Δ-lacunary weakly statistical convergence and weak $N_\theta(\Delta)$-summable of order α

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ABSTRACT

In this section, we are going to introduce the principles of Δ-lacunary weakly statistically convergent sequence of order α (or $WS_\theta^\alpha(\Delta)$-statistically convergent) and Δ-lacunary weak N-summability of order α (or $WN_\theta^\alpha(\Delta)$-summability) for $\alpha \in (0, 1]$. We are also going to give some characteristics of these two principles with some relations between them.

1. Introduction

(Zygmund, 1979) provided the statistical convergence's idea in the first edition published in Warsaw, After that (Fast, 1951) and (Steinhaus, 1951) reintroduced this idea formally and then (Schoenberg, 1959) independently reintroduced this idea.

The statistical convergence depends on the density of the subsets of $\mathbb{N}$.
Let $E \subset \mathbb{N}$. Then $\delta(E)$ denotes the natural density of E which defined by

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : k \in E\}|$$

whenever the limit exists.$|\{k \leq n : k \in E\}|$denotes the number of elements of E not exceeding n.

A sequence$(x_k)$is named statistically convergence to a number l if for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n: |x_k - l| \geq \varepsilon\}| = 0.$$

We write $S - \lim_{k\to\infty} x_k = l$ in this case see(Fridy, 1985). This principle is used to solve many of the problems in trigonometric series, Banach spaces, fuzzy set theory, ergodic theory. Several authors have also researched related topics with the concept of summability theory such as (Rath and Tripathy, 1994), (Şengül and Et, 2014), (Şengül, 2017)

Also many researchers studied related topics with summability theory (Salat, 1980), (Rath and Tripathy, 1994),(Şengül and Et 2014), (das and Pal 2011),Şengül 2017,(Şengül and Et 2018)

(Gadjiev and Orhan 2002) was introduced Thestatistical convergence with degree $0 < \beta < 1$in 2002. Then (Çolak 2010) were studied the statistically convergence of order α and strong p- Cesàro summability of order αin 2010, and Ercan, S., Altin, Y., and Bektaş, Ç.A. (2019) introdused On lacunary weak statistical convergence of order α in (2019).

Lacunary density and lacunary statistical convergence are introduced by fridy and Orhan(Freedman, Sember and Raphael 1978).and(PAC. J. MATH. 1993)as follows:

Let $K \subset \mathbb{N}$. θ-density of set $K$was defined by

$$\delta_\theta(K) = \lim_{r\to\infty} \frac{1}{h_r} |K \cap I_r|.$$

A sequence (x_k) is a lacunary statistically convergent to l if for every $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : |x_k - l| \geq \varepsilon\}| = 0.$$

In this case, we write $\delta_\theta - \lim_{k\to\infty} x_k = l$.The convergence of lacunary sequence and related notaions were studied by some authors (Et and Sengul 2016, Et and Sengul 2015, Et and Sengul 2014 (Rath and Tripathy, 1994),(Das and Mishra 1983).

The set of all lacunary strongly convergent sequence N_θ is given as follows

$$N_\theta = \left\{ x = (x_k) : \lim_{r\to\infty} \left(\frac{1}{h_r^\alpha} \sum_{k\in I_r} |x_k - l| \right) = 0 \text{, for some } l \right\}.$$

A sequence (x_k) in a normed space X is said to be weakly convergent to $l \in X$ provided that $\lim_{k\to\infty} \varphi(x_k - l) = 0$. For each $\varphi \in X^*$, the continuous dual of X. In this case, we write $W - \lim_{k\to\infty} x_k = l$.

