Bayesian and Non-Bayesian Estimators of the Parameters of Weibull Distribution

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Abstract
In this paper, some estimators for the shape and scale parameters of Weibull distribution have been obtained using Maximum likelihood as non-Bayesian estimators, as well as Bayes estimators. Bayesian estimations have been obtained under Scale invariant and Entropy loss functions based on exponential priors. Lindley's approximation has been used effectively in Bayesian estimation. Based on the Monte Carlo simulation method, those estimators are compared depending on the mean squared errors (MSE's).
Keywords: Weibull Distribution; Maximum likelihood estimator; Scale invariant loss function; Entropy loss function; Lindley's approximation.

1.Introduction

The Weibull distribution is one of the most important common continuous probability distributions, it is widely used in reliability and lifetime, Quality Control, weather forecasting, life sciences, and engineering [1]. Although it was first identified and used by Frenchman Maurice Fréchet in 1927 and applied by R. Rosin and E. Rammler in1933, it is named after Waal Obi Weibull in 1939[2].

There are some recent works and literature of Weibull distribution, Al Omari and Ibrahim (2011) conducted a study on Bayesian estimator for Weibull distribution with censored data [3], G. B. Chris and A. I. Noor (2012) indicated the estimated parameters of the Weibull distribution form Bayesian estimator under LINEX loss function is the best comparatively with respect to other methods [4], P. Ivana and S. Zuzana (2014) recommend the Maximum likelihood method to estimate the parameters of Weibull distribution [1].

There are many shapes of Weibull distribution attains for various values of the shape parameter, as special cases, if the shape parameter is equal to one it becomes the Exponential distribution if the shape parameter is equal to two it becomes the Rayleigh distribution, and if x denotes the Weibull variable than (-x) has a type three extreme value distribution [5].

The probability density function (pdf) of two parameters Weibull distribution is defined as:

$$f(x \mid \lambda, \vartheta) = \frac{\lambda}{\vartheta}\left(\frac{x}{\vartheta}\right)^{\lambda-1} e^{-(x/\vartheta)^{\lambda}} \qquad x > 0 \quad ; \quad \lambda, \vartheta > 0 \qquad (1)$$

The corresponding cumulative distribution function (CDF) of the Weibull distribution is given by:

$$F(x \mid \lambda, \vartheta) = 1 - e^{-\left(\frac{x}{\vartheta}\right)^{\lambda}}$$

The Reliability function defined as:

$$R(X ; \lambda, \vartheta) = 1 - F(X ; \lambda, \vartheta) = e^{-\left(\frac{x}{\vartheta}\right)^{\lambda}}$$

2. Estimation Methods

2.1 Non-Bayesian Estimators

Maximum Likelihood Estimation

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n followed Weibull distribution defined by equation (1). The likelihood function of the pdf is

$$L(x; \lambda, \vartheta) = \prod_{i=1}^{n} \left(\frac{\lambda}{\vartheta}\right)\left(\frac{x}{\vartheta}\right)^{\lambda-1} e^{-\left(\frac{x}{\vartheta}\right)^{\lambda}}$$

(2)

Taking the natural logarithm for the likelihood function, gives:

$$\ln(L) = n\ln(\lambda) - n\,\lambda\ln(\vartheta) + (\lambda - 1)\sum_{i=1}^{n} \ln(x_i) - \sum_{i=1}^{n} \left(\frac{x_i}{\vartheta}\right)^{\lambda}$$

Differentiating ln (L) with respect to ϑ and λ and equating to zero, yields:

$$\frac{\partial \ln L(\lambda, \vartheta)}{\partial \vartheta} = -n\left(\frac{\lambda}{\vartheta}\right) + \left(\frac{\lambda}{\vartheta}\right)\sum_{i=1}^{n} \left(\frac{x_i}{\vartheta}\right)^{\lambda} = 0 \qquad (3)$$

$$\frac{\partial \ln L(\lambda, \vartheta)}{\partial \lambda} = \left(\frac{n}{\lambda}\right) - n\ln\vartheta + \sum_{i=1}^{n} \ln x_i - \sum_{i=1}^{n} \left(\frac{x_i}{\vartheta}\right)^{\lambda} \ln\left(\frac{x_i}{\vartheta}\right) = 0 \qquad (4)$$

