On An Elliptic Equation of P-Kirchhoff Type With Convection Term And Singular Weights.

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Abstract: We suppose that Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$. Consider the quasi-linear elliptic equation of p-Kirchhoff type with convection term singular weights,

$$M\left(\int_\Omega \frac{1}{d(x)^\alpha}|\nabla u|^p\right)\Delta_p u = \frac{1}{d(x)^\beta}h(x,u) + \frac{1}{d(x)^\gamma}g(x,\nabla u)$$

in Ω. We proved the existence of solutions to a class of elliptic problems, by using Galerkins approach with a priori estimates.

Keywords: Existence; p-laplacian; singularweight.

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1. Introduction

The main objective of this work is to present some results about the existence of nonnegative solutions to the following problem

$$\begin{cases} -M\left(\int_\Omega \frac{1}{d(x)^\alpha}|\nabla u|^p\right)\Delta_p u = \frac{1}{d(x)^\beta}h(x,u) + \frac{1}{d(x)^\gamma}g(x,\nabla u) & in\,\Omega \\ u = 0 & on\,\partial\Omega \end{cases} \quad (1.1)$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$ and the operator $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$ with $1 < p <= \infty$

and the functional M verifies

$(\boldsymbol{H_1})$ $M : (0,+\infty) \longrightarrow (0,+\infty)$ continuous and $m_0 = inf_{s>0} M(s) > 0$.

We suppose that h is sublinear function and g is bounded by a gradient term. For these functions, we set these hypotheses:

$(\boldsymbol{H_2})$ $h : \Omega \times R \to R$ is locally Holder continuous, there exist positive constants

$a_1 \in L^{p'}\left(\Omega, \frac{1}{d^\beta}\right)$ $b_1 \in L^{\frac{p}{p-(r_1+1)}}\left(\Omega, \frac{1}{d^\beta}\right)$, and $r_1 \in (0, p-1)$ such that

$$h(x,t) \leq a_1 + b_1|t|^{r_1} \quad \forall (x,t) \in \Omega \times \mathbb{R}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

(***H*₃**) $g:\Omega\times R^N\to R$ is locally Holder continuous, there exist positive constants $a_2\in L^{p'}\left(\Omega,\frac{1}{d^{\gamma}}\right)$ $b_2\in L^{\frac{p}{p-r_2}}\left(\Omega,\frac{1}{d^{\gamma}}\right)\cap L^{\infty}(\Omega)$ and $r_2\in(0,p-1)$ such that

$$g(x,\delta)\le a_2+b_2|\delta|^{r_2}\qquad\forall(x,\delta)\in\Omega\times\mathbb{R}^n$$

In recent years, more and more attention has been paid to the existence of solutions for semilinear degenerate elliptic problems. For the Results relating to these problems considered over a bounded smooth domain **Ω** instead of $\mathbb{R}^N$, the corresponding problem was studied, for instance, in [**1**, **2**] and the references cited therein. For degenerate partial differential equations, i.e., equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces [**1**] and references therein.

Problems such as (**1.1**) are knowing in the literature is called the Kirchhoff equation. Equations of the problems (**1.1**) appears in a variety of applications, such as the modelling of electrorheological fluids, elasticity problems, image processing, mathematical description of the processes filtration of an ideal barotropic gas through a porous medium, etc.; see [**8**, **9**, **12**] for more details. Because of the term

$$M\left(\int_{\Omega}\frac{1}{d(x)^{\alpha}}|\nabla u|^p\,dx\right)$$

a problem like (**1.1**) is nonlocal.

The Kirchhoff type equations arise in the description of nonlinear vibrations of an elastic string, see Kirchhoff [**3**]. In recent years, much interest has grown on *p*-Kirchhoff type problems of the kind of semilinear degenerate elliptic kirchhoff type problems of the form (**1.1**) with nonlinearity on the right-hand side which also depends on the gradient of the solution have been extensively studied by many authors, using various methods. The gradient terms make the study of solvability more complicated because of some difficulties arising in the use of methods of calculus of variations.

Let us first note that several authors have started by studing extensively the semi-linear case ($p = 2$); Among these works we quote for example [**8**, **10**, **11**, **12**, **13**]. Later, research intersted in these problem for the case $p > 2$, we refer the reader to [**9**, **14**, **15**, **16**]. In [**16**], A.Ourraoui considered the following problem

$$\begin{cases}-M\left(\int_{\Omega}\frac{1}{d(x)^{\alpha}}|\nabla u|^p\right)\Delta_p u=f(x,u,\nabla u) & in\ \Omega\\ \qquad\qquad u=0 & on\ \partial\Omega\end{cases}\qquad(1.2)$$

the author proved the existence of the positive solution of the problem (**1.2**) by used Galerkins approach with a priori estimates.

