A Common Fixed Point Theorem in Complex Valued Metric Spaces.

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Abstract:
In this paper, we have extended the contractive condition of Karapinar[5] and Noorwali[6] to complex valued metric space.
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1.0 Introduction:

In 2011, Azam, Fisher and Khan [1] introduced the notion of complex valued metric space, which is a generalization of the classical metric space and established sufficient conditions for the existence of a common fixed points for a pair of mappings satisfying a contractive condition. A complex number $z \in \mathbb{C}$ is an ordered pair of real numbers, whose first co-ordinate is called Re(z) and second co-ordinate is called Im(z).

In 1968, Kannan [3, 4] introduced a contraction mapping which is non-continuous and gave a fixed point. Such as if X is a complete metric space and T: X→X is a mapping satisfying

$$d(Tx,Ty) \leq \alpha[d(x,Tx) + d(y,Ty)], \qquad \forall\, x,y \in X \text{ and } \alpha \in [0,1).$$

Then T has a unique fixed point.

Many authors generalized the contractive condition of Kannan in different types of metric spaces. Recently in 2018, one of the generalization done by Karapinar [5] and Noorwali [6] introduced a kannan-type contraction called interpolative Kannan types contraction.

In our main result we have generalized the above author's contraction mapping in complex valued metric space.

It is helpful to recall some definitions in the context of a complex valued metric space.

2.0 Preliminaries:

First we recall some notations and definitions that will be utilized in our subsequent discussion.

Definition 2.1 Let $\mathbb{C}$ be the set of Complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\preccurlyeq$ on $\mathbb{C}$ as follows:

$$z_1 \preccurlyeq z_2 \text{ if and only if } Re(z_1) \leq Re(z_2) \text{ and } Im(z_1) \leq Im(z_2).$$

Consequently, we can infer that $z_1 \preccurlyeq z_2$ if one of the following conditions is satisfied:

(i) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$,

(ii) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$,
(iii) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$,
(iv) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$.

In particular, we write $z_1 \precsim z_2$ if $z_1 \neq z_2$ and one of (i), (ii), and (iii) is satisfied and we write $z_1 \prec z_2$ if only (iii) is satisfied.

We Note that $0 \precsim z_1 \precnsim z_2 \Rightarrow |z_1| < |z_2|$, and $z_1 \precsim z_2, z_2 \prec z_3 \Rightarrow z_1 \prec z_3$.

The following definition is recently introduced by **Azam *et al.*** [1].

Definition 2.2[2] Let X be a nonempty set whereas $\mathbb{C}$ be the set of complex numbers. Suppose that the mapping $d : X \times X \to \mathbb{C}$, satisfies the following conditions:

(d_1): $0 \precsim d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(d_2): $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d_3): $d(x, y) \precsim d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then d is called a complex valued metric on X, and (X, d) is called a complex valued metric space.

Note that $d(x,y) \precsim 1 + d(x,y)$ and so $\left|\frac{d(x,y)}{1+d(x,y)}\right| \le 1$.

Example 2.3 Let $X = [0,1]$ Define the mapping $d: X \times X \to \mathbb{C}$ by $d(z_1, z_2) = e^{i\frac{\pi}{6}}|z_1 - z_2|$, .Then (X, d) is a complex valued metric space.

Lemma 2.4 Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.

Lemma 2.5 Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a cauchy sequence iff $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$. Where m in X.

Definition 2.6 Let (X, d) be a complex valued metric space. Then X is said to be complete if every cauchy sequence in X is convergent in X.

Definition 2.7 A point $x \in X$ is said to be a fixed point of T if $Tx = x$.

Definition 2.8 A point $x \in X$ is said to be a common fixed point of T and S if $Tx = Sx = x$.

3.0 Main Result:

Theorem: Let A and B be two maps on a complete complex valued metric space (X, d) into itself satisfying:

$$d(Ax, By) \precsim \alpha[d(x, Ax)]^{\gamma}[d(y, By)]^{1-\gamma} + \beta[d(x, y)]^{1-\gamma}[d(Ay, By)]^{\gamma}$$

for all x,y in X , $x \neq y$,for some $\alpha, \beta \in [0,1)$ and $\gamma \in (0,1)$ $with$

$\alpha + \beta < 1$.

then A and B have a unique common fixed point in X.

