Research Article

# LYAPUNOV EXPONENTIALLY STABILITY FOR SOME MODELS NONLINEAR PDEs

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## Abstract

The aim of this paper is to analyzing exponential stability of non-linear partial differential equation using Lyapunov second method. We consider different models from heat and wave non-linear equations in addition to 2×2 hyperbolic system with balance laws. We show the effectiveness of the proposed methodology using some examples of different types of nonlinear PDEs.

**Keywords:** Lyapunov function, L$^2$-norm, exponential stability

## Introduction

Lyapunov's second method, also sometimes called Lyapunov's direct method, is one of the effective and powerful classical methods for studying asymptotic behavior and stability of the dynamical systems by ordinary differential equations. In the classical Lyapunov stability theory, we will start by defining the exponential stability in ordinary differential equations. The system

$$\dot{X} = AX$$

Where $X \in R^n$ is called exponentially stable at the equilibrium $X = 0$ if there exist positive constants$M$ and $\alpha$, such that

$$\|X(t)\| \leq Me^{-\alpha t}\|X(0)\|$$

Where $\|.\|$is a vector norm[1].

It should be noted, however, that the vector norms are equivalent in a finite dimension, unfortunately, this is not true in infinite dimension systems like PDEstherefore, we miss generalizing the results of stability. Lyapunov's work in 1892 had a lasting influence on stability studies not only for ordinary differential equations but also for general dynamical systems, especially for partial differential equations [2]. In fact, Lyapunov stability theorem was applied to linear partial equations and remarkable results were obtained [3]. Vast parts of real-world physical systems are described by nonlinear partial differential equations. Such equations arise in various fields of applications, for example, fluid mechanics, gas dynamics, combustion theory, relativity, elasticity, thermodynamics, biology, ecology, neuroscience and many others. In this paper, we apply the second Lyapunovmethod to some models of nonlinear partial differential equations in one dimension in $L^2$-norm. It should be noted that it is not always easy to find a way to apply this method to nonlinear PDEs [4] we have been benefitedthe results of nonlinear energy stability obtained in convective problems, which are very similar to Lyapunov method [5].

## Preliminaries

First of all, let us define some important concepts that we will use in this paper

1. Some functional spaces

$$L^2(\Omega) = \{f(x) | \int_\Omega f^2(x)\, dx < \infty\} \tag{1}$$

All of functions in space $L^2(\Omega)$ have bounded energy

$$H^1(\Omega) = (f(x) | f \in L^2, f' \in L^2\} \tag{2}$$

2. Recall some useful inequalities

**a) Young's inequality:**

$$ab \leq \frac{\varepsilon}{2} a^2 + \frac{1}{2\varepsilon} b^2 \tag{3}$$

**b) Cauchy –Schwarz inequality:**

$$\int_0^1 uw\, dx \leq \left(\int_0^1 u^2\, dx\right)^{1/2} \left(\int_0^1 w^2\, dx\right)^{1/2} \tag{4}$$

**c) Poincare inequality:**

$$\int_0^1 u^2\, dx \leq 2u^2(1) + 4\int_0^1 u_x^2\, dx$$
$$\int_0^1 u^2\, dx \leq 2u^2(0) + 4\int_0^1 u_x^2\, dx \tag{5}$$

For any $u$ continuously differentiable on [0,1]

**d) Sobolev inequality:**

Let $\Omega$ be a bounded domain in $R^3$ with boundary$\partial\Omega$. Then for function $u$ with $u = 0$ on $\partial\Omega$

$$\left(\int_\Omega u^6 dV\right)^{1/3} \leq C\int_\Omega |\nabla u|^2\, dV \tag{6}$$

3. Let$\mathcal{D}_m$ denoted the set of diagonal $m \times m$ real matrices with strictly positive diagonal entries. We introduce the following norm for the matrix $K$

$$\rho(K) \triangleq \inf\{\|\Delta K \Delta^{-1}\|, \Delta \in \mathcal{D}_{2n} \tag{7}$$

