Multi-objective integer linear programming problems withFuzzy parameters in the constraint

Khaled Elsharkawy
Egypt-Cairo-6th October High Institute for Engineering
Basic science department
E-mail: khaledelsharkawy68@gmail.com

Abstract

In this paper we focus on the solution of multi-objective integer linear programming problems with fuzzy parameters (FMOILP). The fuzzy parameters are involved in the constraints, a-Pareto optimality is introduced, in addition a solution algorithm is described to solve the FMOILP and a parametric study is carried out on it, finally a numerical example illustrates various aspects of the results developed in this paper.

Keywords: Multi – criteria decision making, fuzzy multi-objective linear programming problems (FMOILP), Integer programming, Fuzzy parameters.

1. Introduction

In an earlier work [3] we described an algorithm to solve multi-objective integer linear programming problems involving fuzzy parameters in the objective functions;in this paper we will extend the study to solve multi-objective integer linear programming problems having fuzzy parameters in the constraints. The plan of this paper is as follows: in section 2, we formulate multi-objective integer linear programming problems having fuzzy parameters in the constraints (FMOILP), in section 3, a parametric study is carried out on the FMOILP where some basic stability notions are characterized, these notions are the set of feasible parameters, the solvability set and the stability set of the first kind. In section 4, we suggest an algorithm to solve the FMOILP. In section 5, a numerical example is included to clarify the theory and the algorithm. Finally section 6 contains the conclusion.

2. Problem statement and solution concept

In this paper we consider the followingmulti-objective integer linear programming problems having fuzzy parameters in the constraints.

(FMOILP): max $f(x) = [f_1(x), f_2(x), \dots, f_k(x)]$ $(1a)$

Subject to

$$x \in X\,(\approx{a}, \approx{b}) = \{x \in R^n \,/\, g_1(x, \approx{a}_1) = \sum_{j=1}^{n} \approx{a}_{ij} x_j \le \approx{b}_1, i = 1, \dots, m.$$

$$x_j \ge d_j \succ 0, j = 1,..,n. \; where \; x_j \text{ int } eger \; \forall \; j = 1,..,n \qquad (1b)$$

Here $\approx{a}_{ij}$ and $\approx{b}_1$, (i=1,…,m, j=1,..,n) represent fuzzy parameters involved in the constraints, x is an n-vector of the integer decision variables, d_j are certain lower bounds on the decision variables x_j for all j, R^n is the set of all ordered n-tuples of real numbers and it is assumed that the feasible region $X(\overset{\approx}{a},\overset{\approx}{b})$ is a compact polyhedral set.

We now assume that $\overset{\approx}{a}_{ij}$ and $\overset{\approx}{b}_1$, (i=1,…,m, j=1,..,n) in the FMOILP (1a, 1b) are fuzzy numbers whose membership functions are $\mu_{\overset{\approx}{a}_{ij}}(a_{ij})$ *and* $\mu_{\overset{\approx}{b}_1}(b_1)$, (i=1,…,m, j=1,..,n) respectively. Now we introduce the concept of α -level set or α - cut of the fuzzy numbers $\overset{\approx}{a}_{ij}$ and $\overset{\approx}{b}_1$ as follows:

Definition 1. [3, 4]

The α -level set of the fuzzy numbers $\overset{\approx}{a}_{ij}$ and $\overset{\approx}{b}_1$ is defined as the ordinary set $L_\alpha(\overset{\approx}{a},\overset{\approx}{b})$

for which the degree of their

$$L_\alpha(\overset{\approx}{a},\overset{\approx}{b}) = \{(a,b) / \mu_{\overset{\approx}{a}_{ij}}(a_{ij}) \geq \alpha,\ \mu_{\overset{\approx}{b}_1}(b_1) \geq \alpha, (\mathrm{i} = 1,\dots,\mathrm{m},\ \mathrm{j} = 1,..,\mathrm{n})\ \} \qquad (2)$$

For a certain degree α, FMOILP (1a, 1b) can be understood as the following non-fuzzy α -multi-objective mixed integer non-linear programming problem (α -MOMINLP):

