The Stability of Fluctuation Immigration Effect on Two Species Prey-Predator System with Holling Type II Functional Response

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Abstract: This paper examines the fluctuation immigration effect on the dynamic behavior of the prey-predator system, with Holling type II functional response. The mathematical technique to formulate and combine the effect of fluctuating immigration with the prey-predator system was provided. the result of this combined is a non-autonomous system. In order to convert it to an autonomous system, we propose an approximated technique for facilitating mathematical analysis. This system has been examined for local and global stability. as well, it is tested using Kolmogorov criteria in order to verify the susceptibility to coexistence and extinction.

Keywords: prey-predator, functional response, fluctuation immigration effect, stability

1. Introduction

The term "ecosystem decay" was introduced by scientist Thomas [1]. He explained the extinction of species from their native environments. Over the long term, this process leads to the extinction of numerous species. Many reasons lead to ecosystem decay. such as the isolated ecosystem leads to the lack of local species. Furthermore, the absence of natural competition for selection will result in an individual's inability to adjust to environmental changes due to a reduction of individuals fitness. The ecosystem decay has several negative impacts on ecosystems:

• Reducing genetic diversity

•Isolating the habitat of individuals, one from the other

•Disappearing of native populations in ecosystems

•Reducing the habitable population

According to the theory of ecological systems, the rescue effect may be defined as the process of giving assistance to any ecosystem that is threatened with extinction [2]. It is possible to offer this support by either adding new individuals to the decay ecosystem or by providing the current individuals with more suitable living chances in the environment in which they reside.

in decaying ecosystems, the process of immigration (adding new individuals) is a significant element in assisting the ecosystem to achieve a state of stability, which in turn leads to an increase in the continuity of species survival in the ecosystem in question [3].

In ecosystems with two distinct species kinds, such as prey-predator systems. It is obvious that the predator's survival is directly linked to the existence of prey and at the same time, the increased predation rates pose a danger to the overall viability of this ecosystem. As a result, while managing decayed ecosystems, the effect of rescue on the prey is critical. As the inclusion of the rescue effect may stabilize the ecosystem's prey and predator populations. In other words, the rescue effect will be inversely proportional to decay rates.

The rescue effect is caused by the addition of new prey individuals into the prey-predator system can be referred to by immigration in this system. Since this addition will vary depending on the requirements of the environment, we may refer to it as the fluctuating immigration effect, which will be the subject of this study.

Many effects on the prey and predator system have been studied in recent literature. Li and Wu (2017) performed a study on the stability of the prey-predator model under the effect of prey protected areas and its impact on prey growth rate [4]. Pal, et al. (2019) investigated the effect of prey-predator model dynamics in the presence of cooperative predator hunting, and they numerically depicted the findings using MATCONT and MATLAB [5]. Sekerci investigated the effect of climate change on the stability of the prey and predator model in (2020) [6]. Emery and Mills investigated the fluctuation in population growth of the prey-predator model under the effect of predation pressure imposed by the predator on the prey in (2020)[7]. Al Basir, Tiwari, and Samanta investigated stability and bifurcation in the diseased prey-predator model utilizing Holling type II functional response in (2021)[8]. Using the HoIling type I functional response, Jayaprakasha and Baishya investigated the impact of toxicity on the prey-predator model in the same year [9]. Using a type IV HoIling functional response, Lemnaouar et al. investigated the stability of the prey-predator model under the effect of harvesting [10]. Finally, Alebraheem investigates the autonomous prey-predator model produced by Crowley–Martin functional response under the fluctuating immigration effect of prey [11]. Further studies can be seen in [12, 13, 14, 15]

In this paper, we will investigate the local and global stability of the prey-predator system under fluctuation immigration effect with with Holling type II functional response.

A mathematical model of the fluctuation immigration effect was developed and combined with the prey-predator model to create a non-autonomous system. In order to simplify the analysis of this system, an approximate technique for converting it into an autonomous one is given. The efficacy of this approach was further evaluated by adding Kolmogorov criteria of coexistence and extinction. The remainder of this document is formatted as follows.

2.Mathematical Formulation

As previously stated, the effect of providing new individuals varies according to the requirements of the ecosystem. On this basis, it may be mathematically represented using the sinusoidal function

$$\sin(\delta t)$$

Where t is the time and δ is the fluctuation's angular frequency.