Notice that, (3) and (4) are difficult to solve analytically. Therefore, ML estimators for λ and ϑ can be derived numerically by using Newton-Raphson method depending on Hessian matrix, which is thesecond partial derivative of the log-likelihood function. Hessian matrix can be constructed as follows:

Assume that,

$$h_1(\vartheta) = \frac{\partial \ln L(\lambda,\vartheta)}{\partial \vartheta}, \qquad h_2(\lambda) = \frac{\partial \ln L(\lambda,\vartheta)}{\partial \lambda}$$

The partial derivatives of $h_1(\vartheta)$ with respect to unknown parameters ϑ and λ are:

$$\frac{\partial h_1(\vartheta)}{\partial \vartheta} = \frac{\partial^2}{\partial \vartheta^2} L(\lambda, \vartheta) = \frac{n\lambda}{\vartheta^2} - \frac{\lambda(\lambda + 1)}{\vartheta^2}\sum_{i=1}^{n} \left(\frac{x_i}{\vartheta}\right)^{\lambda}$$

$$\frac{\partial h_1(\lambda)}{\partial \lambda} = \frac{\partial^2}{\partial \vartheta\, \partial \lambda} L(\lambda, \vartheta) = -\frac{n}{\vartheta} + \frac{1}{\vartheta}\sum_{i=1}^{n} \left(\frac{x_i}{\vartheta}\right)^{\lambda} + \left(\frac{\lambda}{\vartheta}\right)\sum_{i=1}^{n} \left(\frac{x_i}{\vartheta}\right)^{\lambda} \ln\left(\frac{x_i}{\vartheta}\right)$$

The partial derivatives of $h_2(\lambda)$with respect to unknown parameters ϑ and λ are given by:

$$\frac{\partial h_2(\lambda)}{\partial \lambda} = \frac{\partial^2}{\partial \lambda^2} L(\lambda, \vartheta) = -\frac{n}{\lambda^2} - \sum_{i=1}^{n} \left(\frac{x_i}{\vartheta}\right)^{\lambda} [\ln\left(\frac{x_i}{\vartheta}\right)]^2$$

$$\frac{\partial h_2(\lambda)}{\partial \vartheta} = \frac{\partial^2}{\partial \lambda\, \partial \vartheta} L(\lambda, \vartheta) = -\frac{n}{\vartheta} + \frac{1}{\vartheta}\sum_{i=1}^{n} \left(\frac{x_i}{\vartheta}\right)^{\lambda} + \left(\frac{\lambda}{\vartheta}\right)\sum_{i=1}^{n} \left(\frac{x_i}{\vartheta}\right)^{\lambda} \ln\left(\frac{x_i}{\vartheta}\right)$$

Thus, the Jacobian matrix $J_k(\lambda, \vartheta)$which is a non-singular symmetric matrix defined as follows:

$$J_k(\lambda, \vartheta) = \begin{bmatrix} \frac{\partial h_1(\lambda,\vartheta)}{\partial \vartheta} & \frac{\partial h_1(\lambda,\vartheta)}{\partial \lambda} \\ \frac{\partial h_2(\lambda,\vartheta)}{\partial \vartheta} & \frac{\partial h_2(\lambda,\vartheta)}{\partial \lambda} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

so, its inverse can be written as:

$$J_K^{-1} = \frac{1}{|J|}\begin{bmatrix} b_{22} & -b_{12} \\ -b_{21} & b_{11} \end{bmatrix}$$

Hence, according to the Newton-Raphson method, the maximum likelihood estimators can be obtained as follows

$$\begin{bmatrix} \vartheta_{k+1} \\ \lambda_{k+1} \end{bmatrix} = \begin{bmatrix} \vartheta_k \\ \lambda_k \end{bmatrix} - J_k^{-1}(\lambda_k, \vartheta_k)\begin{bmatrix} h_1(\lambda_k, \vartheta_k) \\ h_2(\lambda_k, \vartheta_k) \end{bmatrix}$$

When $k = 0$, $\vartheta_0$and λ_0 are represented the initial values for ϑ and λ respectively which should be chosen carefully.