(α -MOMINLP): max $f(x) = [f_1(x), f_2(x), \dots, f_k(x)], \qquad (3a)$

$$x \in X(\overset{\approx}{a},\overset{\approx}{b}) = \{x \in R^n / g_1(x,\overset{\approx}{a}_1) = \sum_{j=1}^{n} \overset{\approx}{a}_{ij} x_j \leq \overset{\approx}{b}_1, i = 1,\dots,m.$$

$$x_j \geq d_j \succ 0, j = 1,..,n.\ where\ x_j \text{ int } eger\ \forall\ j = 1,..,n\}. \qquad (3b)$$

$$(a,b) \in L_\alpha(\overset{\approx}{a},\overset{\approx}{b}) \qquad (3c)$$

In the (α -MOMINLP) above the parameters (a,b) are treated as decision variables rather than constraints. Based on the definition of α -level set of the fuzzy numbers we introduce the concept of α -pareto optimal solution to the(α -MOMINLP) [3] in the following definition.

Definition 2.

A point $x^* \in$ X (a, b) is said to be an α -pareto optimal solution to the (α -MOMINLP) [3], if and only if there does not exist another x $\in$ X (a, b) $\in L_\alpha(\overset{\approx}{a},\overset{\approx}{b})$, such that $f_r(x) \geq f_r(x^*)$,

R=1,…, k with strict inequality holding for at least one r where the corresponding values of parameters (a^*, b^*) are called α -level optimal parameters.

Problem (α -MOMINLP) [3] will be treated using the weighting method [2], i.e. by considering the following mixed-integer non-linear problem with single objective:

$$p(w): \quad \max \sum_{r=1}^{k} w_r f_r(x), \qquad (4a)$$
$$s.t \quad x \in X(a,b), \qquad (4b)$$
$$l_{ij}^{(0)} \le a_{ij} \le L_{ij}^{(0)}, (i=1,..m, j=1,..,n), \ (4c)$$
$$h_1^{(0)} \le b_1 \le H_1^{(0)}, i=1,...,m) \qquad (4d)$$
$$where \ w_r \ge 0, (r=1,2,..,k) \, and \sum_{r=1}^{k} w_r = 1.$$

It should be noted that the constraint (3c) in problem (α -MOMINLP) (3) has been replaced by the equivalent two constraints (4c) and (4d), where $l_{ij}^{(0)}, h_1^{(0)}, L_{ij}^{(0)}$ and $H_1^{(0)}$ are lower and upper bounds on a_{ij} and b_1 respectively.

The non-linearity in the constraints can be treated using the following transformation:

$$y_{ij} = a_{ij} x_j, \ (i=1,..m, j=1,..,n), \ (5)$$

Consequently, the problem under consideration becomes:

$$p^{'}(w): \quad \max \sum_{r=1}^{k} w_r f_r(x), \qquad (6a)$$
$$s.t \quad \sum_{j=1}^{n} y_{ij} \le b_i, \qquad (6b)$$
$$x_j \ge d_j \succ 0, \ j=1,.....,n, \ (6c)$$
$$x_j \text{ int } eger \, \forall \, j=1,...,n \ (6d)$$
$$l_{ij}^{(0)} x_j \le y_{ij} \le L_{ij}^{(0)} x_j, (i=1,..m, j=1,..,n), \ (6e)$$
$$h_1^{(0)} \le b_1 \le H_1^{(0)}, i=1,...,m) \qquad (6f)$$
$$where \ w_r \ge 0, (r=1,2,..,k) \, and \sum_{r=1}^{k} w_r = 1.$$

Problem $p^{'}(w)$ [6] is a single-objective mixed integer linear problem which can be solved easily using the branch and bound method (5).