A sequence (x_k) in normed space X is called weakly statistically convergent to $l \in X$ provided that,

2.1Main results

In this section we provide the principles of WS_θ^α-statistical convergence and weak N_θ-summability of order α for $\alpha \in (0, 1]$ in this section. We are also going to give some characteristics of these principles with some relations between them. In this study, we write X to denote a normed linear space and we write X^* to denote continuous dual and also$\theta = (k_r)$ denotes a lacunary sequence such that $h_r = k_r - k_{r-1} \to \infty$as$r \to \infty$, $I_r = (k_{r-1}, k_r]$ and $k_0 = 0$. Also, the ratiok_r/k_{r-1} abbreviated by q_r.

Definition 2.1 Suppose $\alpha \in (0,1]$ and $\theta = (k_r)$ is a lacunary sequence. A sequence $(x_k) \in X$ is named Δ-lacunary weakly statistically convergent sequence of order α (or $WS_\theta^\alpha(\Delta)$-statistically convergent) to l if, for any $\varphi \in X^*$

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} |\{k \in I_r : |\varphi(\Delta x_k - l)| \geq \varepsilon\}| = 0 \qquad (2.1)$$

for every $\varepsilon > 0$. Let $I_r = (k_{r-1}, k_r]$ and h_r^α denotes the αth power $(h_r)^\alpha$ of h_r, that is, $h^\alpha = (h_r^\alpha) = (h_1^\alpha, h_2^\alpha, \ldots, h_r^\alpha, \ldots)$. In this case, we write $WS_\theta^\alpha(\Delta) - \lim_{k\to\infty} x_k = l$. The space of these sequences is denoted by $WS_\theta^\alpha(\Delta)$. Instead of $WS_\theta^\alpha(\Delta)$ we write $WS_\theta(\Delta)$if $\alpha = 1$. And instead of $WS_\theta^\alpha(\Delta)$ we write $WS(\Delta)$ if $\alpha = 1$ and $\theta = (2^r)$.

Definition 2.2 Suppose $\alpha \in (0,1]$ and $\theta = (k_r)$ is a lacunary sequence.A sequence $(x_k) \in X$ is named weakly $N_\theta(\Delta)$-summable of order α to l if

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} \sum_{k\in I_r} |\varphi(\Delta x_k - l)| = 0$$

and it is denoted by $WN_\theta^\alpha(\Delta) - \lim_{k\to\infty} x_k = l$. The class of $WN_\theta^\alpha(\Delta)$-summable sequences of order α is symbolized by $WN_\theta^\alpha(\Delta)$. Instead of $WN_\theta^\alpha(\Delta)$ we write $WN_\theta(\Delta)$ if $\alpha = 1$. We write $W^\alpha(\Delta)$ when $\theta = (k_r)$. Instead of $W^\alpha(\Delta)$ we write $W_0^\alpha(\Delta)$ if $l = 0$. The class of weakly $N_\theta(\Delta)$-summable sequence of order α to 0 is symbolized by $WN_{\theta,0}^\alpha(\Delta)$.

Definition 2.3 Suppose $\alpha \in (0,1]$ and $\theta = (k_r)$ is a lacunary sequence. A sequence $(x_k) \in X$ is named Δ-strongly lacunary statistically convergent of order $\alpha$to l, if

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} |\{k \in I_r : \|\Delta x_k - l\| \geq \varepsilon\}| = 0$$

for every $\varepsilon > 0$. We write $S_\theta^\alpha(\Delta) - \lim_{k\to\infty} x_k = l$ in this situation.

Definition 2.2 Suppose $\alpha \in (0,1]$ and $\theta = (k_r)$ is a lacunary sequence.A sequence $(x_k) \in X$ is named an $WS_\theta^\alpha(\Delta)$-Cauchy sequence if there is a subsequence $(x_{k'(r)})$ of (x_k) such that $k'(r) \in I_r$ for every r, $W(\Delta) - \lim_{r\to\infty} x_{k'(r)} = l$

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} \left|\{k \in I_r : \left|\varphi(\Delta x_k - \Delta x_{k'(r)})\right| \geq \varepsilon\}\right| = 0$$

for each $\varphi \in X^*$ and every $\varepsilon > 0$. This definition is similar to the definition WS_θ-Cauchy for the case $\alpha = 1$.

