beta}\mathbf{f}(z))'}{\mathbf{H}^{\gamma,\delta}_{\lambda+1,\alpha,\beta}g(z)} - \xi\right) = p(z) + \frac{zp'(z)}{(1-\eta)q(z)+\lambda+\eta} \prec \psi(z). \tag{2.7}$$

Since $\lambda > 0$ and $q \prec \phi$, $\mathrm{Re}\{(1-\eta)q(z)+\lambda+\eta\} > 0$. Hence, applying Lemma 2, we have $p \prec \psi$, so that $\mathbf{f} \in \mathbf{C}^{\gamma,\delta}_{\lambda,\alpha,\beta}(\eta,\xi;\phi,\psi)$. Similarly the second inclusion can be obtained by using (1.6).

Similarly, we can prove the following theorem.

Theorem 4. Let $\mathrm{Re}\{\frac{\gamma}{\delta}\},\ \lambda > 0$ and $\phi,\psi \in S.$ Then

$$\mathsf{C}^{\gamma,\delta}_{\lambda+1,\alpha,\beta}(\eta,\xi;\phi,\psi) \subset \mathsf{C}^{\gamma,\delta}_{\lambda,\alpha,\beta}(\eta,\xi;\phi,\psi) \subset \mathsf{C}^{\gamma+1,\delta}_{\lambda,\alpha,\beta}(\eta,\xi;\phi,\psi).$$

By using (1.8), we can prove the following theorem.

Theorem 5. Let Let $\mathrm{Re}\{\frac{\beta}{\alpha}\},\ \lambda > 0$ and $\phi,\psi \in S.$ Then

$$\mathbf{S}^{\gamma,\delta}_{\lambda,\alpha,\beta+1}(\eta;\phi) \subset \mathbf{S}^{\gamma,\delta}_{\lambda,\alpha,\beta}(\eta;\phi),\ \mathbf{K}^{\gamma,\delta}_{\lambda,\alpha,\beta+1}(\eta;\phi) \subset \mathbf{K}^{\gamma,\delta}_{\lambda,\alpha,\beta}(\eta;\phi), \tag{2.8}$$

$$\mathbf{C}^{\gamma,\delta}_{\lambda,\alpha,\beta+1}(\eta,\xi;\phi,\psi) \subset \mathbf{C}^{\gamma,\delta}_{\lambda,\alpha,\beta}(\eta,\xi;\phi,\psi),$$ and

$$\mathsf{C}^{\gamma,\delta}_{\lambda,\alpha,\beta+1}(\eta,\xi;\phi,\psi) \subset \mathsf{C}^{\gamma,\delta}_{\lambda,\alpha,\beta}(\eta,\xi;\phi,\psi).$$

Taking $\phi(z) = \frac{1+Az}{1+Bz} \quad (-1 \le B < A \le 1;\ z \in \mathsf{U}),$ in (2.8), we have

Corollary 2. Let $\mathrm{Re}\{\frac{\beta}{\alpha}\},\ \lambda > 0.$ Then

$$\mathbf{S}^{\gamma,\delta}_{\lambda,\alpha,\beta+1}(\eta;A,B) \subset \mathbf{S}^{\gamma,\delta}_{\lambda,\alpha,\beta}(\eta;A,B),$$ and

$$\mathbf{K}^{\gamma,\delta}_{\lambda,\alpha,\beta+1}(\eta;A,B) \subset \mathbf{K}^{\gamma,\delta}_{\lambda,\alpha,\beta}(\eta;A,B).$$

Inclusion properties involving the integral operator F_μ

The generalized Libera integral operator F_μ (cf. [2], [9] and [10]) is defined by

$$F_\mu(\mathbf{f}) = F_\mu(\mathbf{f})(z) = \frac{\mu+1}{z^\mu}\int_0^z t^{\mu-1}\mathbf{f}(t)dt \quad (f \in \mathbf{A};\ \mu > -1) \tag{3.1}$$

and satisfies

$$z(\mathbf{H}^{\gamma,\delta}_{\lambda,\alpha,\beta}F_\mu(\mathbf{f})(z))' = (\mu+1)\mathbf{H}^{\gamma,\delta}_{\lambda,\alpha,\beta}\mathbf{f}(z) - \mu\mathbf{H}^{\gamma,\delta}_{\lambda,\alpha,\beta}F_\mu(\mathbf{f})(z). \tag{3.2}$$