Clearly, if (x^*, y^*, b^*) is the optimal solution of problem $p^{'}(w)$ [6],then (x^*, y^*, b^*) becomes the optimal solution of problemp (w) [4], where

$$a_{ij}^{*} = \frac{y_{ij}^{*}}{x_j^{*}}, \ (i=1,..m, j=1,..,n),$$

3. Parametric study on the (FMOILP)

Problem p (w) [4] can be rewritten in the following parametric form:

$$P(w,l,L,h,H): \quad \max \sum_{r=1}^{k} w_r f_r(x), \qquad (7a)$$

$$s.t \qquad x \in X\,(w,l,L,h,H), \qquad (7b)$$

$$Where \;\; X\,(l,L,h,H) = \{(x,a,b) \in R^{n(1+m)=m} \;/\; x \in X(a,b), x_j \le c_j,$$
$$l_{ij} \le a_{ij} \le L_{ij}, h_i \le b_i \le H_i, (i=1,..m, j=1,..,n)\}, \; (7c)$$

And l_{ij}, L_{ij}, h_i, H_i are assumed to be parameters in the parametric problem P (w, l, L, h, H) [7] rather than constants. The constraints $x_j \le c_j$ are additional constraints that have been added to the original set of constraints of problem P (w) [4] in order to get its optimal solution.

3.1. The set of feasible parameters

Definition 3.

The set of feasible parameters of problem (FMOILP) (1a)-(1b) which is denoted by A, is defined by

= {(w, l, L, h, H) $\in R^{2m(n+1)+k} / X(l,L,h,H) \neq \phi\}$ (8) A

3.2 The solvability set

Definition 4

The solvability set of problem (FMOILP) (1a)-(1b) which is denoted by β, is defined by

= {(w, l, L, h, H) $\in$ A/ problem (FMOILP) has α-pareto optimal solutions}. (9) β

3.3 The stability set of the first kind of problem (FMOILP)

Suppose that $(w^*, l^*, L^*, h^*, H^*) \in B$ with a corresponding α-pareto optimal

Solution $x^* of$ problem (FMOILP) (1a)-(1b) together with (a^*, b^*) the α-level optimal parameters, then the stability set of the first kind of problem (FMOILP) (1a)-(1b) corresponding to x^*, which is defined by S (x^*), is defined by

S (x^*)= {(w, l, L, h, H) $\in$ B/ x^* is an α-pareto optimal solution problem (FMOILP) (1a)-(1b)} (10)

3.4 Utilization of the Kuhn-Tucker conditions corresponding to problem p (w, l, L, h, H) [7]

The Kuhn-Tucker necessary optimality conditions corresponding to problem p (w, l, L, h, H) [7]j at the solution (x^*, a^*, b^*) will have the form:

$$\sum_{r=1}^{k} w_r \frac{\partial f_r(x)}{\partial x_j} - \sum_{i=1}^{m} \mu_i \frac{\partial g_i(x,a_i)}{\partial x_j} + U_j - \eta_j = 0, \quad (j = 1,..,n),$$
$$-\mu_i \frac{\partial g_i(x,a_i)}{\partial a_{ij}} - V\,ij + G_{ij} = 0, \; (i = 1,,,.,m,\; j = 1,...,n),$$
$$\mu_i - \zeta_i + Q_i = 0, \; (i = 1,,,.,m),$$
$$g_i(x,a_i) \le b_i, \quad (i = 1,,,.,m),$$
$$x_j \ge d_j, \quad (j = 1,...,n),\; x_j \le c_j, \quad (j = 1,...,n),$$
$$a_{ij} \le L_{ij}, \quad (i = 1,,,.,m,\; j = 1,...,n),$$
$$a_{ij} \ge l_{ij}, \quad (i = 1,,,.,m,\; j = 1,...,n),$$
$$b_i \le H_i, \; (i = 1,,,.,m), b_i \ge h_i, \; (i = 1,,,.,m),$$
$$\mu_i[g_i(x,a_i) - b_i] = 0, (i = 1,...,m),$$
$$U_j(-x_j + d_j) = 0, (j = 1,..,n), \eta_j(x_j - c_j) = 0, (j = 1,..,n)$$
$$V_{ij}(a_{ij} - L_{ij}) = 0, G_{ij}(-a_{ij} + l_{ij}) = 0, \; (i = 1,,,.,m,\; j = 1,...,n),$$
$$\zeta_i(b_i - H_i) = 0, \; Q_i(-b_i + h_i) = 0, (i = 1,,,.,m),$$
$$\mu_i, U_j, \eta_j, V_{ij}, G_{ij}, \zeta_i, Q_i \ge 0, \qquad (11).$$

Where all the above relations are evaluated at (x^*, a^*, b^*) and $\mu_i, U_j, \eta_j, V_{ij}, G_{ij}, \zeta_i, Q_i$ are the lagrange multipliers.