It is clear that the $WS_\theta^\alpha(\Delta)$-statistically convergent is defined for $\alpha \in (0,1]$. But for $\alpha > 1$ it is not defined. For this, if we take the functional $\varphi(x) > 0$on X, then for any number l we have

$$\lim_{r\to\infty}\frac{1}{h_r^{\alpha}}|\{k\in I_r:|\varphi(\Delta x_k-l)|\geq\varepsilon\}|=0$$

for every $\varepsilon>0$. This means that the $WS_\theta^\alpha(\Delta)-\lim_{k\to\infty}x_k$is not unique for $\alpha>1$.

Theorem 2.5 Let $\alpha\in(0,1]$and (x_k),$(y_k)\in X$.

(i) If $WS_\theta^\alpha(\Delta)-\lim_{k\to\infty}x_k=l$, then l must be unique.

(ii) If $WS_\theta^\alpha(\Delta)-\lim_{k\to\infty}x_k=l_0$and c being a scalar, then $WS_\theta^\alpha(\Delta)-\lim_{k\to\infty}cx_k=cl_0$.

(iii) If $WS_\theta^\alpha(\Delta)-\lim_{k\to\infty}x_k=l_0$and $WS_\theta^\alpha(\Delta)-\lim_{k\to\infty}y_k=l_1$, $WS_\theta^\alpha(\Delta)-\lim_{k\to\infty}(x_k+y_k)=l_0+l_1$.

Theorem 2.6 (Ercan et al., 2019) Let $\theta=(k_r)$and $\theta'=(s_r)$ be lacunary sequences such that for each $r\in\mathbb{N}$, $I_r\subset J_r$ and $0<\alpha\leq\beta\leq1$.

(i) If

$$\lim_{r\to\infty}\inf\frac{h_r^\alpha}{\ell_r^\beta}>0, \qquad (2.2)$$

then$WS_{\theta'}^\beta(\Delta)\subseteq WS_\theta^\alpha(\Delta)$.

(ii) If

$$\lim_{r\to\infty}\frac{\ell_r}{h_r^\beta}=1, \qquad (2.3)$$

then $WS_\theta^\alpha(\Delta)\subseteq WS_{\theta'}^\beta(\Delta)$.

Proof:

(i) Assuming $I_r\subset J_r$ for each $r\in\mathbb{N}$ and let (2.2) be satisfied. For any $\varepsilon>0$,

$$\{k\in J_r:|\varphi(\Delta x_k-l)|\geq\varepsilon\}\supseteq\{k\in I_r:|\varphi(\Delta x_k-l)|\geq\varepsilon\}$$

and so

$$\frac{1}{\ell_r^\beta}|\{k\in J_r:|\varphi(\Delta x_k-l)|\geq\varepsilon\}|\geq\frac{h_r^\alpha}{\ell_r^\beta}\frac{1}{h_r^\alpha}|\{k\in I_r:|\varphi(\Delta x_k-l)|\geq\varepsilon\}|$$

for each $r\in\mathbb{N}$, where $h_r=k_r-k_{r-1}$, $I_r=(k_{r-1},k_r]$, $j_r=(s_{r-1},s_r]$ and $\ell_r=s_r-s_{r-1}$. By using (2.2) and taking the limit as $r\to\infty$ on both sides in the last inequality we obtain that $WS_\theta^\beta(\Delta)\subseteq WS_\theta^\alpha(\Delta)$.