To make this addition always positive, with the possibility to adjust its degree, it will be expressed as follows:

$$1+\epsilon\sin(\delta t)$$

Where ϵ is the fluctuation degree. In addition, to regulate the quantity of immigration the following option will be introduced.

$$i(1+\epsilon\sin(\delta t))$$

Where, i is a parameter that represents the number immigrated prey to the ecosystem.

By combined $i(1+\epsilon\sin(\delta t))$ with the Holling type II functional responseprey-predator model [16] we get the following non-autonomous prey-predator model under fluctuation immigration effect with Holling type II functional response

$$\begin{aligned}\frac{dX}{dt} &= rX\left(1-\frac{X}{k}\right)-\frac{\alpha XY}{1+h\alpha X}+i(1+\epsilon\sin(\delta t))\\ \frac{dY}{dt} &= -uY+\frac{e\alpha XY}{1+h\alpha X}-\frac{e\alpha}{1+h\alpha X}Y^2\end{aligned}\qquad (1)$$

where X and Y denotes to the density of prey and predator populations at time t , r is the growth rate of prey, α is the measure efficiency of the searching and the capture of predator Y , u is the death rates of predator$Y$, $h$represent handling and digestion rates of predators, $\frac{\alpha X}{1+h\alpha X}$ is Holling type II functional responses to the predator Y [17], e is represent the efficiency of converting consumed prey into predator births, and k represents the carrying capacity of X. It is worth to mention that all parameters and initial conditions of the model (1) are assumed to be positive values.

To convert the model () to autonomous on we will use the following approximated method

Let,

$$-1\leq \sin(\delta t)\leq 1$$

By multiply the inequality by ϵ,

$$-\epsilon\leq \sin(\delta t)\leq \epsilon$$

And adding 1 to the inequality

$$1-\epsilon\leq \sin(\delta t)\leq 1+\epsilon$$

This approximation is viewed as fluctuating throughout time, but under favorable or adverse conditions

Then, the autonomous Holling type II functional responseprey-predator system with the fluctuation immigration effect where $r=1$can written as follow

$$\begin{aligned}\frac{dX}{dt} &= X\left(1-\frac{X}{k}\right)-\frac{\alpha XY}{1+h\alpha X}+i(1\pm\epsilon)\\ \frac{dY}{dt} &= -uY+\frac{e\alpha XY}{1+h\alpha X}-\frac{e\alpha}{1+h\alpha X}Y^2\end{aligned}\qquad (2)$$

3.The Boundedness of the Model

The following theorem shows the boundedness of the system (2).

Theorem(1): In R_+^2 the non-dimensional autonomous system (2) is bounded.

Proof: The equation of prey in system is bounded through

$$\frac{dX}{dt}\leq X\left(1-\frac{X}{k}\right)$$

The solution of

$$\frac{dX}{dt}=\frac{k}{\left(\frac{k}{X_0}-1\right)e^{-t}+1}$$

Therefore, $X(t) \leq k + i(1 \pm \epsilon), \forall t > 0.$

Now let $D(t) = X(t) + Y(t)$ (1)

By driving the both side of the Eq. (1) we get

$$\frac{dD}{dt} = \frac{dX}{dt} + \frac{dY}{dt} \qquad (2)$$

By substituting the prey and predator equations in Eq. (2) we have

$$\frac{dD}{dt} = X\left(1 - \frac{X}{k}\right) - \frac{\alpha XY}{1 + h\alpha X} + i(1 \mp \epsilon) + \left(-uY + \frac{e\alpha XY}{1 + h\alpha X} - \frac{e\alpha}{1 + h\alpha X} Y^2\right)$$

Let $I = i(1 \mp \epsilon)$ and since the solution is non-negative in R and all parameters are positive it can be assumed that

$$\frac{dD}{dt} \leq X\left(1 - \frac{X}{k}\right) - \frac{\alpha XY}{1 + h\alpha x} + I + \left(-uY + \frac{e\alpha XY}{1 + h\alpha X} - \frac{e\alpha}{1 + h\alpha X} Y^2\right) \qquad (3)$$

The maximum of $X\left(1 - \frac{X}{k}\right)$ is$\frac{k}{4}$ By substituting it in Eq. (3) we get

$$\frac{dD}{dt} \leq \frac{k}{4} + I + (-uY + \frac{e\alpha XY}{1 + h\alpha X} - \frac{e\alpha}{1 + h\alpha X} Y^2)$$

$$\frac{dD}{dt} \leq \frac{k}{4} + I\left(-uY + \frac{e\alpha XY}{1 + h\alpha X} - \frac{e\alpha}{1 + h\alpha X} Y^2\right) + D(t) - D(t) \qquad (4)$$