Proof: for any arbitrary point $x_0 \in X$, $\{x_n\}$ be a sequence defined as

$$x_0;\ x_1 = Ax_0;\ x_2 = Bx_1;\ x_3 = Ax_2; \ldots, x_{2n-1} = Ax_{2n-2};\ x_{2n} = Bx_{2n-1}$$

Then,

$$d(x_{2n-1}, x_{2n}) = d(Ax_{2n-2}, Bx_{2n-1})$$

$$\preccurlyeq \alpha[d(x_{2n-2},Ax_{2n-2})]^{\gamma}[d(x_{2n-1},Bx_{2n-1})]^{1-\gamma}$$
$$+\beta[d(x_{2n-2},x_{2n-1})]^{1-\gamma}[d(Ax_{2n-2},Bx_{2n-1})]^{\gamma}$$
$$= \alpha[d(x_{2n-2},x_{2n-1})]^{\gamma}[d(x_{2n-1},x_{2n})]^{1-\gamma}$$
$$+\beta[d(x_{2n-2},x_{2n-1})]^{1-\gamma}[d(x_{2n-1},x_{2n})]^{\gamma}$$

Or, $[d_{2n-1}] \preccurlyeq \alpha[d_{2n-2}]^{\gamma}[d_{2n-1}]^{1-\gamma} + \beta[d_{2n-2}]^{1-\gamma}[d_{2n-1}]^{\gamma}$

where $d_{2n-1} = d(x_{2n-1},x_{2n})$

$d_{2n-2} = d(x_{2n-2},x_{2n-1})$ And so on.

Further,

$$d(x_{2n},x_{2n+1}) = d(Bx_{2n-1},Ax_{2n})$$
$$= d(Ax_{2n},Bx_{2n-1})$$
$$\preccurlyeq \alpha[d(x_{2n},Ax_{2n})]^{\gamma}[d(x_{2n-1},Bx_{2n-1})]^{1-\gamma}$$
$$+\beta[d(x_{2n},x_{2n-1})]^{1-\gamma}[d(Ax_{2n},Bx_{2n-1})]^{\gamma}$$
$$= \alpha[d(x_{2n},x_{2n+1})]^{\gamma}[d(x_{2n-1},x_{2n})]^{1-\gamma}$$
$$+\beta[d(x_{2n},x_{2n-1})]^{1-\gamma}[d(x_{2n+1},x_{2n})]^{\gamma}$$

Or, $[d_{2n}] \preccurlyeq \alpha[d_{2n}]^{\gamma}[d_{2n-1}]^{1-\gamma} + \beta[d_{2n-1}]^{1-\gamma}[d_{2n}]^{\gamma}$

Or, $[d_{2n}] \preccurlyeq (\alpha+\beta)[d_{2n}]^{\gamma}[d_{2n-1}]^{1-\gamma}$

Or, $[d_{2n}]^{1-\gamma} \preccurlyeq (\alpha+\beta)[d_{2n-1}]^{1-\gamma}$

$|[d_{2n}]| \leq (\alpha+\beta)^{\frac{1}{1-\gamma}} |[d_{2n-1}]|$

Or,$|[d_{2n}]| \leq h^{2n} |[d_0]|$, where $(\alpha+\beta)^{\frac{1}{1-\gamma}} = h < 1.$

As given $\alpha + \beta < 1.$

As we have, $|d_{2n}| \leq h|d_{2n-1}| \leq h^2|d_{2n-2}| \leq \cdots \leq h^{2n}|d_0|$

Now we have require to show Cauchy sequence, then

$$|d(x_p,x_{p+q})| \leq |d(x_p,x_{p+1}) + d(x_{p+1},x_{p+2}) + \cdots + d(x_{p+q-1},x_{p+q})|.$$
$$\leq [h^p + h^{p+1} + \cdots + h^{p+q-1}]|d(x_0,x_1)|$$
$$= \frac{h^p}{1-\lambda}|d(x_0,x_1)|.$$
$$\to 0 \ \ as\ p,q \to \infty.$$

Hence, $\{x_n\}$ is a cauchy sequence. As the X is complete. Let it converges to some u in X. now we shall prove that u is a unique common fixed point of A and B. for this first we prove that u=Au=Bu.