## NonlinearPDE

**Example.1:**Consider the Burger equationwhich is more easily accessible to reader who has background in Lyapunov exponential stability of linear PDE. (Burgers' equation is a nonlinear PDE in progress in different fields of mathematics, such as fluid mechanics, nonlinear acoustics, traffic flow and gas dynamic)

$$u_t = \mu u_{xx} + uu_x \tag{8}$$
$$u(0,t) = u(1,t) = 0$$

Consider Lyapunov function

$$V = \int_0^1 u^2\, dx \tag{9}$$

Taking time derivative along the trajectory of (8), we get

$$\dot{V} = \int_0^1 uu_t\, dx = \mu \int_0^1 uu_{xx}\, dx + \int_0^1 u^2 u_x\, dx$$
$$= \mu \left[ uu_x|_0^1 - \int_0^1 u_x^2 \right] - \frac{u^3}{3}\Big|_0^1 \tag{10}$$
$$= -\mu \int_0^1 u_x^2\, dx \tag{11}$$

By using (5), we get

$$\dot{V} \leq -\frac{\mu}{4}\int_0^1 u^2\,dx \leq -\frac{\mu}{2}V \qquad (12)$$

Proving exponential stability of the system (8) in $L^2$-norm
Exact solution for Burger equation

$$u(x,t) = 2\tanh(t + x - 1) + 1$$

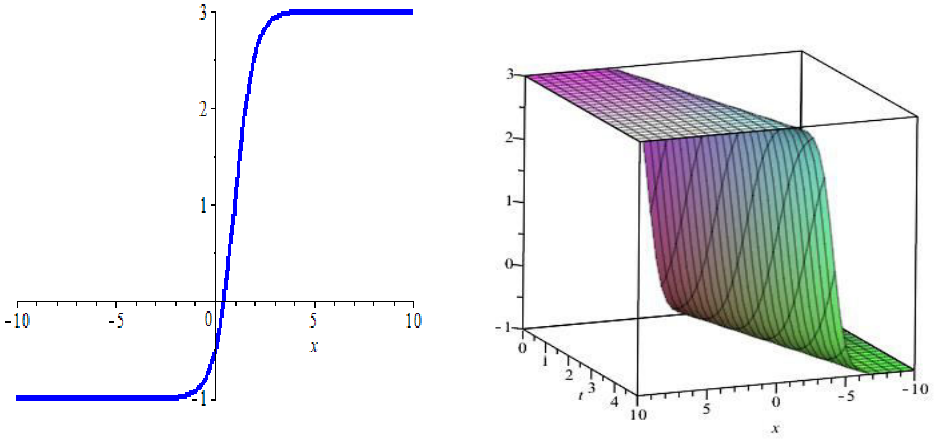

Figure 1: The 3D and 2D graphics for the Burger equation

**Example 2:** consider Burger equation in another boundary condition [3]

$$u_t = \mu u_{xx} + u u_x \qquad (13)$$

$$u(0,t) = 0$$

$$u_x(1,t) = -\frac{1}{6}\,(u(1) + u^3(1))$$

Let we recall the equation (10), and use the boundary condition given in this example, then we have

$$\dot{V} = \mu\left[u u_x|_0^1 - \int_0^1 u_x^2\right] - \frac{u^3}{3}\bigg|_0^1$$

$$= -\frac{1}{6}u^2(1) - \frac{1}{6}u^4 - \frac{1}{3}u^3(1) - \int_0^1 u_x^2\,dx$$

$$\dot{V} \leq \frac{1}{2}V \qquad (14)$$

Then the system (13) is exponentially stable in $L^2$-norm

**Example 3:**Let we consider the following diffusion equation, but here with additional term i.e. quadratic nonlinear term. [5]

$$u_t = \mu u_{xx} - u u_x + \beta u^2 \qquad (15)$$

$$u(0,t) = u(1,t) = 0$$

By using Lyapunov function (9), and taking time derivative, we get

$$\dot{V} = \int_0^1 u u_{xx}\,dx - \int_0^1 u^2 u_x\,dx + \beta\int_0^1 u^3\,dx$$

$$= u u_x|_0^1 - \int_0^1 u_x^2\,dx - \frac{u^3}{3}\bigg|_0^1 + \beta\int_0^1 u^3\,dx$$

$$= -\int_0^1 u_x^2\,dx + \beta\int_0^1 u^3\,dx \tag{16}$$

From Cauchy-Schwarz inequality (4), we get

$$\int_0^1 u^3\,dx = \int_0^1 u^2 u\,dx \le \left(\int_0^1 u^4\,dx\right)^{1/2}\left(\int_0^1 u^2\,dx\right)^{1/2}$$

Sobelov inequality (6) give us

$$\int_0^1 u^4\,dx \le \frac{1}{4}\left(\int_0^1 u_x^2\,dx\right)^2$$

Then

$$\int_0^1 u^3\,dx \le \frac{1}{2}\left(\int_0^1 u_x^2\,dx\right)\left(\int_0^1 u^2\,dx\right)^{1/2} \tag{17}$$