The first three together with the last seven relations of system (11) represent a poly-tope in $\mu\, U\, \eta VG\zeta Q$ Space for which its vertices can be determined using any algorithm based on

The simplex methodBalinski [1]. According to whether any of the variables

Then the set $T(x^*)$ can be determined. This set is the set of $\mu_i, U_j, \eta_j, V_{ij}, G_{ij}, \zeta_i, Q_i \ge 0,$ parameters (w,l,L,h,H) for which the Kuhn-Tucker necessary optimality conditions corresponding to problem p(w, l, L, h, H) (7) are utilized at (x^*, a^*, b^*).

Clearly $T(x^*) \subset S(x^*)$

4. Solution algorithm

In this section, we describe a solution algorithm for solving problem (FMOILP) (1a)-(1b) in finite steps. This algorithm can be summarized in the following manner:

Step 1. Set a certain degree $\alpha = \alpha \in [0,1]$.

Step 2. Elicit a membership function for each fuzzy number in the formulated problem(FMOILP) (1a)-(1b),(see [3]).

Step 3. Convert the (FMOILP) (1a)-(1b) in the form of problem $(\alpha - MOMINLP)$ (3).

Step 4. Formulate problem p (w) (4) corresponding to problem $(\alpha - MOMINLP)$ (3).

Step 5. Formulate problem $p^{,}(w)$ (6) using transformation (5).

Step 6. Choose $w = w^* \in R^k, where\ w_r^* \geq 0, \sum_{r=1}^{k} w_r^* = 1$ and solve problem $p^{,}(w^*)$ using branch and bound method [5]. Call x^* be the optimal solution of problem $p^{,}(w^*)$. This solution is an α-pareto optimal solution problem $(\alpha - MOMINLP)$ (3), where

$$y_{ij}^* = a_{ij}^* x_j^*, (i = 1,,,., m, j = 1,..., n).$$

Step 7. Determine the set $T(x^*)$ by utilizing the Kuhn-Tucker necessary optimality conditions (11) corresponding o problem p(w, l, L, h, H) (7).

5. An illustrative Example

In this section we provide a numerical simple example to clarify the developed theory and the algorithm. The problem under consideration is the following two objective integer linear programming problem involving fuzzy parameters in the constraints:

$$(FMOILP): \quad \max F(x) = \{f_1(x), f_2(x)\}$$

$$Subject\ to\ x \in X(\tilde{\tilde{a}}, \tilde{\tilde{b}})$$

Let the

$$Where\ x \in X(\tilde{\tilde{a}}, \tilde{\tilde{b}}) = \{x \in R^2 / \tilde{\tilde{a}}_{11} x_1 + \tilde{\tilde{a}}_{12} x_2 \leq \tilde{\tilde{b}}_1, \tilde{\tilde{a}}_{21} x_1 + 4x_2 \leq 9,$$
$$x_1 \geq 1, x_2 \geq 1 and \text{ int } eger\}$$
$$and\ f_1(x) = 6x_1 + x_2, f_2(x) = -x_1 + 2x_2.$$

fuzzy parameters are characterized by the following fuzzy numbers:

$$\tilde{\tilde{a}}_{11} = (2,4,6,7),\ \tilde{\tilde{a}}_{12} = (0,1,3,5)$$
$$\tilde{\tilde{a}}_{21} = (0,2,4,6),\ \tilde{\tilde{b}}_1 = (1,3,3,4)$$

Assume that the membership functions corresponding to the fuzzy numbers are of the form:

$$\mu_{\tilde{\tilde{\lambda}}}(\lambda) = \begin{cases} 0 \quad , & \lambda \leq p_1 \\ 1 - (\dfrac{\lambda - p_2}{p_1 - p_2})^2 \ , & p_1 \leq \lambda \leq p_2, p_2 \leq \lambda \leq p_3 \\ 1 - (\dfrac{\lambda - p_3}{p_4 - p_3})^2 \ , & p_3 \leq \lambda \leq p_4 \\ 0 \quad , & \lambda \geq p_4 \end{cases}$$