The main result is stated below

Theorem 1.1. *Suppose* (***H*₁**) − (***H*₃**). *Then, the problem* ***(1.1)*** *admits at least one nontrivial weak solution.*

2.Preliminaries

Let $p > 1$ and for $\alpha\in\mathbb{R}$ we let

$$L^p\left(\Omega,\frac{1}{d^{\alpha}}\right)=\left\{u=u(x)\quad x\in\Omega/\quad\int_{\Omega}\frac{1}{d^{\alpha}}|u|^p\mathrm{dx}<\infty\right\}$$

to be the weighted Lebesgue space with the norm

$$\|u\|_{p,\alpha}=\left(\int_{\Omega}\frac{1}{d^{\alpha}}|u|^p\mathrm{dx}\right)^{\frac{1}{p}}$$

and

$$W^{1,p}\left(\Omega,\frac{1}{d^\alpha}\right)=\left\{u\in L^p\left(\Omega,\frac{1}{d^\alpha}\right)\quad/\quad\int_\Omega\frac{1}{d^\alpha}\left[|\nabla u|^p+|u|^p\right]\mathrm{dx}<\infty\right\}$$

to be the weighted Sobolev space with the norm

$$\|u\|_{1,p;\alpha}=\left(\int_\Omega\frac{1}{d^\alpha}\left[|\nabla u|^p+|u|^p\right]\mathrm{dx}\right)^{\frac{1}{p}}$$

We also define $W_0^{1,p}\left(\Omega,\frac{1}{d^\alpha}\right)\subset W^{1,p}\left(\Omega,\frac{1}{d^\alpha}\right)$ to be a closure of the set $C_0^\infty(\Omega)$

(smooth functions with compact support in Ω) with respect to the norm $\|.\|_{1,p;\alpha}$.

Lemma 2.1. *[Continuous embedding]*
Let $\alpha\geq 0$, $0\leq\beta\leq\alpha+p$. *Then*

$$W_0^{1,p}\left(\Omega,\frac{1}{d^\alpha}\right)\hookrightarrow W_0^{1,p}\left(\Omega,\frac{1}{d^\beta}\right)$$

Proof. Follows from Kufner [[**4**], Sec. 8.8 on p. 57].

Lemma 2.2. *[Continuous embedding]*
Let $\alpha\geq 0$, $1<p<n$. *Then*

$$W_0^{1,p}\left(\Omega,\frac{1}{d^\alpha}\right)\hookrightarrow W_0^{1,p}(\Omega)\hookrightarrow L^{p*}(\Omega)\qquad p*=\frac{np}{n-p}$$

Proof. The first embedding in **2.1** holds due to $\alpha\geq 0$ and Ω bounded.

The second one is a well-known fact (see, e.g., Pick et al. [**5**]).

Lemma 2.3. *[Compact embedding]*
Let $0\leq\gamma<\alpha+p$. *Then*

$$W_0^{1,p}\left(\Omega,\frac{1}{d^\alpha}\right)\hookrightarrow L^p\left(\Omega,\frac{1}{d^\gamma}\right)\hookrightarrow L^{p*}(\Omega)$$

Proof. Follows from P. Drbek, J. Hernndez [[**1**], Sec. 2 on Proposition 2.3].

We denote $X=W_0^{1,p}\left(\Omega,\frac{1}{d^\alpha}\right)$ and and introduce an equivalent norm

$$\|u\|_X=\left(\int_\Omega\frac{1}{d^\alpha}|\nabla u|^p\mathrm{dx}\right)^{\frac{1}{p}}$$

Then $\left(X,\|.\|_{1,p;\alpha}\right)$ is a uniformly convex (and hence reflexive) Banach space.

Lemma 2.4. *[See* [**7**]*] Let F:* $\mathbb{R}^n\longrightarrow\mathbb{R}^n$ *be a continuous function with* $(F(x),x)\geq 0$, *for all x verifying*

$|x|=R>0$, *where* $(,)$ *is the usual inner product of* $\mathbb{R}^n$. *Then there exists* $\gamma\in B_R(0)$ *such that* $F(\gamma)=0$.