By putting (17) in (16), we get

$$\begin{aligned}\dot V &\le -\int_0^1 u_x^2\,dx + \frac{\beta}{2}\left(\int_0^1 u_x^2\,dx\right)\left(\int_0^1 u^2\,dx\right)^{1/2}\\ &\le -\int_0^1 u_x^2\,dx\left(1-\frac{\beta}{2}\|u\|\right) \qquad (18)\end{aligned}$$

By Poincare inequality (5), we have

$$\dot V \le -\frac{1}{2}\,V\left(1-\frac{\beta}{2}\|u\|\right) \tag{19}$$

If we want to prove exponential stability condition, we shall assume that

$$\|u_0\| \le 2\beta^{-1}$$

Then$\dot V \le -\frac{A}{2}V$, and the system (15) is exponentially stable in $L^2$-norm.

**Example 4:**consider the fisher equation [6], (Fisher's equation is a nonlinear parabolic equation firstly proposed by fisher to model the progression gene in an infinite-dimensional homeland [7]. Moreover, Fisher's equation has been used as a basis for a wide variety of models for the spatial diffusion of gene in population, chemical wave diffusion, flame diffusion, ramifying Brownian motion process and even nuclear reactor theory.

$$\begin{gathered}u_t = u_{xx} + \alpha u - \beta u^2 \qquad (20)\\ u(0) = 0\,, u(1) = U(t)\end{gathered}$$

Where $\alpha, \beta$ are positive constant and $U(t)$ is control

By using Lyapunov (9) and taking time derivative of it along the trajectory of (20), we get

$$\begin{aligned}\dot V &= \int_0^1 u u_{xx}\,dx + \alpha\int_0^1 u^2\,dx - \beta\int_0^1 u^3\,dx\\ &= u u_x|_0^1 - \int_0^1 u_x^2\,dx + \alpha\int_0^1 u^2\,dx - \beta\int_0^1 u^3\,dx\\ &= u(1)u_x(1) - \int_0^1 u_x^2\,dx + dx - \beta\int_0^1 u^3\,dx \qquad (21)\end{aligned}$$

If $U(t) = 0$, and by using Sobolev inequality (6), we get

$$\begin{aligned}\dot V &\le -\int_0^1 u_x^2\,dx + \alpha\int_0^1 u^2\,dx - \frac{\beta}{2}\left(\int_0^1 u_x^2\,dx\right)\left(\int_0^1 u^2\,dx\right)^{1/2}\\ &\le -\int_0^1 u_x^2\,dx\left(1+\frac{\beta}{2}\|u\|\right) + \alpha\int_0^1 u^2\,dx\end{aligned}$$

Now using Poincar'e inequality (5)

$$\dot{V} \leq -\frac{1}{4}\int_0^1 u^2\,dx\left(1+\frac{\beta}{2}\|u\|\right)+\alpha\int_0^1 u^2\,dx$$
$$\leq -\int_0^1 u^2\,dx\left(\frac{1}{4}\left(1+\frac{\beta}{2}\|u\|\right)-\alpha\right)$$
$$\leq -2V\left(\frac{1}{4}+\frac{\beta}{8}\|u\|-\alpha\right)$$

So,thesystem (20) is exponentially stable in $L^2$-norm if $\alpha \leq \frac{1}{4}$.

Exact solution for Fisher's equation

$$u(x,t)=\frac{1}{4}\tanh\left(\frac{5}{12}t+\frac{1}{12}\sqrt{6}x+1\right)^2+\frac{1}{2}\tanh\left(\frac{5}{12}t+\frac{1}{12}\sqrt{6}x+1\right)-\frac{3}{4}$$