(ii) Assume that $(x_k)\in WS_\theta^\alpha(\Delta)$ and (2.3) is satisfied. Since $I_r\subset J_r$, then for every $\varepsilon>0$, we have

$$\frac{1}{\ell_r^\beta}|\{k\in J_r:|\varphi(\Delta x_k-l)|\geq\varepsilon\}|$$

$$=\frac{1}{\ell_r^\beta}|\{s_{r-1}<k\leq k_{r-1}:|\varphi(\Delta x_k-l)|\geq\varepsilon\}|$$

$$+\frac{1}{\ell_r^{\beta}}|\{k_r < k \le s_r : |\varphi(\Delta x_k - l)| \ge \varepsilon\}|$$

$$+\frac{1}{\ell_r^{\beta}}|\{k_{r-1} < k \le k_r : |\varphi(\Delta x_k - l)| \ge \varepsilon\}|$$

$$\le \left(\frac{\ell_r}{h_r^{\beta}} - 1\right) + \frac{1}{h_r^{\alpha}}|\{k \in I_r : |\varphi(\Delta x_k - l)| \ge \varepsilon\}|$$

for all $r \in \mathbb{N}$. Since $\lim_{r\to\infty}\frac{\ell_r}{h_r^{\beta}} = 1$ by (2.3) the first term and since $(x_k) \in WS_\theta^\alpha(\Delta)$ the second term tends to 0 as $r \to \infty$ $\left(\left(\frac{\ell_r}{h_r^{\beta}} - 1\right) \ge 0\right)$. This implies that $WS_\theta^\alpha(\Delta) \subseteq WS_{\theta'}^\beta(\Delta)$. We have the following results from Theorem 2.6.

Corollary 2.7 Suppose $\theta = (k_r)$ and $\theta' = (s_r)$ are lacunary sequences such that $I_r \subseteq J_r$ for each $r \in \mathbb{N}$. If (2.2) holds, then,

(i) $WS_{\theta'}^\alpha(\Delta) \subseteq WS_\theta^\alpha(\Delta)$ for $0 < \alpha \le 1$,
(ii) $WS_{\theta'}(\Delta) \subseteq WS_\theta^\alpha(\Delta)$ for $0 < \alpha \le 1$,
(iii) $WS_{\theta'}(\Delta) \subseteq WS_\theta(\Delta)$.

If (2.3) holds, then

(i) $WS_\theta^\alpha(\Delta) \subseteq WS_{\theta'}^\alpha(\Delta)$ for $0 < \alpha \le 1$,
(ii) $WS_\theta^\alpha(\Delta) \subseteq WS_{\theta'}(\Delta)$ for $0 < \alpha \le 1$,
(iii) $WS_\theta(\Delta) \subseteq WS_{\theta'}(\Delta)$.

Corollary 2.8 Suppose $\alpha, \beta \in (0,1]$ such that$\alpha < \beta$.Then $WS_\theta^\alpha(\Delta) \subseteq WS_\theta^\beta(\Delta)$.

Corollary 2.9 Suppose $\theta = (k_r)$ is a lacunary sequence and $0 < \alpha \le 1$. Then $WS_\theta^\alpha(\Delta) \subseteq WS_\theta(\Delta)$.

Corollary 2.10

(i) $WS_\theta^\alpha(\Delta) = WS_\theta^\beta(\Delta)$ if $\alpha = \beta$,
(ii) $WS_\theta^\alpha(\Delta) = WS_\theta(\Delta)$ if $\alpha = 1$.

Theorem 2.11 Suppose $\theta = (k_r)$is a lacunary sequence, $0 < \alpha \le 1$ and $\lim_{r\to\infty}\inf\frac{h_r^\alpha}{h_r} > 0$. If there exists $K = \{k_1 < k_2 < \cdots\} \subset \mathbb{N}$ such that $\delta_\theta(\Delta)(K) = 1$ and $\lim_{k\in K}\varphi(\Delta x_k - l) = 0$ for any $\varphi \in X^*$, then (x_k) is $WS_\theta^\alpha(\Delta) -$ convergent to l.