The absolute value for the difference between the new value for ϑ and λ in new iterative value with previous value for ϑ and λ in last iterative represent the error term denoted by ε, which is assumed a very small value. The error term is formulated as:

$$\begin{bmatrix} \varepsilon_{k+1}(\vartheta) \\ \varepsilon_{k+1}(\lambda) \end{bmatrix} = \begin{bmatrix} \vartheta_{k+1} \\ \lambda_{k+1} \end{bmatrix} = \begin{bmatrix} \vartheta_k \\ \lambda_k \end{bmatrix}$$

2.2. Bayesian Estimators

In this suction, Bayesian estimators are obtained based on two different loss functions which are Scale invariant and Entropy loss functions. The Bayesian estimators are derived with assuming the exponential prior for each of λ and ϑ i.e.,

$\lambda \sim Exponential\ (1/\delta)$ with the following pdf

$$g_1(\lambda) = \delta e^{-\delta\lambda} \qquad \delta > 0 \tag{5}$$

$\vartheta \sim Exponential\ (1/\gamma)$ with the pdf defined as

$$g_2(\vartheta) = \gamma e^{-\gamma\vartheta} \qquad \gamma > 0 \tag{6}$$

λ and γ are assumed independent of each other. Therefore, the joint exponential prior is:

$g(\lambda, \vartheta) = g_1(\lambda)\ g_2(\vartheta) = \delta\, e^{-\delta\lambda}\gamma e^{-\gamma\vartheta}$

The joint posterior distribution of ϑ and λ is given by

$$\pi(\lambda,\vartheta;\underline{X})=\frac{e^{-(\gamma\vartheta+\delta\lambda)}\left(\frac{\lambda}{\vartheta^{\lambda}}\right)^{n}\prod_{i=1}^{n}x_i^{\lambda-1}\exp\left(-\frac{\sum_{i=1}^{n}x_i^{\lambda}}{\vartheta^{\lambda}}\right)}{\int_0^{\infty}\int_0^{\infty}e^{-(\gamma\vartheta+\delta\lambda)}\left(\frac{\lambda}{\vartheta^{\lambda}}\right)^{n}\prod_{i=1}^{n}x_i^{\lambda-1}\exp\left(-\frac{\sum_{i=1}^{n}x_i^{\lambda}}{\vartheta^{\lambda}}\right)d\lambda d\vartheta}$$

2.2.1. The Entropy loss function (ELF)

The Entropy loss function is Asymmetric Loss Function, it was first introduced by James and Stein [6] for the estimation of the Dispersion (i.e., Variance-Covariance) matrix of the Multivariate normal distribution. Dey et al. [7] considered this loss function for simultaneous estimation of scale parameters and their reciprocals, for p independent gamma distributions. Rukhin and Ananda [8] considered the estimation problem of the variance of a Multivariate Normal vector under the Entropy loss and Quadratic loss, it is defined as:

$$L(\hat{\theta},\theta)=\left(\frac{\hat{\theta}}{\theta}\right)-\ln\left(\frac{\hat{\theta}}{\theta}\right)-1$$

The risk function $R_E(\hat{\theta},\theta)$ can be derived as

$$R_E(\hat{\theta},\theta)=\int_0^{\infty}\left[\left(\frac{\hat{\theta}}{\theta}\right)-\ln\left(\frac{\hat{\theta}}{\theta}\right)-1\right]\pi(\theta,\underline{X})\,d\theta$$

Let

$\frac{\partial R_S(\hat{\theta},\theta)}{\partial\hat{\theta}}=0$, yields:

$$\frac{\partial R_S(\hat{\theta},\theta)}{\partial\hat{\theta}}=\int_0^{\infty}\frac{1}{\theta}\pi(\theta,\underline{X})\,d\theta-\frac{1}{\hat{\theta}}\int_0^{\infty}\pi(\theta,\underline{X})\,d\theta=0$$

Hence,

$$\hat{\theta}_E=\left[E\left(\frac{1}{\theta}|X\right)\right]^{-1} \tag{7}$$

a-Bayesian estimation for ϑunderELF

Bayes estimator for ϑ, under ELF can be obtained as follows:

Assumed that w (λ,ϑ) be any function for λ, ϑ. Therefore:

$$E[w(\lambda,\vartheta)]=\int_0^{\infty}\int_0^{\infty}w(\lambda,\vartheta)\,\pi(\lambda,\vartheta)\,d\lambda\,d\vartheta$$

$$=\int_0^{\infty}\int_0^{\infty}w(\lambda,\vartheta)\frac{L(x_1,x_2,\dots,x_n;\lambda,\vartheta)\pi(\lambda,\vartheta)\,d\lambda\,d\vartheta}{\int_0^{\infty}\int_0^{\infty}L(x_1,x_2,\dots,x_n;\lambda,\vartheta)\pi(\lambda,\vartheta)\,d\lambda\,d\vartheta}$$

$$=\frac{\int_0^{\infty}\int_0^{\infty}w(\lambda,\vartheta)L(x_1,x_2,\dots,x_n;\lambda,\vartheta)\pi(\lambda,\vartheta)\,d\lambda\,d\vartheta}{\int_0^{\infty}\int_0^{\infty}L(x_1,x_2,\dots,x_n;\lambda,\vartheta)\pi(\lambda,\vartheta)\,d\lambda\,d\vartheta}$$

Assuming that

$W(\lambda,\vartheta)=\frac{1}{\vartheta}$

Thus,

$$E\left[\frac{1}{\vartheta}\ \middle|\ \underline{X}\right]=\frac{\int_0^\infty\int_0^\infty \frac{1}{\vartheta}e^{-(\gamma\vartheta+\delta\lambda)}\left(\frac{\lambda}{\vartheta^\lambda}\right)^n\prod_{i=1}^n x_i^{\lambda-1}\exp\left(-\frac{\sum_{i=1}^n x_i^\lambda}{\vartheta^\lambda}\right)d\lambda\, d\vartheta}{\int_0^\infty\int_0^\infty e^{-(\gamma\vartheta+\delta\eta)}\left(\frac{\lambda}{\vartheta^\lambda}\right)^n\prod_{i=1}^n x_i^{\lambda-1}\exp\left(-\frac{\sum_{i=1}^n x_i^\lambda}{\vartheta^\lambda}\right)d\lambda\, d\vartheta}$$

Observed that, it is difficult to obtain the solution of the ratio of two integrals. Hence, the solution will be approximately by using the Lindley's approximation [9], as follows:

$$E\left[\frac{1}{\vartheta}\ \middle|\ \underline{X}\right]\approx\frac{1}{\hat{\vartheta}}+\frac{1}{2}(w_{11}\sigma_{11})+p_1w_1\sigma_{11}+\frac{1}{2}(L_{30}w_1\sigma_{11}^2)+\frac{1}{2}(L_{12}w_1\sigma_{11}\sigma_{22}) \qquad (8)$$

Where,

$$w_1=\frac{\partial w(\lambda,\vartheta)}{\partial\vartheta}=\frac{\partial}{\partial\vartheta}\left(\frac{1}{\vartheta}\right)=\frac{-1}{\vartheta^2}$$

(9)

$$w_{11}=\frac{\partial^2 w(\lambda,\vartheta)}{\partial\vartheta^2}=\frac{2}{\vartheta^3}$$

(10)

$$L_{ij}=\frac{\partial^{i+j}}{\partial\vartheta^i\,\partial\lambda^j}\ln L(\lambda,\vartheta) \qquad i,j=0,1,2,3$$

$$=\frac{\partial^{i+j}}{\partial\vartheta^i\,\partial\lambda^j}\left[n\ln(\lambda)-n\lambda\ln(\vartheta)+(\lambda-1)\sum_{i=1}^n\ln(x_i)-\sum_{i=1}^n\left(\frac{x_i}{\vartheta}\right)^\lambda\right]$$

$$L_{12}=\frac{\partial^3\ln L(\lambda,\vartheta)}{\partial\vartheta\,\partial\lambda^2}=\frac{\lambda}{\vartheta}\sum_{i=1}^n\left(\frac{x}{\vartheta}\right)^\lambda\left(\ln\left(\frac{x_i}{\vartheta}\right)\right)^2+\frac{2}{\vartheta}\sum_{i=1}^n\left(\frac{x_i}{\vartheta}\right)^\lambda\ln\left(\frac{x_i}{\vartheta}\right) \qquad (11)$$

$$L_{20}=\frac{\partial^2\ln L(\lambda,\vartheta)}{\partial\vartheta^2}=\frac{n\lambda}{\vartheta^2}-\frac{\lambda(\lambda+1)}{\vartheta^2}\sum_{i=1}^n\left(\frac{x_i}{\vartheta}\right)^\lambda \quad , \quad L_{02}=\frac{\partial^2\ln L(\lambda,\vartheta)}{\partial\lambda^2}$$