3.Proof of the main result

We recall that $u\in X$ is a weak solution to the problem (**1.1**) if it verifies

$$M\left(\|u\|_X^p\right)\int_\Omega\frac{1}{d^\alpha}|\nabla u|^{p-2}\nabla u\nabla v dx-\int_\Omega\frac{1}{d^\beta}h(x,u)vdx-\int_\Omega\frac{1}{d(x)^\gamma}g(x,\nabla u)vdx=0\qquad\forall v\in X$$

Proof of Theorem 1.1.
Let $\kappa=\{e_1,e_2,...,e_n,...\}\subset X$ such that

$$X=\overline{\mathbf{span}\{e_1,e_2,...,e_n\}}$$

Define $\vartheta_n=\{e_1,e_2,...,e_n\}$. It is known that ϑ_n and $\mathbb{R}^n$ are isomorphic and for $\xi\in\mathbb{R}^n$, we have an unique $v\in\vartheta_n$

by the identification:

$$\phi : \xi \longrightarrow \sum_{i=1}^{n} \xi_i e_i = v$$

Define the function $F = (F_1, F_2, ..., F_n) : \mathrm{R}^n \longrightarrow \mathrm{R}$ by

$$F_i(u) = M\left(\|u\|_X^p\right)\int_\Omega \frac{1}{d^\alpha}|\nabla u|^{p-2}\nabla u \nabla e_i dx - \int_\Omega \frac{1}{d^\beta} h(x,u)e_i dx - \int_\Omega \frac{1}{d(x)^\gamma} g(x,\nabla u)e_i dx \qquad u \in \vartheta_i$$

We show the existence of weak solutions $u_n \in \vartheta_n$ for the problem

$$M\left(\|u_n\|_X^p\right)\int_\Omega \frac{1}{d^\alpha}|\nabla u_n|^{p-2}\nabla u_n \nabla e_i dx = \int_\Omega \frac{1}{d^\beta} h(x,u_n)e_i dx + \int_\Omega \frac{1}{d(x)^\gamma} g(x,\nabla u_n)e_i dx \qquad \forall\left(u_n \in \vartheta_i\right)(3.1)$$

For $u_n \in \vartheta_n$, we have that

$$\begin{aligned}(F(u_n), u_n) &= M\left(\int_\Omega \frac{1}{p(x)}\frac{1}{d^\alpha}|\nabla u_n|^{p(x)}\right)\int_\Omega \frac{1}{d^\alpha}|\nabla u_n|^{p(x)} \\ &\quad - \int_\Omega \frac{1}{d^\beta} h(x,u_n)u_n \mathrm{dx} - \int_\Omega \frac{1}{d^\gamma} g(x,\nabla u_n)u_n \\ &\geq m_0\|u_n\|_X^p - \int_\Omega \frac{1}{d^\beta} h(x,u_n).u_n \mathrm{dx} - \int_\Omega \frac{1}{d^\gamma} g(x,\nabla u_n).u_n \mathrm{dx}.\end{aligned}$$

According to condition **(H_2)** and **(H_3)** we have

$$\begin{aligned}\int_\Omega \frac{1}{d^\beta} h(x,u_n)u_n &\leq |a_1|_{p';\beta}|u_n|_{p;\beta} + \|b_1\|_{\frac{p}{p-(r_1+1)};\beta}|u_n|_{p;\beta}^{r_1+1} \\ &\leq k_1|a_1|_{p';\beta}\|u_n\|_X + k_2|b_1|_{\frac{p}{p-(r_1+1)};\beta}\|u_n\|_X^{r_1+1}.\end{aligned}$$

And

$$\begin{aligned}\int_\Omega \frac{1}{d^\gamma} g(x,\nabla u_n)u_n &\leq |a_2|_{p';\gamma}|u_n|_{p;\gamma} + \|b_2\|_{\frac{p}{p-r_2};\gamma}|u_n|_{p;\gamma}^{r_2+1} \\ &\leq k_3|a_2|_{p';\gamma}\|u_n\|_X + k_4|b_2|_{\frac{p}{p-r_2};\gamma}\|u_n\|_X^{r_2+1}.\end{aligned}$$

Then

$$\begin{aligned}(F(u_n), u_n) &\geq m_0\|u_n\|_X^p - \int_\Omega \frac{1}{d^\beta} h(x,u_n).u_n \mathrm{dx} - \int_\Omega \frac{1}{d^\gamma} g(x,\nabla u_n).u_n \\ &\geq m_0\|u_n\|_X^p - k_1|a_1|_{p';\beta}\|u_n\|_X - k_2|b_1|_{\frac{p}{p-(r_1+1)};\beta}\|u_n\|_X^{r_1+1} \\ &\quad - k_3|a_2|_{p';\gamma}\|u_n\|_X - k_4|b_2|_{\frac{p}{p-r_2};\gamma}\|u_n\|_X^{r_2+1}.\end{aligned}$$