Theorem 6. If $\mathbf{f}$ belongs to $\mathbf{S}^{\gamma,\delta}_{\lambda,\alpha,\beta}(\eta;\phi),$ then $F_\mu(\mathbf{f})$ belongs to

$\mathbf{S}_{\lambda,\alpha,\beta}^{\gamma,\delta}(\eta;\phi) \quad (\mu \geq 0).$

Proof: Let $\mathbf{f} \in \mathbf{S}_{\lambda,\alpha,\beta}^{\gamma,\delta}(\eta;\phi)$ and set

$$p(z)=\frac{1}{1-\eta}\left(\frac{z(\mathbf{H}_{\lambda,\alpha,\beta}^{\gamma,\delta}F_{\mu}(\mathbf{f})(z))'}{\mathbf{H}_{\lambda,\alpha,\beta}^{\gamma,\delta}F_{\mu}(\mathbf{f})(z)}-\eta\right), \tag{3.3}$$

then p is analytic in $\mathbb{U}$ with $p(0)=1$. Using (3.2) and (3.3), we obtain

$$(\mu+1)\frac{\mathbf{H}_{\lambda,\alpha,\beta}^{\gamma,\delta}\mathbf{f}(z)}{\mathbf{H}_{\lambda,\alpha,\beta}^{\gamma,\delta}F_{\mu}(\mathbf{f})(z)}=(1-\eta)p(z)+\mu+\eta. \tag{3.4}$$

Taking the logarithmic differentiation on both sides of (3.4) and multiplying by z, we have

$$\frac{1}{1-\eta}\left(\frac{z(\mathbf{H}_{\lambda,\alpha,\beta}^{\gamma,\delta}\mathbf{f}(z))'}{\mathbf{H}_{\lambda,\alpha,\beta}^{\gamma,\delta}\mathbf{f}(z)}-\eta\right)=p(z)+\frac{zp'(z)}{(1-\eta)p(z)+\mu+\eta}.$$

Hence, by virtue of Lemma 1, we conclude that $p \prec \phi$, which implies $F_{\mu}(\mathbf{f}) \in \mathbf{S}_{\lambda,\alpha,\beta}^{\gamma,\delta}(\eta;\phi).$

Theorem 7. If $\mathbf{f} \in \mathbf{K}_{\lambda,\alpha,\beta}^{\gamma,\delta}(\eta;\phi),$ then $F_{\mu}(\mathbf{f}) \in \mathbf{K}_{\lambda,\alpha,\beta}^{\gamma,\delta}(\eta;\phi)\ (\mu \geq 0).$

Proof: By applying Theorem 6, we have

$$\begin{aligned}
\mathbf{f}(z) \in \mathbf{K}_{\lambda,\alpha,\beta}^{\gamma,\delta}(\eta;\phi) &\Leftrightarrow z\mathbf{f}'(z) \in \mathbf{S}_{\lambda,\alpha,\beta}^{\gamma,\delta}(\eta;\phi)\\
&\Rightarrow F_{\mu}(z\mathbf{f}'(z)) \in \mathbf{S}_{\lambda,\alpha,\beta}^{\gamma,\delta}(\eta;\phi)\\
&\Leftrightarrow z(F_{\mu}(\mathbf{f})(z))' \in \mathbf{S}_{\lambda,\alpha,\beta}^{\gamma,\delta}(\eta;\phi)\\
&\Leftrightarrow F_{\mu}(\mathbf{f})(z) \in \mathbf{K}_{\lambda,\alpha,\beta}^{\gamma,\delta}(\eta;\phi),
\end{aligned}$$

which proves Theorem 7.

From Theorems 6 and 7, we have

Corollary 3. If $\mathbf{f}$ belongs to $\mathbf{S}_{\lambda,\alpha,\beta}^{\gamma,\delta}(\eta;A,B)$ (or $\mathbf{K}_{\lambda,\alpha,\beta}^{\gamma,\delta}(\eta;A,B)$), then $F_{\mu}(\mathbf{f})$ belongs to $\mathbf{S}_{\lambda,\alpha,\beta}^{\gamma,\delta}(\eta;A,B)$ (or $\mathbf{K}_{\lambda,\alpha,\beta}^{\gamma,\delta}(\eta;A,B)$) $(\mu \geq 0)$.