Proof: Assume that $\lim_{r\to\infty}\inf\frac{h_r^\alpha}{h_r} > 0$ holds and there exists a set $K = \{k_1 < k_2 < \cdots\} \subset \mathbb{N}$ with $\delta_\theta(\Delta)(K) = 1$ and $\lim_{k\in K}\varphi(\Delta x_k - l) = 0$for $\varphi \in X^*$. Then there is a number $N \in \mathbb{N}$ such that $|\varphi(\Delta x_k - l)| < \varepsilon$ for all $k \ge N$ and for any $\varphi \in X^*$.

If we take $M_\varepsilon = \{k \in I_r : |\varphi(\Delta x_k - l)| \ge \varepsilon\} \subseteq \mathbb{N} - \{k_{N+1}, k_{N+2}, k_{N+3}, \ldots\}$, then $\delta_\theta(\Delta)(M_\varepsilon) = 0$. Using Corollary 2.7 (ii) we get that (x_k) is $WS_\theta^\alpha(\Delta)$-convergent to l.

Theorem 2.12 Let $x = (x_k) \in w$. If $W(\Delta) - \lim_{k\to\infty} x_k = l$, then $WS_\theta(\Delta) - \lim_{k\to\infty} x_k = l$.

Proof: Let $W(\Delta)-\lim_{k\to\infty} x_k = l$, then for each $\varepsilon > 0$ and any $\varphi \in X^*$ if there is a number $N \in \mathbb{N}$ such that

$$|\varphi(\Delta x_k - l)| < \varepsilon$$

for each $k \geq N$. So the set $M(\varepsilon) = \{k \in I_r : |\varphi(\Delta x_k - l)| \geq \varepsilon\}$ is finite; for which $\delta_\theta(\Delta)\big(M(\varepsilon)\big) = 0$. This shows that $WS_\theta(\Delta) - \lim_{k\to\infty} x_k = l$.

In general, the opposite of the above result is not correct. To show that we can use Example 3.16.

Theorem 2.13 Assume that $\theta = (k_r)$ is a lacunary sequence, $0 < \alpha \leq 1$, $x = (x_k) \in w$ and $\lim_{r\to\infty} \inf \frac{h_r^\alpha}{h_r} > 0$. If $W(\Delta) - \lim_{r\to\infty} x_k = l$, then $WS_\theta^\alpha(\Delta) - \lim_{r\to\infty} x_k = l$.

It can be easily proved from corollary 2.7 (ii) and theorem 2.12

Theorem 2.12 Let$0 < \alpha \leq 1$. Then$S_\theta^\alpha(\Delta) \subset WS_\theta^\alpha(\Delta)$ for any lacunary sequence $\theta = (k_r)$ with the same limit in X.

Proof: Let $(x_k) \in X$ such that $x_k \xrightarrow{S_\theta^\alpha(\Delta)} l$. Then for each $\varepsilon > 0$,

$$\frac{1}{h_r^\alpha}|\{k \in I_r : \|\Delta x_k - l\| \geq \varepsilon\}| = 0.$$

Now, for each $\varepsilon > 0$ and any $\varphi \in X^*$, the expression

$$\frac{1}{h_r^\alpha}|\{k \in I_r : |\varphi(\Delta x_k - l)| \geq \varepsilon\}| \leq \frac{1}{h_r^\alpha}|\{k \in I_r : \|\varphi\|\|\Delta x_k - l\| \geq \varepsilon\}|$$

$$= \frac{1}{h_r^\alpha}\left|\left\{k \in I_r : \|\Delta x_k - l\| \geq \frac{\varepsilon}{\|\varphi\|}\right\}\right|,$$

gives immediately $\frac{1}{h_r^\alpha}|\{k \in I_r : |\varphi(\Delta\, x_k - l)| \geq \varepsilon\}| = 0$.

The converse is not true, in general. To show that we can use Example 3.19.

Theorem 2.15 Let $\theta = (k_r)$be a lacunary sequence and $0 < \alpha \leq 1$.If $\lim\inf q_r > 1$, then, $WS^\alpha(\Delta) \subset WS_\theta^\alpha(\Delta)$.