The Eq. (4) can be written as follows

$$\frac{dD}{dt} + D(t) \leq \frac{k}{4} + I + \left(-uY + \frac{e\alpha XY}{1 + h\alpha X} - \frac{e\alpha}{1 + h\alpha X} Y^2\right) + X$$

Since $\mathrm{X} \leq k$ then

$$\frac{dD}{dt} + D(t) \leq \frac{k}{4} + k + I + \left(-uY + \frac{e\alpha kY}{1 + h\alpha k} - \frac{e\alpha}{1 + h\alpha k} Y^2\right)$$

But the maximum of Y is,

$$max(Y) = \left(-u + \frac{e\alpha k}{1 + h\alpha k}\right)^2 \left(\frac{1 + h\alpha k}{4e\alpha}\right)$$

Therefore,

$$\frac{dD}{dt} + D(t) \leq L$$

Where, $L = \frac{k}{4} + k + I\left(-u + \frac{e\alpha k}{1+h\alpha k}\right)^2 \left(\frac{1+h\alpha k}{4e\alpha}\right)$

That mean $D \leq L - e^{-t}$where $t \to \infty$then $D(t) \leq L$

Thus, the system (2) is bounded.

4.Equilibrium Points

It is observed that the system (2) have 4 equilibrium points, which are obtained by setting $\frac{dX}{dt} = \frac{dY}{dt} = 0$. The equilibrium points in the system (2) are reached as follows:

By substituting $X = 0$ and $Y = 0$ in system (2) we get that $E_1(X, Y)$ is equal to $E_1(0,0)$. Where $i(1 \pm \epsilon) = 0$ if $X = 0$.

As well, by substituting $X = 0$ in system (2) we obtained either $Y = 0$ which is neglected since it led us to $E_1(0,0)$, or $Y = \frac{-u}{e\alpha}$ which is also neglected because it is negative. Thus $E_2\left(0, \frac{-u}{e\alpha}\right)$ is negative equilibrium point.

Now, to get $E_3(X, Y)$. By substituting $Y = 0$ in system (2) and multiplying both side of prey equation by $(1 + h\alpha X)$ we have the following cubic equation,

$$X^3 - \left(k - \frac{1}{h\alpha}\right)X^2 - \left(\frac{k}{h\alpha} + ki(1 \mp \in)\right)X - \frac{ki(1 \mp \in)}{h\alpha} = 0 \qquad (5)$$

It concluded that $\left(\frac{-1}{h\alpha}\right)$ is a root of Eq. (5). Then, by dividing Eq. (5) on $\left(X - \frac{1}{h\alpha}\right)$ we get, $\left(X^2 - kX - ki(1 \mp \in)\right)$ then the Eq. (5) became

$$\left(X - \frac{1}{h\alpha}\right)\left(X^2 - kX - ki(1 \mp \in)\right) = 0 \qquad (6)$$

By solving the quadratic part of Eq. (6) we have

$$X = \frac{k \mp \sqrt{k^2 + 4ki(1 \mp \in)}}{2}$$

We take just the positive root. Therefore the third equilibrium point is

$$E_3\left(\frac{k + \sqrt{k^2 + 4ki(1 \mp \in)}}{2}, 0\right)$$

Which is positive without any condition on its parameters.

For $E_4(X, Y)$. The value of X from the predator equation will written as:

$$X = \frac{u + e\alpha Y}{e\alpha - h\alpha u} \qquad (7)$$

As well, from the prey equation by setting $I = i(1 \mp \in)$, the value of Y is

$$Y = \frac{\frac{-h\alpha}{k}X^3 + \left(h\alpha - \frac{1}{k}\right)X^2 + (1 + h\alpha I)X + I}{\alpha X} \qquad (8)$$

By substituting Eq. (8) in Eq. (7) we have.

$$\left(\frac{-eh\alpha}{k}\right)X^3 + \left(eh\alpha - \frac{e}{k} - e\alpha + h\alpha u\right)X^2 + (u + e + eh\alpha I)X + eI = 0 \qquad (9)$$

After rearranging the Eq. (9) we get

$$\left(\frac{-eh\alpha}{k}\right)X^3 - \left(\frac{e}{k} + e\alpha - eh\alpha - h\alpha u\right)X^2 + (u + e + eh\alpha I)X + eI = 0$$

It is clear that the sequence of signs is $(-,-,+,+)$. The signs is switch one time then by Descarte's rule of signs there is at least one positive root.