$Let\ \alpha = 0.36, then\ we\ get$

$$2.4 \leq a_{11} \leq 6.8,\ 0.2 \leq a_{12} \leq 4.6$$
$$0.4 \leq a_{21} \leq 5.6,\ 1.4 \leq b_1 \leq 3.8$$

The non-fuzzy $(\alpha - MOMINLP)$ can be written as:

$$(\alpha - MOMINLP):\ \max F(x) = \{6x_1 + x_2, -x_1 + 2x_2\}$$

$$s.t.\ a_{11}\,x_1 + a_{12}\,x_2 \le b_1, a_{21}\,x_1 + 4x_2 \le 9, x_1 \ge 1, x_2 \ge 1$$
$$2.4 \le a_{11} \le 6.8,\ 0.2 \le a_{12} \le 4.6$$
$$0.4 \le a_{21} \le 5.6,\ 1.4 \le b_1 \le 3.8,\quad x_1, x_2\ are\ \text{int}\ egers.$$

Using the weighting method [2], then the problem $(\alpha - MOMINLP)$ becomes:

$$p(w):\ \max F(x) = w_1(6x_1 + x_2) + w_2(-x_1 + 2x_2)$$

$$s.t.\ a_{11}\,x_1 + a_{12}\,x_2 \le b_1, a_{21}\,x_1 + 4x_2 \le 9, x_1 \ge 1, x_2 \ge 1$$
$$2.4 \le a_{11} \le 6.8,\ 0.2 \le a_{12} \le 4.6$$
$$0.4 \le a_{21} \le 5.6,\ 1.4 \le b_1 \le 3.8,\quad x_1, x_2\ are\ \text{int}\ egers.$$
$$0 \le w_1 \le 1,\ 0 \le w_2 \le 1$$

Then the problem is solved.

6. Conclusion

In this paper we focus 0n the solution of multi-objective integer linear programming problems with fuzzy parameters (FMOILP). The fuzzy parameters are involved in the constraints, a-Pareto optimality is introduced, in addition a solution algorithm is described to solve the FMOILP and a parametric study is carried out on it, finally a numerical example illustrates various aspects of the results developed in this paper.

REFERENCES

[1]. P. Czyza, "Multicriteria agricultural problem solving under uncertainty", Foundations of Control Engineering, vol.14,(1989),pp. 61-80.

[2]. H.J. Zimmermann , "Fuzzy programming and linear programming with several objective functions", Fuzzy sets and System vol. 1,(1978), pp.45- 55.

[3]. PeidroMulaPoler R. and José-Luis Verdegay, "Fuzzy optimization for supply chainplanning under supply", demand and process uncertainties Fuzzy Sets and Systems,vol. 160,(2009),pp. 2640-2657.

[4]. V. Iskander, "A computational comparison between two evaluation criteria in fuzzy multi-objective linear programs using possibility programming", Computers and Mathematics with Applications,vol.55,(2008),pp. 2506-2511

[5]. Arikan, F. 2013. A fuzzy solution approach for multi objective supplier selection. *Expert Systems with Applications,* vol. 40. pp. 947-952

[6]. R. Slowinski, "An interactive method for multi-objective linear programming with fuzzy parameters and its application to water supply planning". In: Kacprzyk, J. and Orlovski, S.A., Editors (1987)

[7]. O.M. Saad , "Stability on multi-objective linear programming problems with fuzzy parameters", Fuzzy Sets and Systems,vol. 74, (1995),pp. 207-215.

[8]. J. Kindler,Rationalizing, "water requirements with aid of fuzzy allocation model". J.Water Res. Plng. and Mgmt., ASCE,vol. 118,(1992),pp. 308 -323

[9]. Jadidi, O., Cavalieri, S., and Zolfaghari, S. 2015. An improved multi-choice goal programming approach for supplier selection problems. *Applied Mathematical Modelling,* vol. 39, no. 14. pp. 4213-4222.