Proof: Assuming$\lim\inf q_r > 1$, then we can choose $\delta > 0$ such that $1 + \delta \leq q_r$ for sufficiently large r, so that

$$\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta} \text{and } \frac{1}{k_r^\alpha} \geq \frac{\delta^\alpha}{(1+\delta)^\alpha}\frac{1}{h_r^\alpha}.$$

If $(x_k) \in WS^\alpha(\Delta)$, then for sufficiently large r and for any $\varepsilon > 0$, and we have

$$\frac{1}{k_r^\alpha}|\{k \leq k_r : |\varphi(\Delta x_k - l)| \geq \varepsilon\}| \geq \frac{1}{k_r^\alpha}|\{k \in I_r : |\varphi(\Delta x_k - l)| \geq \varepsilon\}|$$

$$\geq \frac{\delta^\alpha}{(1+\delta)^\alpha}\frac{1}{h_r^\alpha}|\{k \in I_r : |\varphi(\Delta x_k - l)| \geq \varepsilon\}|.$$

So, the sufficiency is proved.

Theorem 2.16 Assume that $\theta = (k_r)$be a lacunary sequence and$0 < \alpha \leq 1$. If $\limsup q_r < \infty$, then $WS_\theta^\alpha(\Delta) \subset WS(\Delta)$.

Proof: If $\limsup q_r < \infty$, then we can choose a number $H > 0$ such that $H > q_r$for all r. Let $(x_k) \in WS_\theta^\alpha(\Delta)$, and let $N_r = |\{k \in I_r : |\varphi(\Delta x_k - l)| \geq \varepsilon\}|$. Then by using (2.1), given $\varepsilon > 0$, there is an $r_0 \in \mathbb{N}$ such that for all $r > r_0$,

$$\frac{N_r}{h_r^\alpha} < \varepsilon \text{ and} \frac{N_r}{h_r} < \varepsilon$$

where $0 < \alpha \leq 1$. The rest of proof is seen in(Lemma 3) from (Fridy and Orhan, 1993).

Theorem 2.17 Assume that $\theta = (k_r)$ is a lacunary and$0 < \alpha \leq 1$. A sequence (x_k) is $WS_\theta^\alpha(\Delta)$-Cauchy sequence if and only if (x_k) is an $WS_\theta^\alpha(\Delta)$-convergent.

Theorem 2.18 Assume that $\theta = (k_r)$ is a lacunary sequence and$0 < \alpha \leq 1$. If$\liminf_r q_r > 1$, then $W^\alpha(\Delta) \subset WN_\theta^\alpha(\Delta)$.

The proof is similar to Theorem 3.23.

Theorem 2.19 Assume that $\theta = (k_r)$is a lacunary sequence and$0 < \alpha \leq 1$. If $\limsup_r \frac{k_r}{k_{r-1}^\alpha} < \infty$, then $WN_\theta(\Delta) \subset W^\alpha(\Delta)$.

Theorem 2.20 Suppose $\theta = (k_r)$ and $\theta' = (s_r)$are lacunary sequences such that $I_r \subseteq J_r$ for $r \in \mathbb{N}$and $0 < \alpha \leq \beta \leq 1$.

(i) $WN_{\theta'}^\beta(\Delta) \subset WN_\theta^\beta(\Delta)$if (2.2) holds.

(ii) $WN_\theta^\alpha(\Delta) \subset WN_{\theta'}^\beta(\Delta)$if (2.3) holds.

Proof:

(i) Omitted.