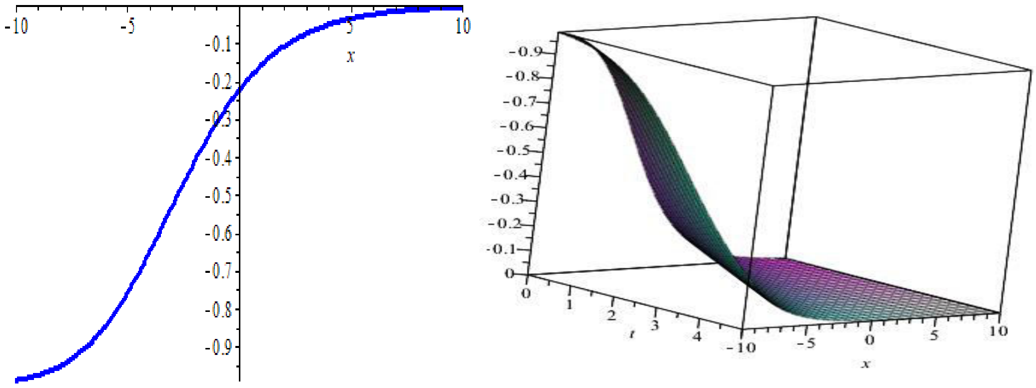

Figure 2: The 3D and 2D graphics for the Fisher's equation

**Example 5:**consider MKdV-Burgersequation [7]

$$u_t=-u_{xxx}+\varepsilon u_{xx}-6uu_x \qquad (22)$$
$$u(0)=u_x(1)=0$$
$$u_{xx}(1)=k_1u^3(1)+k_2u(1)$$

Where $\varepsilon, k_1$and $k_2$ are positive constants

By using Lyapunov (9) and taking time derivative of it along the trajectoryof (22), we get

$$\dot{V}=-\int_0^1 uu_{xxx}\,dx+\varepsilon\int_0^1 uu_{xx}\,dx-6\int_0^1 u^2u_x\,dx \qquad (23)$$

$$\int_0^1 uu_{xxx}\,dx=uu_{xx}|_0^1-\int_0^1 u_xu_{xx}\,dx=-u(1)u_{xx}(1)+\left.\frac{u_x^2}{2}\right|_0^1$$

$$\int_0^1 uu_{xxx}\,dx=-k_1u^4(1)-k_2u^2(1)-\frac{u_x^2(0)}{2}(24)$$

$$\int_0^1 uu_{xx}\,dx=-\int_0^1 u_x^2\,dx \qquad (25)$$

$$\int_0^1 u^2u_x\,dx=\frac{u^3(0)}{3} \qquad (26)$$

By putting (24), (25) and (26) in (23), we get

$$\dot{V}=-u^2(1)k_2-k_2u^4(1)-\frac{1}{2}u_x^2(0)-\varepsilon\int_0^1 u_x^2-2u^3(1)$$

By using Poincar'e inequality (5), we have

$$\dot{V} \leq -\frac{\varepsilon}{4}\int_0^1 u^2\,dx = -\frac{\varepsilon}{2}V$$

Then the system (22) is exponentially stable in $L^2$-norm.

## 2×2 Hyperbolic System with balance laws

**Example 6:** double-pipe heat exchanger is governed, based on the thermal energy balance equations by the following PDE system [8]

$$\begin{cases} u_t + uu_x = \alpha_1(w-u) \\ w_t + ww_x = \alpha_2(u-w) \end{cases} \qquad (27)$$

Let $\alpha_1(w-u) = \delta(u,w)$ and $\alpha_1(u-w) = \gamma(u,w)$

We can write the system (27) in matrix form

$$\begin{bmatrix} u_t \\ w_t \end{bmatrix} + \begin{bmatrix} u & 0 \\ 0 & w \end{bmatrix}\begin{bmatrix} u_x \\ w_x \end{bmatrix} = \begin{bmatrix} \delta \\ \gamma \end{bmatrix} \qquad (28)$$

Let we define the vector $T \triangleq (u,w)^\top$ then system (28) can be written in

$$T_t + F(T)T_x = E(T) \qquad (29)$$

Where

$$F(T) \triangleq \begin{bmatrix} u & 0 \\ 0 & w \end{bmatrix}, \qquad E(T) = \begin{bmatrix} \delta(u,w) \\ \gamma(u,w) \end{bmatrix}$$

A constant state $T^*$ which is satisfies the condition $E(T^*) = 0$ is an equilibrium state (or steady state) for the system (29)

Now it is well known that for any system in the form (27), there exists change of coordinates (Riemann coordinates) $Z = \rho(T)$ which enable us to rewrite the system (1) in the characteristic form [9]

$$\partial t \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} + \begin{bmatrix} c_1(Z) & 0 \\ 0 & c_2(Z) \end{bmatrix} \partial x \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = Y(Z) \qquad (30)$$