$$=-\frac{n}{\lambda^2}-\sum_{i=1}^n\left(\frac{x_i}{\vartheta}\right)^\lambda\left(\ln\left(\frac{x_i}{\vartheta}\right)\right)^2$$

$$L_{30}=\frac{\partial^3 L(\lambda,\vartheta)}{\partial\vartheta^3}=-\frac{2n\lambda}{\vartheta^3}+\frac{\lambda(\lambda+1)(\lambda+2)}{\vartheta^3}\sum_{i=1}^n\left(\frac{x_i}{\vartheta}\right)^\lambda$$

(12)

$$\sigma_{11}=-\frac{1}{L_{20}}=\frac{-\vartheta^2}{n\lambda-\lambda(\lambda+1)\sum_{i=1}^n\left(\frac{x_i}{\vartheta}\right)^\lambda}$$

(13)

$$\sigma_{22}=-\frac{1}{L_{02}}=\frac{\lambda^2}{n+\lambda^2\sum_{i=1}^n\left(\frac{x_i}{\vartheta}\right)^\lambda\left(\ln\left(\frac{x_i}{\lambda}\right)\right)^2}$$

(14)

We have,

$g(\lambda,\vartheta)=\gamma\,\delta\ e^{-(\gamma\vartheta+\delta\lambda)}$

$P=\ln g(\lambda,\vartheta)=\ln(\gamma)+\ln(\delta)-\gamma\,\vartheta-\delta\,\lambda$

$p_1 = \frac{\partial p}{\partial \vartheta} =$
$-\gamma$
(15)
Substituting (9), (10), (11), (12), (13), (14) and (15) into (8), yields the approximated solution of $E\left[\frac{1}{\vartheta} \mid \underline{X}\right]$.
After Substitute (8) into (7) gives the Bayesian estimation for ϑ under ELF denoted by $\hat{\vartheta}_E$
b- Bayesian estimation for λ under ELF
Similarly, Bayes estimator for λ, under ELF can be derived as the following:
Assume that, $W(\lambda, \vartheta) = \frac{1}{\lambda}$
The Lindley's approximation for $E\left[\frac{1}{\lambda} \mid \underline{X}\right]$ is given by:
$$E\left[\frac{1}{\lambda} \mid \underline{X}\right] \approx \frac{1}{\hat{\lambda}} + \frac{1}{2}(w_{22}\sigma_{22}) + p_2 w_2 \sigma_{22} + \frac{1}{2}(L_{03} w_2 \sigma_{22}^2) + \frac{1}{2}(L_{21} w_2 \sigma_{11} \sigma_{22})$$
(16)
Where,
$$w_1 = \frac{\partial w(\lambda, \vartheta)}{\partial \vartheta} = \frac{\partial}{\partial \vartheta}\left(\frac{1}{\lambda}\right)$$
$$= 0 = w_{11}$$
$$w_2 = \frac{\partial w(\lambda, \vartheta)}{\partial \lambda} = \frac{-1}{\lambda^2}$$
$$w_{22} = \frac{\partial^2 w(\lambda, \vartheta)}{\partial \lambda^2} = \frac{2}{\lambda^3}$$
$$L_{03} = \frac{\partial^3 L(\lambda, \vartheta)}{\partial \lambda^3} = \frac{2n}{\lambda^3} - \sum_{i=1}^{n} \left(\frac{x_i}{\vartheta}\right)^{\lambda} \left(\ln\left(\frac{x_i}{\vartheta}\right)\right)^3$$
$$L_{21} = \frac{\partial^3 L(\lambda, \vartheta)}{\partial \vartheta^2 \partial \lambda} = \frac{n}{\vartheta^2} - \frac{2\lambda + 1}{\vartheta^2} \sum_{i=1}^{n} \left(\frac{x_i}{\vartheta}\right)^{\lambda} - \frac{\lambda(\lambda + 1)}{\vartheta^2} \sum_{i=1}^{n} \left(\frac{x_i}{\vartheta}\right)^{\lambda} \ln\left(\frac{x_i}{\vartheta}\right)$$
$P = \ln g(\lambda, \vartheta) = \ln(\gamma) + \ln(\delta) - \gamma\vartheta - \delta\lambda$
$$p_2 = \frac{\partial p}{\partial \lambda} = \frac{\partial}{\partial \lambda}(\ln(\gamma) + \ln(\delta) - \gamma\vartheta - \delta\lambda) = -\delta$$