Assume that the positive root of X is $\hat{X}$ therefore,

$$Y = \frac{\frac{-h\alpha}{k}\hat{X}^3 + \left(h\alpha - \frac{1}{k}\right)\hat{X}^2 + (1 + h\alpha I)\hat{X} + I}{\alpha\hat{X}}$$

$$= \frac{\frac{-h\alpha}{k}\hat{X}^3 + h\alpha\hat{X}^2 - \frac{1}{k}\hat{X}^2 + \hat{X} + h\alpha I\hat{X} + I}{\alpha\hat{X}}$$

$$= \frac{\frac{-h\alpha}{k}\hat{X}^3 + h\alpha I\hat{X} - \frac{1}{k}\hat{X}^2 + I + h\alpha\hat{X}^2 + \hat{X}}{\alpha\hat{X}}$$

$$= \frac{h\alpha\hat{X}\left(\frac{-\hat{X}^2}{k} + I\right) + \left(\frac{-\hat{X}^2}{k} + I\right) + \hat{X}\left(h\alpha\hat{X} + 1\right)}{\alpha\hat{X}}$$

$$= \frac{\left(\frac{-\hat{X}^2}{k} + I\right)\left(h\alpha\hat{X} + 1\right) + \hat{X}\left(h\alpha\hat{X} + 1\right)}{\alpha\hat{X}}$$

$$= \frac{\left(\frac{-\hat{X}^2}{k} + I + \hat{X}\right)\left(h\alpha\hat{X} + 1\right)}{\alpha\hat{X}} = \hat{Y}$$

Therefore, $\hat{Y}$ is positive if $\hat{X}^2 < Ik + \hat{X}k$

5.Local Stability

The Jacobian matrix of the system (2) is given as follow

$$J(X,Y) = \begin{bmatrix} 1 - \frac{2X}{k} - \frac{\alpha Y}{1 + h\alpha X} + \frac{\alpha^2 hXY}{(1 + h\alpha X)^2} & \frac{-\alpha X}{1 + h\alpha X} \\ \frac{e\alpha Y}{1 + h\alpha X} - \frac{e\alpha^2 hXY}{(1 + h\alpha X)^2} + \frac{e\alpha^2 hY^2}{(1 + h\alpha X)^2} & -u + \frac{e\alpha X}{1 + h\alpha X} - \frac{2e\alpha Y}{1 + h\alpha X} \end{bmatrix} \quad (10)$$

The following theorems will demonstrate the stability of the positive equilibrium points.

Theorem (2): $E_1(0,0)$ in system (2) is unstable saddle point.

Proof: the Jacobian matrix of the system (2) after substituting $X = Y = 0$ is

$$J(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & -u \end{bmatrix}$$

It is clear that λ_1, λ_2 are opposite signs. Therefore, $E_1(0,0)$ unstable saddle point.

Theorem (3): the equilibrium point $E_3\left(\frac{k+\sqrt{k^2+4ki(1\mp\epsilon)}}{2}, 0\right)$ is stable if

- $2\Upsilon > k$
- $u > \frac{e\alpha\Upsilon}{1+h\alpha\Upsilon}$

Proof: let $\Upsilon = \frac{k+\sqrt{k^2+4ki(1\mp\epsilon)}}{2}$, that mean the equilibrium point is became $E_3(\Upsilon,0)$.

By substituting $E_3(\Upsilon,0)$ in Eq. (10) we have

$$J(\Upsilon,0) = \begin{bmatrix} 1-\frac{2\Upsilon}{k} & \frac{-\alpha\Upsilon}{1+h\alpha\Upsilon} \\ 0 & -u+\frac{e\alpha\Upsilon}{1+h\alpha\Upsilon} \end{bmatrix}$$

From the Jacobian matrix $\lambda_1 = 1-\frac{2\Upsilon}{k}$ which is negative if $2\Upsilon > k$.

Also, $\lambda_2 = -u+\frac{e\alpha\Upsilon}{1+h\alpha\Upsilon}$. The value of λ_2 be negative if $u > \frac{e\alpha\Upsilon}{1+h\alpha\Upsilon}$.