Since $r_i + 1 < p, \quad i = 1, 2$, there exist positive numbers ρ and R such that

$$\left(F(u_n), u_n\right) \geq \rho > 0 \qquad \text{on} \qquad \|u\|_{1,p;\alpha} = R.$$

F is continuous, by lemma **2.4**, the equation (3.1) has a solution u_n in $\vartheta_n \subset X$ with $\|u\|_{1,p;\alpha} \leq R$.

we may assume that there exists $u \in X$ such that

$$u_n \rightharpoonup u \text{ in } X,$$

$$u_n \to u \qquad a.e. \qquad x \in \Omega$$

By using the Dominated Convergence Theorem, we get

$$h(x,u_n)wdx \to h(x,u)wdx \qquad for \quad w \in \vartheta_n$$

In view of condition (**H_3**), we have

$$\|g(x,\nabla u_n)\|_{\frac{p}{r_2},\gamma} \le \left(\int_\Omega \frac{1}{d^\gamma}|a_2+b_2|\nabla u_n|^{r_2}|^{\frac{p}{r_2}}\mathrm{dx}\right)^{\frac{r_2}{p}}$$

$$\Rightarrow \|g(x,\nabla u_n)\|_{\frac{p}{r_2},\gamma} \le \left(\int_\Omega \frac{1}{d^\gamma}|a_2|^{\frac{p}{r_2}}\mathrm{dx}\right)^{\frac{r_2}{p}} + \left(\int_\Omega \frac{1}{d^\gamma}|b_2|^{\frac{p}{r_2}}|\nabla u_n|^p\mathrm{dx}\right)^{\frac{r_2}{p}}$$

$$\Rightarrow \|g(x,\nabla u_n)\|_{\frac{p}{r_2},\gamma} \le \left(\|a_2\|_{\frac{p}{r_2},\gamma} + m_5||b_2|^{\frac{p}{r_2}}|_\infty.\|u\|_X^{r_2}\right)^{\frac{r_2}{p}}$$

implique

$$\|g(x,\nabla u_n)\|_{\frac{p}{r_2},\gamma} \le \left(\|a_2\|_{\frac{p}{r_2},\gamma} + m_5||b_2|^{\frac{p}{r_2}}|_\infty.R^{r_2}\right)^{\frac{r_2}{p}} < \infty$$

where $m_5 = \underset{x\in\Omega_\sigma}{Max}\, d^{\alpha-\gamma}$.

From the reflexivity of $L^{\frac{r_2}{P}}\left(\Omega,\frac{1}{d^\gamma}\right)$, passing to a subsequence if necessary;

there is $g\left(x,\nabla u\right) \in L^{\frac{r_2}{P}}\left(\Omega,\frac{1}{d^\gamma}\right)$ such that

$$\int_\Omega \frac{1}{d(x)^\gamma} g(x,\nabla u_n)\varphi dx \to \int_\Omega \frac{1}{d(x)^\gamma} g(x,\nabla u)\varphi dx \quad \forall \varphi \in L^q\left(\Omega,\frac{1}{d^\gamma}\right)$$

with $\frac{r_2}{p}+\frac{1}{q}=1$.

We have

$$M\left(\|u_n\|_X^p\right)\int_\Omega \frac{1}{d^\alpha}|\nabla u_n|^{p-2}\nabla u_n \nabla\varphi dx = \int_\Omega \frac{1}{d^\beta}h(x,u_n)\varphi dx + \int_\Omega \frac{1}{d(x)^\gamma} g(x,\nabla u_n)\varphi dx \qquad \forall \varphi \in X$$

on the other hand we have $u_n \rightharpoonup u$, thus when $n \to +\infty$, we get

$$\int_\Omega \frac{1}{d^\beta}\left(h(x,u_n)-h(x,u)\right)\left(u_n-u\right)dx \to 0 \qquad (3.3)$$

Then

$$< \Delta_p u_n - \Delta_p u, u_n - u >= \int_\Omega \frac{1}{d^\alpha}\left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)\nabla\left(u_n-u\right)dx \to 0$$

From Lemma 1 in [**6**], we have

$$\gamma_0|u_n-u|^p < \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)\nabla(u_n-u)$$

we obtain

$$\|u_n - u\|_X \to 0 \qquad (3.4)$$

and then

$$u_n \to u \quad in \quad X\,(3.5)$$

M is continuous, this implies that

$$M\left(\int_\Omega \frac{1}{d^\alpha}|\nabla u_n|^p\mathrm{dx}\right) \longrightarrow M\left(\int_\Omega \frac{1}{d^\alpha}|\nabla u|^p\mathrm{dx}\right)$$

So we obtain that *u* is a weak solution to the problem (**1.1**), and from (H_1), we obtain $u \neq 0$.

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