2.2.2. The Scale Invariant Squared Error Loss Function (SISELF)

The Scale invariant squared error loss function is symmetric and continuous and loss function [10], it was suggested by De Groot (1970). It is defined as:
$$L(\hat{\theta}, \theta) = \left(1 - \frac{\hat{\theta}}{\theta}\right)^2$$
The risk function $R_S(\hat{\theta}, \theta)$ can be derived as follows
$$R_S(\hat{\theta}, \theta) = E[L(\hat{\theta}, \theta)] = \int_0^{\infty} L(\hat{\theta}, \theta)\, \pi(\theta, \underline{X}) d\theta = \int_0^{\infty} \left(1 - \frac{\hat{\theta}}{\theta}\right)^2 \pi(\theta, \underline{X})\, d\theta$$
To obtain the value of $\hat{\theta}$ that minimize the risk function under SISELF, assume that:

$\frac{\partial R_S(\hat{\theta},\theta)}{\partial\hat{\theta}} = 0$, yields:

$$2\int_0^\infty \left(1-\frac{\hat{\theta}}{\theta}\right)\left(-\frac{1}{\theta}\right)\pi(\theta,\underline{X})d\theta = 0$$

$$\hat{\theta}\int_0^\infty \frac{1}{\theta^2}\pi(\theta,\underline{X})d\theta - \int_0^\infty \frac{1}{\theta}\pi(\theta,\underline{X}) = 0$$

$$\hat{\theta}_S = \frac{E[\frac{1}{\theta}|\underline{X}]}{E[\frac{1}{\theta^2}|\underline{X}]} \quad (17)$$

a-Bayesian estimation for ϑunderSISELF

To obtain Bayesian estimation for ϑ, under SISELF, assume that, W (λ, ϑ) = $\frac{1}{\vartheta^2}$

According to Lindley's approximation:

$$E\left[\frac{1}{\vartheta^2}|\underline{X}\right] \approx \frac{1}{\hat{\vartheta}^2} + \frac{1}{2}(w_{11}\sigma_{11}) + p_1 w_1\sigma_{11} + \frac{1}{2}(L_{30}w_1\sigma_{11}^2) + \frac{1}{2}(L_{12}w_1\sigma_{11}\sigma_{22}) \quad (18)$$

Where,

$$w_1 = \frac{\partial w(\lambda,\ \vartheta)}{\partial\vartheta} = \frac{\partial w}{\partial\vartheta}\left(\frac{1}{\vartheta^2}\right) = \frac{-2}{\vartheta^3}$$

$$w_{11} = \frac{\partial^2 w(\lambda,\vartheta)}{\partial\vartheta^2} = \frac{6}{\vartheta^4}$$

Bayesian estimation ofϑ, under SISELF which is denoted by $\hat{\vartheta}_S$ can be getting after substituting (8) and (18) into (17).

b- Bayesian estimation for λ underELF

To obtain Bayesian estimation for λ, under SISELF, assume that, W (λ, ϑ) = $\frac{1}{\lambda^2}$

According to Lindley's approximation:

$$E\left[\frac{1}{\lambda^2}|\underline{X}\right] \approx \frac{1}{\hat{\lambda}^2} + \frac{1}{2}(w_{11}\sigma_{11}) + p_1 w_1\sigma_{11} + \frac{1}{2}(L_{30}w_1\sigma_{11}^2) + \frac{1}{2}(L_{12}w_1\sigma_{11}\sigma_{22}) \quad (19)$$

Where,

$$w_1 = \frac{\partial w(\lambda,\ \vartheta)}{\partial\vartheta} = \frac{\partial w}{\partial\vartheta}\left(\frac{1}{\lambda^2}\right) = \frac{-2}{\lambda^3}$$

$$w_{11} = \frac{\partial^2 w(\lambda,\vartheta)}{\partial\vartheta^2} = \frac{6}{\lambda^4}$$

Bayesian estimation ofϑ, under SISELF which is denoted by $\hat{\lambda}_S$ can be getting after substituting (16) and (19) into (17).