Theorem 8. If $\mathbf{f}$ belongs to $\mathbf{C}_{\lambda,\alpha,\beta}^{\gamma,\delta}(\eta,\xi;\phi,\psi)$, then $F_{\mu}(\mathbf{f})$ belongs to $\mathbf{C}_{\lambda,\alpha,\beta}^{\gamma,\delta}(\eta,\xi;\phi,\psi)$ $\quad(\mu \geq 0)$.

Proof: Let $\mathbf{f} \in \mathbf{C}_{\lambda,\alpha,\beta}^{\gamma,\delta}(\eta,\xi;\phi,\psi)$, then, $g \in \mathbf{S}_{\lambda,\alpha,\beta}^{\gamma,\delta}(\eta;\phi)$ such that

$$\frac{1}{1-\xi}\left(\frac{z(\mathbf{H}_{\lambda,\alpha,\beta}^{\gamma,\delta}\mathbf{f}(z))'}{\mathbf{H}_{\lambda,\alpha,\beta}^{\gamma,\delta}g(z)}-\xi\right) \prec \psi(z).$$

Thus, we set

$$p(z)=\frac{1}{1-\xi}\left(\frac{z(\mathbf{H}_{\lambda,\alpha,\beta}^{\gamma,\delta}F_{\mu}(\mathbf{f})(z))'}{\mathbf{H}_{\lambda,\alpha,\beta}^{\gamma,\delta}F_{\mu}(g)(z)}-\xi\right),$$

where p is analytic in U with $p(0)=1$. Since $g \in \mathbf{S}_{\lambda,\alpha,\beta}^{\gamma,\delta}(\eta;\phi)$, we see from Theorem 6 that $F_{\mu}(g) \in \mathbf{S}_{\lambda,\alpha,\beta}^{\gamma,\delta}(\eta;\phi)$. Using (3.2), we have

$$[(1-\xi)p(z)+\xi]\mathbf{H}_{\lambda,\alpha,\beta}^{\gamma,\delta}F_{\mu}(g)(z)+\mu\mathbf{H}_{\lambda,\alpha,\beta}^{\gamma,\delta}F_{\mu}(\mathbf{f})(z)$$
$$=(\mu+1)\mathbf{H}_{\lambda,\alpha,\beta}^{\gamma,\delta}\mathbf{f}(z).$$

Then, by a simple calculation,

$$(\mu+1)\frac{z(\mathbf{H}_{\lambda,\alpha,\beta}^{\gamma,\delta}\mathbf{f}(z))'}{\mathbf{H}_{\lambda,\alpha,\beta}^{\gamma,\delta}F_{\mu}(g)(z)}$$
$$=[(1-\xi)p(z)+\xi][(1-\eta)q(z)+\mu+\eta]+(1-\xi)zp'(z),$$

where

$$q(z)=\frac{1}{1-\eta}\left(\frac{z(\mathbf{H}_{\lambda,\alpha,\beta}^{\gamma,\delta}F_{\mu}(g)(z))'}{\mathbf{H}_{\lambda,\alpha,\beta}^{\gamma,\delta}F_{\mu}(g)(z)}-\eta\right).$$

Hence, we have

$$\frac{1}{1-\xi}\left(\frac{z(\mathbf{H}_{\lambda,q,s}(\alpha_1)\mathbf{f}(z))'}{\mathbf{H}_{\lambda,q,s}(\alpha_1)g(z)}-\xi\right)=p(z)+\frac{zp'(z)}{(1-\eta)q(z)+\mu+\eta}.$$

The remaining part of this proof is similar to that of Theorem 3, so we omit it.

Theorem 9. If $\mathbf{f}\in \mathrm{C}^{\gamma,\delta}_{\lambda,\alpha,\beta}(\eta,\xi;\phi,\psi)$, then $F_{\mu}(\mathbf{f})\in \mathrm{C}^{\gamma,\delta}_{\lambda,\alpha,\beta}(\eta,\xi;\phi,\psi) \quad (\mu\geq 0)$.

Proof: As in Theorem 7. The proof follows by using Theorem 8.

Remark 1. If we take $\mathrm{Re}(\alpha)=0$ and $\lambda=0$ in the above Theorems, we get results that depend on $\mathbf{f}(z)$ instead of $H^{\gamma,\delta}_{\lambda,\alpha,\beta}\mathbf{f}(z)$.

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