$$d(u,Au) \preccurlyeq d(u,x_{2n}) + d(x_{2n},Au)$$

$$\preccurlyeq d(u,x_{2n}) + \alpha[d(u,Au)]^{\gamma}[d(x_{2n-1},Bx_{2n-1})]^{1-\gamma}$$
$$+\beta[d(u,x_{2n-1})]^{1-\gamma}[d(Au,Bx_{2n-1})]^{\gamma}$$
$$= d(u,x_{2n}) + \alpha[d(u,Au)]^{\gamma}[d(x_{2n-1},x_{2n})]^{1-\gamma}$$
$$+\beta[d(x_{2n-1},u)]^{1-\gamma}[d(Au,x_{2n})]^{\gamma}.$$

Letting $n \to \infty$ we have,

$d(u,Au) \leq 0.$ Which implies that $d(u,Au) = 0$ i.e. Au=u.

Similarly Bu=u. therefore, u is the common fixed point of both A and B.

To show that u is unique. If possible let v be another common fixed point of A and B. so Av=Bv=v.

Thus by given condition

$$d(u,v) = d(Au,Bv)$$
$$\preccurlyeq \alpha[d(u,Au)]^{\gamma}[d(v,Bv)]^{1-\gamma}$$
$$+\beta[d(u,v)]^{1-\gamma}[d(Au,Bv)]^{\gamma}$$
$$= \alpha[d(u,u)]^{\gamma}[d(v,v)]^{1-\gamma}$$
$$+\beta[d(u,v)]^{1-\gamma}[d(u,v)]^{\gamma}.$$

i.e.$d(u,v) \leq \beta d(u,v)$, which is a contradiction since $\beta < 1$ and hence our supposition was wrong so u=v. **Proved.**

Remarks: In theorem

1. By putting $\beta = 0$, we get Noorwali [6] and
2. By putting $\beta = 0$ and a=b we get Karapinar[5].

Corollary: Let X be a complex valued metric space and S and T be self-mappings on X. if S and T satisfy:

$$d(Sx,Ty) \preccurlyeq \lambda[d(x,sx)]^{\alpha}[d(y,Ty)]^{1-\alpha} \tag{3.1}$$

for all x, y $\in X$. Such that $Sx \neq x, Ty \neq y$, where $\lambda \in [0,1)$and $\alpha \in (0,1)$.

Then S and T have a unique common fixed point.

Proof: Let $x_0 \in X$, we define sequence $\{x_n\}$ in X as follows:

$x_{2n+1} = Sx_{2n}$, and $x_{2n+2} = Tx_{2n+1}$ for n= 0,1,2,…if there exists $n \in \{0,1,2,3..\}$ such that $x_n = x_{n+1} = x_{n+2}$ then x_n is a common fixed point of S and T. so we assume that there does not exist three consecutive identical terms in the sequence $\{x_n\}$ and that $x_0 \neq x_1$.

Now using (3.1), we have

$$d(x_{2n+1},x_{2n+2}) = d(Sx_{2n},Tx_{2n+1})$$
$$\preccurlyeq \lambda[d(x_{2n},Sx_{2n})]^{\alpha}[d(x_{2n+1},Tx_{2n+1})]^{1-\alpha}$$
$$= \lambda[d(x_{2n},x_{2n+1})]^{\alpha}[d(x_{2n+1},x_{2n+2})]^{1-\alpha}$$

Or, $[d(x_{2n+1},x_{2n+2})]^{\alpha} \preccurlyeq \lambda[d(x_{2n},x_{2n+1})]^{\alpha}$

Or,$d(x_{2n+1},x_{2n+2}) \preccurlyeq \lambda^{\frac{1}{\alpha}} d(x_{2n},x_{2n+1})$
Or, $d(x_{2n+1},x_{2n+2}) \preccurlyeq \lambda d(x_{2n},x_{2n+1})$
Hence,
$|d(x_{2n+1},x_{2n+2})| \leq \lambda |d(x_{2n},x_{2n+1})| + \lambda^2 |d(x_{2n-1},x_{2n})| + \cdots + \lambda^{2n+1}|d(x_0,x_1)|$
Or, $|d(x_{2n+1},x_{2n+2})| \leq \lambda^{2n+1}|d(x_0,x_1)|$ (3.2)