Therefore $E_3(\Upsilon,0)$ is stable if $2\Upsilon > k$ and $u > \frac{e\alpha\Upsilon}{1+h\alpha\Upsilon}$.

Theorem (4): the non-trivial equilibrium point $E(\hat{X},\hat{Y})$ is stable if the following conditions are satisfied:

- $2\hat{X} > k$
- $2\hat{Y} > \hat{X}$
- $\left(1+h\alpha\hat{X}\right)^2 < \alpha\hat{Y}$

Proof: we use the Jacobian matrix of the system (2) to analyze the stability of equilibrium point $E(\hat{X},\hat{Y})$:

$$J(\hat{X},\hat{Y}) = \begin{bmatrix} 1-\frac{2\hat{X}}{k}-\frac{\alpha\hat{Y}}{1+h\alpha\hat{X}}+\frac{\alpha^2 h\hat{X}\hat{Y}}{\left(1+h\alpha\hat{X}\right)^2} & \frac{-\alpha\hat{X}}{1+h\alpha\hat{X}} \\ \frac{e\alpha\hat{Y}}{1+h\alpha\hat{X}}-\frac{e\alpha^2 h\hat{X}\hat{Y}}{\left(1+h\alpha\hat{X}\right)^2}+\frac{e\alpha^2 h\hat{Y}^2}{\left(1+h\alpha\hat{X}\right)^2} & -u+\frac{e\alpha\hat{X}}{1+h\alpha\hat{X}}-\frac{2e\alpha\hat{Y}}{1+h\alpha\hat{X}} \end{bmatrix}$$

$$J(\hat{X},\hat{Y}) = \begin{bmatrix} 1-\frac{2\hat{X}}{k}-\frac{\alpha\hat{Y}+\alpha^2 h\hat{X}\hat{Y}-\alpha^2 h\hat{X}\hat{Y}}{\left(1+h\alpha\hat{X}\right)^2} & \frac{-\alpha\hat{X}}{1+h\alpha\hat{X}} \\ \frac{e\alpha\hat{Y}+e\alpha^2 h\hat{X}\hat{Y}-e\alpha^2 h\hat{X}\hat{Y}}{\left(1+h\alpha\hat{X}\right)^2}+\frac{e\alpha^2 h\hat{Y}^2}{\left(1+h\alpha\hat{X}\right)^2} & -u+\frac{e\alpha\hat{X}}{1+h\alpha\hat{X}}-\frac{2e\alpha\hat{Y}}{1+h\alpha\hat{X}} \end{bmatrix}$$

$$J(\hat{X},\hat{Y}) = \begin{bmatrix} 1-\frac{2\hat{X}}{k}-\frac{\alpha\hat{Y}}{\left(1+h\alpha\hat{X}\right)^2} & \frac{-\alpha\hat{X}}{1+h\alpha\hat{X}} \\ \frac{e\alpha\hat{Y}}{\left(1+h\alpha\hat{X}\right)^2}+\frac{e\alpha^2 h\hat{Y}^2}{\left(1+h\alpha\hat{X}\right)^2} & -u+\frac{e\alpha\hat{X}}{1+h\alpha\hat{X}}-\frac{2e\alpha\hat{Y}}{1+h\alpha\hat{X}} \end{bmatrix}$$

The determinant of $J(\hat{X},\hat{Y})$ is

$$|J(\hat{X},\hat{Y})| = -u + \frac{e\alpha\hat{X}}{1+h\alpha\hat{X}} - \frac{2e\alpha\hat{Y}}{1+h\alpha\hat{X}} + \frac{2u\hat{X}}{k} - \frac{2e\alpha\hat{X}^2}{k(1+h\alpha\hat{X})} + \frac{4e\alpha\hat{X}\hat{Y}}{k(1+h\alpha\hat{X})} + \frac{\alpha u\hat{Y}}{(1+h\alpha\hat{X})^2} - \frac{e\alpha^2\hat{X}\hat{Y}}{(1+h\alpha\hat{X})^3} + \frac{2e\alpha^2\hat{Y}^2}{(1+h\alpha\hat{X})^3} + \frac{e\alpha^2\hat{X}\hat{Y}}{(1+h\alpha\hat{X})^3} + \frac{e\alpha^3 h\widehat{\hat{X}Y}^2}{(1+h\alpha\hat{X})^3}$$