Where

$c_i(Z) \triangleq \lambda_i(\beta^{-1}(Z))$ and $Y(Z) \triangleq (\frac{\partial\beta}{\partial T}(\beta^{-1}(Z))Y(\beta^{-1}(Z))$

Let we put $Y(Z) = HZ$ then the system (30) can be written as

$$Z_t + LZ_x = HZ \qquad (31)$$

Where

$$Z \triangleq (Z_1, Z_2)^T, L = dig\{c_1, c_2\}$$

With boundary condition

$$N_0 Z(0,1) + N_1 Z(1,t) = 0 \qquad (32)$$

The system (31) is the linear approximation of the system (30) around the origin

Consider the Lyapunov function[10]

$$V = \int_0^1 Z^T P(x) Z\ dx \quad (33)$$

Where the matrix $P(x)$ is defined as

$P(x) \triangleq diag\{p_i e^{-\sigma_i \mu x}, i = 1,2,\dots,2n\}$, With $\varepsilon > 0, p_i > 0$ are positive real numbers and $\sigma_i = sign(c_i)$.

Taking time derivative of function $V$ along the solutions of (31)

$$\dot{V} = \int_0^1 (\partial_t Z^T P(x) Z + Z^T P(x) \partial_t Z)\,dx$$

$$= -\int_0^1 (\partial_x Z^T L P(x) Z + Z^T P(x) L \partial_x Z - Z^T H^T P(x) Z - Z^T P(x) H Z) dx$$

$$= -\int_0^1 \partial_x (Z^T G(x) Z)\, dx + \int_0^1 Z^T (H^T P(x) + P(x) H) Z\, dx$$

Where

$G(x) \triangleq diag\{p_i|c_i|e^{-\sigma_i \mu x}, i = 1, \dots, 2n\}$, is positive diagonal matrix.

Using integration by parts, we have

$$\dot{V} = -\int_0^1 \partial_x [Z^T G(x) Z]\, dx - \int_0^1 Z^T (\mu G(x) - H^T P(x) - P(x) H) Z\, dx$$

$$= -Z^T G(x) Z|_0^1 - \int_0^1 Z^T (\mu G(x) - H^T P(x) - P(x) H) Z\, dx$$

$$= -[Z^T(1,t) G(1) Z(1,t) - Z^T(0,t) G(0) Z(0,t)] - \int_0^1 Z^T (\mu R(x) - H^T P(x) - P(x) H) Z\, dx$$

The system (31)-(32) is exponentially stable if there exist $\mu > 0$ and $p_i > 0, i = 1, \dots, 2n$ satisfy the following two conditions:

1. $Z^T(0,t) G(0) Z(0,t) - Z^T(1,t) G(1) Z(1,t)$ is positive definite according to linear boundary condition $N_0 Z(0,t) + N_1 Z(1,t) = 0$
2. $\forall x \in (0,1)$ the matrix $\mu M(x) - H^T P(x) - P(x) H$ is positive definite

The boundary condition which satisfy the condition (1) is [11]

$N_r\big(Z^+(0,t), Z^+(1,t), Z^-(0,t), Z^-(1,t)\big) = 0$ (34)

Assume that the map $N_r$ is differentiable in a neighborhood of the orgin

The linearization of the boundary condition (34) about the origin is

$$\begin{bmatrix} Z^+(0,t) \\ Z^-(1,t) \end{bmatrix} = \begin{bmatrix} K_{00} & K_{01} \\ K_{10} & K_{11} \end{bmatrix} \begin{bmatrix} Z^+(1,t) \\ Z^-(0,t) \end{bmatrix} \qquad (35)$$

Again, the linear approximation of system (31) around the origin

$\begin{bmatrix} \partial_t Z^+ + L^+ \partial_x Z^+ \\ \partial_t Z^- - L^- \partial_x Z^- \end{bmatrix} = MZ$ (36)

Consider the following Lyapunov function

$$V = \int_0^1 \left[\left(Z^{+^T} P_0 Z^+\right) e^{-\mu x} + (Z^{-^T} P_1 Z^-) e^{\mu x}\right] dx \qquad (37)$$