4. Simulation Study

In this section, we employed the Monte–Carlo simulation to compare the performance of different estimates (Maximum likelihood and Bayes Estimators under Scale invariant squared error loss function and Entropy loss function) for unknown shape and scale parameters of Weibull distribution based on the mean squared errors (MSE's) which can be written as:

$$\text{MSE}(\hat{\theta}) = \frac{\sum_{i=1}^{I} \left(\hat{\theta}_i - \theta\right)^2}{I}$$

Where, I is the number of replications.

We generated I = 5000 samples from two parameters Weibull distribution with different sizes (n = 20, 50, and 100). With assuming $\vartheta = 0.7, 1.4$ and $\lambda = 0.5, 1.5$.

The values of the prior's parameter of ϑ have been chosen as $\delta = 0.5, 1.5$, and for λ ' s prior parameter $\gamma = 0.5, 1.5$.

5. Discussion and Conclusion

The expected values and (MSE's) for estimating ϑ and λ are tabulated in Tables (1-8).

The results of the Tables can be summarized by the following points

1. The performance of Bayes estimates under Scale invariant squared error loss function for the parameter ϑ are the best, since they give smallest mean square error, as indicated for all combinations of initial values of parameters. While the performance of Bayes estimates under Entropy loss function for the parameter λ are the best, since they give smallest mean square error.
2. It is clear that, the results for ϑ (expected values and MSE's) at $\lambda, \vartheta = 1.5$ are the best as the corresponding result when $\lambda, \vartheta = 0.5$.
3. It is observed that, MSE's of all estimators of shape parameter is increasing with the increase of the value of the shape parameter. Also, MSE values for all estimates of scale parameter are increasing with the increase of the scale parameter value in all cases.
4. In general, the preference of Bayesian methods is the best compared to Maximum likelihood estimators.
5. Finally, all estimates of the two parameters of Weibull distribution shows that the averages are close to the default values for all sample sizes and that the average estimation of parameters are always approaching the true values with the increase of the sample size.

Table 1: Expected values and MSE's for $\hat{\lambda}$ when $\lambda = 0.5$ and $\vartheta = 0.7$

n	Criterion	$\hat{\lambda}_{ML}$	$\hat{\lambda}_E$		$\hat{\lambda}_S$	
			γ =0.5, δ=0.5	**γ =1.5, δ =1.5**	**γ =0.5, δ=0.5**	**γ =1.5, δ =1.5**
20	Mean	0.536731	0.478716	0.472610	0.453667	0.459517
	MSE	0.011540	0.000469	0.000776	0.002215	0.001690
50	Mean	0.514674	0.496608	0.493936	0.484495	0.487114
	MSE	0.003707	0.000013	0.000040	0.000249	0.000172
100	Mean	0.507513	0.503162	0.501784	0.496685	0.498050
	MSE	0.001679	0.000010	0.000004	0.000012	0.000005

Table 2: Expected values and MSE's for $\hat{\lambda}$ when $\lambda = 1.5$ and $\vartheta = 0.7$

n	Criterion	$\hat{\lambda}_{ML}$	$\hat{\lambda}_E$		$\hat{\lambda}_S$	
			γ =0.5, δ=0.5	**γ =1.5, δ =1.5**	**γ =0.5, □=0.5**	**□ =1.5, □ =1.5**
20	Mean	1.605435	1.407234	1.352672	1.363062	1.418063
	MSE	0.103945	0.008687	0.022077	0.019052	0.006763
50	Mean	1.541936	1.466723	1.442692	1.444852	1.468918
	MSE	0.033172	0.001114	0.003319	0.003073	0.000971
100	Mean	1.522649	1.487981	1.475620	1.476272	1.488638
	MSE	0.015092	0.000145	0.000600	0.000568	0.000130

Table 3: Expected values and MSE's for $\hat{\lambda}$ when $\lambda = 0.5$ and $\vartheta = 1.4$

n	Criterion	$\hat{□}_{□□}$	$\hat{□}_{□}$		$\hat{□}_{□}$	
			□ =0.5, □=0.5	**□ =1.5, □ =1.5**	**□ =0.5, □=0.5**	**□ =1.5, □ =1.5**
20	Mean	0.536587	0.478772	0.472683	0.453781	0.459618
	MSE	0.011534	0.000467	0.000772	0.002204	0.001682
50	Mean	0.514739	0.496636	0.493972	0.484550	0.487162
	MSE	0.003711	0.000013	0.000039	0.000247	0.000171
100	Mean	0.507493	0.503180	0.501808	0.496722	0.498081
	MSE	0.001680	0.000010	0.000004	0.000012	0.000005