(ii) Assuming that $(x_k) \in WN_\theta^\alpha(\Delta)$ and that (2.3) holds. Since φ is bounded there exists $M > 0$such that $|\varphi(\Delta x_k - l)| \leq M$ for each k. We can find inclusion from.

$$\frac{1}{\ell_r^\beta}\sum_{k\in J_r}|\varphi(\Delta x_k - l)| = \frac{1}{\ell_r^\beta}\sum_{k\in J_r - I_r}|\varphi(\Delta x_k - l)| + \frac{1}{\ell_r^\beta}\sum_{k\in I_r}|\varphi(\Delta x_k - l)|$$

$$\leq \left(\frac{\ell_r}{h_r^\beta} - 1\right)M + \frac{1}{h_r^\alpha}\sum_{k\in I_r}|\varphi(\Delta x_k - l)|$$

for every $r \in \mathbb{N}$. Therefore $(x_k) \in WN_{\theta'}^\beta(\Delta)$.

The result below is obtained from Theorem 2.20.

Corollary 2.21 Suppose $\theta = (k_r)$and $\theta' = (s_r)$are two lacunary sequences with the condition $I_r \subseteq J_r$ for every $r \in \mathbb{N}$. If (2.2) holds, then we have the following inclusions

(i) $WN_{\theta'}^{\alpha}(\Delta) \subseteq WN_{\theta}^{\alpha}(\Delta)$, for each $0 < \alpha \leq 1$,

(ii) $WN_{\theta'}(\Delta) \subseteq WN_{\theta}^{\alpha}(\Delta)$, for each $0 < \alpha \leq 1$,

(iii) $WN_{\theta'}(\Delta) \subseteq WN_{\theta}(\Delta)$.

If (2.3) holds, then

(i) $WN_{\theta}^{\alpha}(\Delta) \subset WN_{\theta'}^{\alpha}(\Delta)$, for each $\alpha \in (0, 1]$,

(ii) $WN_{\theta}^{\alpha}(\Delta) \subset WN_{\theta'}(\Delta)$, for each $\alpha \in (0, 1]$,

(iii) $WN_{\theta}(\Delta) \subset WN_{\theta'}(\Delta)$.

Theorem 2.22 Suppose $\theta = (k_r)$ and $\theta' = (s_r)$are lacunary sequences such that $I_r \subseteq J_r$ for all $r \in \mathbb{N}$, α and $\beta$are fixed real numbers such that $0 < \alpha \leq \beta \leq 1$.

(i) $WN_{\theta'}^{\beta}(\Delta) \subseteq WS_{\theta}^{\alpha}(\Delta)$ if (2.2) holds,

(ii) $WS_{\theta}^{\alpha}(\Delta) \subseteq WN_{\theta'}^{\beta}(\Delta)$if (2.3) holds.

The proof is similar to Theorem 3.27.

Corollary 2.23 Assume that $\theta = (k_r)$and $\theta' = (s_r)$ are two lacunary sequences such that $I_r \subset J_r$ for all $r \in \mathbb{N}$. If (2.2) holds, then

(i) $WN_{\theta'}^{\alpha}(\Delta) \subseteq WS_{\theta}^{\alpha}(\Delta)$,

(ii) $WN_{\theta'}(\Delta) \subseteq WS_{\theta}^{\alpha}(\Delta)$,

(iii) $WN_{\theta'}(\Delta) \subseteq WS_{\theta}(\Delta)$.

If (2.3) holds, then

(i) $WS_{\theta}^{\alpha}(\Delta) \subseteq WN_{\theta'}^{\alpha}(\Delta)$,

(ii) $WS_{\theta}^{\alpha}(\Delta) \subseteq WN_{\theta'}(\Delta)$,

(iii) $WS_{\theta}(\Delta) \subseteq WN_{\theta'}(\Delta)$.

3. Conclusions

In this paper, we have given the natural density, statistical convergence, weak convergence, weak statistical convergence, some concepts of statistical convergence of order α for$0 < \alpha \leq 1$, and strong p-Cesàro summability of order α for $0 < \alpha \leq 1$.and we established the Δ-lacunary weak statistical convergence of order α and weak $N_{\theta}^{\alpha}(\Delta)$-summability for $0 < \alpha \leq 1$. We have also given some inclusion relations between these concepts.

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