Where $P_0 \in \mathcal{D}_n, P_1 \in \mathcal{D}_n$ and $\mu > 0$. Taking the time derivative of $V$ we have

$$\dot{V} = \int_0^1 -\partial_x (Z^{+^T} P_0 L^+ Z^+) e^{-\mu x}\, dx + \int_0^1 -\partial_x (Z^{-^T} P_1 L^- Z^-) e^{-\mu x}\, dx$$

$$+ \int_0^1 Z^T (M^T P(x) + P(x) M) Z\, dx$$

By using integration by parts we have

$$\dot{V} = \dot{V}_1 + \dot{V}_2$$

Where

$$\dot{V}_1 \triangleq -\left[Z^{+^T} P_0 L^+ Z^+ e^{-\mu x}\right]_0^1 + \left[Z^{-^T} P_1 L^- Z^- e^{\mu x}\right]_0^1$$

$$\dot{V}_2 \triangleq \int_0^1 Z^T (-\mu P(x) L + M^T P(x) + P(x) M) Z\, dx$$

Where

$P(x) \triangleq diag\{P_0 e^{-\mu x}, P_1 e^{\mu x}\}$ and $L \triangleq diag\{L^+, L^-\}$

Let $Z_0^- \triangleq Z^-(0,t), Z_1^+ \triangleq Z^+(1,t)$

1) Analysis of $\dot{V}_1$ terms:

By using boundary condition (35), we have

$$\dot{V}_1 = -\left[Z^{+^T} P_0 L^+ Z^+ e^{-\mu x}\right]_0^1 + \left[Z^{-^T} P_1 L^- Z^- e^{\mu x}\right]_0^1$$

$$= -\left(Z^{+^T} P_0 L^+ Z^+ e^{-\mu x} + Z^{-^T} P_1 L^- Z^- e^{\mu x}\right) + \left(Z_1^{+^T} K_{00}^T + Z_0^{-^T} K_{01}^T\right) P_0 L^+ (K_{00} Z_1^+ + K_{01} Z_0^-)$$

$$+ \left(Z_1^{+^T} K_{00}^T + Z_0^{-^T} K_{01}^T\right) P_0 L^+ (K_{00} Z_1^+ + K_{01} Z_0^-) e^{\mu}$$

**Theorem (1):**if $\rho(K) < 1$, there exist $\mu > 0$ such that, if $\|M\| < \varepsilon$, then the linear hyperbolic system (35)-(36) is exponentially stable.[12]

Now since $\rho(K) < 1$, there exist $L_0 \in \mathcal{D}_n, L_1 \in \mathcal{D}_n$ and $\Delta \triangleq diag\{L_0, L_1\}$, such that

$$\|\Delta K \Delta^{-1}\| < 1 \qquad (38)$$

We selected the matrices $P_0$ and $P_1$ such that $P_0 L^+ = \mathcal{D}_0^2$ and $P_1 L^- = \mathcal{D}_1^2$

Let $q_0 \triangleq \mathcal{D}_0 Z_0^-$ , $q_1 \triangleq \mathcal{D}_1 Z_1^+$and $q^T \triangleq (q_0^T, q_1^T)$ Then, by using the inequality (38), we get

$$\left(Z_1^{+^T} K_{00}^T + Z_0^{-^T} K_{01}^T\right) P_0 L^+ (K_{00} Z_1^+ + K_{01} Z_0^-)$$

$$+ \left(Z_1^{+^T} K_{00}^T + Z_0^{-^T} K_{01}^T\right) P_0 L^+ (K_{00} Z_1^+ + K_{01} Z_0^-) e^{\mu}$$

$$= \|\Delta K \Delta^{-1} q\|^2 < \|q\|^2$$

$$= Z_1^{+^T} P_0 L^+ Z_1^+ + Z_0^{-^T} P_1 L^- Z_0^-$$

By selecting $\mu$ small enough such that $\dot{V}_1$ is negative definite.

2) Analysis of $\dot{V}_2$ term:

It is clear that for any $\mu > 0$ there exist $\varepsilon$ and $\alpha$ are two positive constants such that $\|M\| < \varepsilon$ $\Rightarrow$ $\dot{V}_2 \leq -\alpha V \Rightarrow \dot{V} = \dot{V}_2 + \dot{V}_2 \leq -\alpha V$then the linear system (35)-(36) is exponentially stable in$L^2$-norm.

## Conclusion

In this paper, stability analysis for non-linear partial differential equation scrutinized by Lyapunov direct method. We have introduced affair of locating sufficient boundary condition for exponential stability of some models of PDE in$L^2$-norm. We also consider the system of balance laws as example of hyperbolic systems and deduced the exponential stability of the steady-state in the linear case for the given example, but the same Lyapunov function cannot be used directly to analyze the stability of the nonlinear case in $L^2$-norm( as shown in detail in [13]) .

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