Similarly,
$$\begin{aligned} d(x_{2n},x_{2n+1}) &= d(Tx_{2n-1},Sx_{2n}) \\ &= d(Sx_{2n},Tx_{2n-1}) \\ &\preccurlyeq \lambda [d(x_{2n},Sx_{2n})]^{\alpha}[d(x_{2n-1},Tx_{2n-1})]^{1-\alpha} \\ &= \lambda [d(x_{2n},x_{2n+1})]^{\alpha}[d(x_{2n-1},x_{2n})]^{1-\alpha} \end{aligned}$$
Or, $[d(x_{2n},x_{2n+1})]^{1-\alpha} \preccurlyeq \lambda [d(x_{2n-1},x_{2n})]^{1-\alpha}$
Or, $d(x_{2n},x_{2n+1}) \preccurlyeq \lambda^{\frac{1}{1-\alpha}} d(x_{2n-1},x_{2n})$.
Or, $|d(x_{2n},x_{2n+1})| \preccurlyeq \lambda^{\frac{1}{1-\alpha}} |d(x_{2n-1},x_{2n})| \preccurlyeq \lambda |d(x_{2n-1},x_{2n})|$.
Hence,
$|d(x_{2n},x_{2n+1})| \leq \lambda |d(x_{2n-1},x_{2n})| \leq \lambda^2 |d(x_{2n-2},x_{2n-1})| \leq \cdots + \lambda^{2n}|d(x_0,x_1)|$ (3.3)
From (3.2) and (3.3) we get,
$|d(x_n,x_{n+1})| \leq \lambda^n |d(x_0,x_1)|$ (3.4)
Now we shall prove that the sequence $\{x_n\}$ is a Cauchy sequence.
Let p,q from the set {0,1,,2,…}.
$$\begin{aligned} |d(x_p,x_{p+q})| &\leq |d(x_p,x_{p+1}) + d(x_{p+1},x_{p+2}) + \cdots + d(x_{p+q-1},x_{p+q})|. \\ &\leq [\lambda^p + \lambda^{p+1} + \cdots + \lambda^{p+q-1}]|d(x_0,x_1)| \\ &\leq [\lambda^p + \lambda^{p+1} + \cdots + \lambda^{p+q-1} + \cdots]|d(x_0,x_1)|. \\ &= \frac{\lambda^p}{1-\lambda}|d(x_0,x_1)|. \\ &\to 0 \ \ as\ p \to \infty. \end{aligned}$$
Hence $\{x_n\}$ is a Cauchy sequence. As X is complete, so there exists u in X such that $\lim_{n\to\infty} x_n = u$. Now we shall show that Su=u. if possible let Su$\neq u$. Then we have $|d(Su,u)| > 0$. Then we get
$$\begin{aligned} d(Su,u) &\preccurlyeq d(Su,x_{2n+2}) + d(x_{2n+2},u) \\ &\preccurlyeq d(x_{2n+2},u) + d(Su,Tx_{2n+1}) \\ &\preccurlyeq d(x_{2n+2},u) + \lambda [d(u,Su)]^{\alpha}[d(x_{2n+1},Tx_{2n+1})]^{1-\alpha} \end{aligned}$$
Letting n→ ∞ we get $|d(Su,u)| = 0$ i.e. Su=u.
Similarly,
$$\begin{aligned} d(u,Tu) &\preccurlyeq d(u,x_{2n+1}) + d(x_{2n+1},Tu) \\ &\preccurlyeq d(u,x_{2n+1}) + d(Sx_{2n},Tu) \\ &\preccurlyeq d(u,x_{2n}) + \lambda [d(x_{2n},Sx_{2n})]^{\alpha}[d(u,Tu)]^{1-\alpha} \end{aligned}$$

Letting n→∞ we get $|d(Tu,u)| = 0$ i.e. Tu=u.

To prove that u is unique common fixed point of S and T. if possible suppose that v is the another common fixed point of S and T. then

$d(u,v) = d(Su,Tv) \precsim \lambda[d(u,Su)]^{\alpha}[d(v,Tv)]^{1-\alpha}$

which gives $|d(u,u)| = 0$ i.e. u=v.

Thus u is unique common fixed point of S and T. **Proved.**

Remarks:-This theorem is free from the condition of different types of contraction. Authors suggest to the readers to extend the mappings as well as in generalized contractive condition.

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