By simplifying the previous determinant we get

$$|J(\hat{X},\hat{Y})| = u\left(\frac{2\hat{X}}{k} - 1\right) + \frac{2e\alpha\hat{Y}}{1+h\alpha\hat{X}}\left(\frac{\alpha\hat{Y}}{(1+h\alpha\hat{X})^2} - 1\right) + \frac{2e\alpha\hat{X}}{1+h\alpha\hat{X}}(2\hat{Y}-\hat{X}) + \frac{e\alpha\hat{X}}{1+h\alpha\hat{X}} + \frac{\alpha u\hat{Y}}{(1+h\alpha\hat{X})^2} + \frac{e\alpha^3 h\widehat{\hat{X}Y}^2}{(1+h\alpha\hat{X})^3}$$

Therefore, $|J(\hat{X},\hat{Y})| > 0$ if $2\hat{X} > k$, $2\hat{Y} > \hat{X}$, and $(1+h\alpha\hat{X})^2 < \alpha\hat{Y}$.

Now, we must to find the trace of $J(\hat{X},\hat{Y})$

$$trace\left(J(\hat{X},\hat{Y})\right) = 1 - \frac{2\hat{X}}{k} - \frac{\alpha\hat{Y}}{(1+h\alpha\hat{X})^2} - u + \frac{e\alpha\hat{X}}{1+h\alpha\hat{X}} - \frac{2e\alpha\hat{Y}}{1+h\alpha\hat{X}}$$

$$trace\left(J(\hat{X},\hat{Y})\right) = -\left(\frac{2\hat{X}}{k} - 1\right) - \frac{\alpha\hat{Y}}{(1+h\alpha\hat{X})^2} - u - \frac{e\alpha}{1+h\alpha\hat{X}}(2\hat{Y}-\hat{X})$$

Therefore, $trace\left(J(\hat{X},\hat{Y})\right) < 0$ if $2\hat{X} > k$ and $2\hat{Y} > \hat{X}$.

Implies that the non-trivial equilibrium point $E(\hat{X},\hat{Y})$ is stable.

6.Global Stability

Theorem (5): The system (2) has periodic solution in R_+^2

Proof:Let,

$$N_1(X,Y) = X\left(1 - \frac{X}{k}\right) - \frac{\alpha XY}{1+h\alpha X} + i(1\mp\epsilon)$$

$$N_2(X,Y) = -uY + \frac{e\alpha XY}{1+h\alpha X} - \frac{e\alpha}{1+h\alpha X}Y^2$$

By multiplying the Dulac function $G(X,Y) = \frac{1}{XY}$ by $N_1(X,Y)$ and $N_2(X,Y)$ we get

$$GN_1 = \frac{1}{XY}\left(X\left(1 - \frac{X}{k}\right) - \frac{\alpha XY}{1+h\alpha X} + i(1\mp\epsilon)\right)$$

$$= \frac{1}{Y} - \frac{1}{kY}X - \frac{\alpha}{1+h\alpha X} + \frac{i(1\mp\epsilon)}{Y}X^{-1}$$

By taking the divertive of GN_1 according to X we have,

$$\frac{\partial GN_1}{\partial X} = -\frac{1}{kY} + \frac{h\alpha^2}{(1+h\alpha X)^2} - \frac{i(1\mp\epsilon)}{YX^2}$$

Also for $N_2(X,Y)$

$$GN_2 = \frac{1}{XY}\left(-uY + \frac{e\alpha XY}{1+h\alpha X} - \frac{e\alpha}{1+h\alpha X}Y^2\right)$$
$$= \frac{-u}{X} + \frac{e\alpha}{1+h\alpha X} - \frac{e\alpha Y}{X(1+h\alpha X)}$$

By taking the divertive of GN_2 according to Y we have,

$$\frac{\partial GN_2}{\partial X} = -\frac{e\alpha}{X(1+h\alpha X)}$$

It is noticed that there is a changing in signs of

$$\Delta\,(GN_1, GN_2) = \frac{-u}{X} + \frac{e\alpha}{1+h\alpha X} - \frac{e\alpha Y}{X(1+h\alpha X)} - \frac{e\alpha}{X(1+h\alpha X)}$$

Then, by Bendixson-Dulac criterion, the system (2) has periodic solution.

Theorem (6): System (2) is not globally stable.

Proof: since system (2) has a periodic solution in R_+^2. Then, by Poincare-Bendixon theorem system (2) is not globally asymptotically stable.

7.Kolmogorov Analysis

The Kolmogorov analysis is investigated to show the conditions on persistence and extinction in the two-dimensional case. The Kolmogorov analysis is used in this chapter to determine the conditions on equilibrium points of systems (2). It is more complicated than the system in chapter two, the existence theorem guarantees the existence of the equilibrium point inside the interior of the positive quadrant.

The system (2) will written as follow:

$$\begin{aligned} XM(X,Y) &= X\left(1 - \frac{X}{k} - \frac{\alpha Y}{1+h\alpha X} + \frac{i(1\pm\epsilon)}{X}\right) \\ YN(X,Y) &= Y\left(-u + \frac{e\alpha X}{1+h\alpha X} - \frac{e\alpha y}{1+h\alpha X}\right) \end{aligned} \qquad (3)$$

Where $M(X,Y)$ and $N(X,Y)$ represents the growth rate of the species that corresponds to the current density of prey$X$ and predator Y, respectively.

Theorem (7): if the number of prey is fixed and the number of predators is increasing then the growth rate of predators and preywill decrease.

Proof:From the first condition of Kolmogorov by assuming the number of prey is fixed we have

$$\frac{\partial M}{\partial Y} = \frac{-\alpha}{1+h\alpha X}$$
$$\frac{\partial N}{\partial Y} = \frac{-e\alpha}{1+h\alpha X}$$

It is noticed that $\frac{\partial M}{\partial Y}$ and $\frac{\partial N}{\partial Y}$ are negative. Because, e and α are positive parameters according to condition (1) in [11] the growth rate of prey and predator in system (3) will decrees.

Theorem (8): The carrying capacity of the system (3) is $\frac{k \pm k\sqrt{1+i(1\pm\epsilon)}}{2}$.

Proof: let $P > 0$ such that

$$M(P,0) = 1 - \frac{P}{k} - \frac{\alpha(0)}{1+h\alpha X} + \frac{i(1\pm\epsilon)}{P} = 0$$

$$P - \frac{1}{k}P^2 - i(1\pm\epsilon) = 0$$

By solving this quadratic equation we obtain

$$P = \frac{k \pm k\sqrt{1+i(1\pm\epsilon)}}{2}$$

Where P is the carrying capacity of the system (3).

Theorem (9): The least number of prey that maintains the lowest rate of growth for the predator in the system (3) is $\tau = \frac{u}{e\alpha - uh\alpha}$.

Proof: let $\tau > 0$ be the least number of prey population in the system (3) then,

$$N(\tau,0) = -u + \frac{e\alpha\tau}{1+h\alpha X} = 0$$

Therefore, $\tau = \frac{u}{e\alpha - uh\alpha}$ is the least number of prey.

Theorem (10): in the system (3) the prey coexist with predators if

$$1 + i(1\pm\epsilon) > \left(\frac{2u}{e\alpha k - uh\alpha k} - 1\right)^2$$

Proof: by assuming that carrying capacity is greater than the number of prey

$$P > \tau \qquad ()$$

Now, substituting the value of P and τ in ().

$$\frac{k \pm k\sqrt{1+i(1\pm\epsilon)}}{2} > \frac{u}{e\alpha - uh\alpha}$$

Accordingly, we have

$$\pm\sqrt{1+i(1\pm\epsilon)} > \frac{2u}{e\alpha k - uh\alpha k} - 1$$

By squaring both side we get

$$1 + i(1\pm\epsilon) > \left(\frac{2u}{e\alpha k - uh\alpha k} - 1\right)^2$$

8.Conclusions

In this paper, the local and global stability of the prey and predator model was studied under the effect of fluctuating immigration. It was concluded that these systems are locally stable under certain conditions as mentioned in theorem (3,4). We also noticed that this system is not globally stable because it contains periodic solutions as we explained in theorem (5,6). Furthermore, we concluded through the analysis of the conditions of coexistence and extinction for Kolmogorov that the the carrying capacity of the system (3) is $\frac{k \pm k\sqrt{1+i(1\pm\epsilon)}}{2}$. There is also a great possibility of coexistence between prey and predator $1 + i(1 \pm \epsilon) > \left(\frac{2u}{e\alpha k - uh\alpha k} - 1\right)^2$ as shown in the theorem (9).

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