Table 4: Expected values and MSE's for $\hat{\lambda}$ when $\lambda = 1.5$ and $\vartheta = 1.4$

n	Criterion	$\hat{□}_{□□}$	$\hat{□}_{□}$		$\hat{□}_{□}$	
			□ =0.5, □=0.5	**□ =1.5, □ =1.5**	**□ =0.5, □=0.5**	**□ =1.5, □ =1.5**
20	Mean	1.604768	1.407571	1.353449	1.363767	1.418318
	MSE	0.103543	0.008629	0.021860	0.018870	0.006724
50	Mean	1.542262	1.466873	1.443038	1.445181	1.469048
	MSE	0.033334	0.001104	0.003281	0.003038	0.000963
100	Mean	1.522649	1.488060	1.475810	1.476455	1.488709
	MSE	0.015092	0.000144	0.000591	0.000559	0.000128

Table 5: Expected values and MSE's for $\hat{\vartheta}$ when $\vartheta = 0.7$ and $\lambda = 0.5$

n	Criterion	$\hat{□}_{□□}$	$\hat{□}_{□}$		$\hat{□}_{□}$	
			□ =0.5, □=0.5	**□ =1.5, □ =1.5**	**□ =0.5, □=0.5**	**□ =1.5, □ =1.5**
20	Mean	0.84895	0.84748	0.71541	0.67457	0.59707
	MSE	0.13325	0.04659	0.00218	0.00243	0.01149
50	Mean	0.74866	0.75742	0.71179	0.69439	0.65718
	MSE	0.04358	0.00467	0.00037	0.00013	0.00194

100	Mean	0.71694	0.73177	0.71015	0.70153	0.68196
	MSE	0.02174	0.00118	0.00014	0.00001	0.00035

Table 6: Expected values and MSE's for $\hat{\vartheta}$ when $\vartheta = 0.7$ and $\lambda = 1.5$

n	Criterion	$\hat{□}_{□□}$	$\hat{□}_{□}$		$\hat{□}_{□}$	
			□ =0.5, □=0.5	**□ =1.5, □ =1.5**	**□ =0.5, □=0.5**	**□ =1.5, □ =1.5**
20	Mean	0.71229	0.73882	0.72565	0.72095	0.70795
	MSE	0.01059	0.00165	0.00071	0.00048	0.00007
50	Mean	0.70111	0.72094	0.71600	0.71407	0.70916
	MSE	0.00474	0.00045	0.00026	0.00020	0.00008
100	Mean	0.70024	0.71528	0.71290	0.71195	0.70958
	MSE	0.00245	0.00023	0.00017	0.00014	0.00009

Table 7: Expected values and MSE's for $\hat{\vartheta}$ when $\vartheta = 1.4$ and $\lambda = 0.5$

n	Criterion	$\hat{□}_{□□}$	$\hat{□}_{□}$		$\hat{□}_{□}$	
			□ =0.5, □=0.5	**□ =1.5, □ =1.5**	**□ =0.5, □=0.5**	**□ =1.5, □ =1.5**
20	Mean	1.69505	1.53565	1.17961	1.24897	1.07400
	MSE	0.53235	0.04746	0.05171	0.02431	0.11321
50	Mean	1.49522	1.45703	1.30108	1.34020	1.22530
	MSE	0.17433	0.00573	0.01024	0.00372	0.03245
100	Mean	1.43336	1.43115	1.35202	1.37349	1.30571
	MSE	0.08715	0.00130	0.00239	0.00073	0.00938

Table 8: Expected values and MSE's for $\hat{\vartheta}$ when $\vartheta = 1.4$ and $\lambda = 1.5$

n	Criterion	$\hat{□}_{□□}$	$\hat{□}_{□}$		$\hat{□}_{□}$	
			□ =0.5, □=0.5	**□ =1.5, □ =1.5**	**□ =0.5, □=0.5**	**□ =1.5, □ =1.5**
20	Mean	1.42478	1.45324	1.40438	1.41853	1.37209
	MSE	0.04233	0.00316	0.00005	0.00040	0.00096
50	Mean	1.40206	1.42659	1.40774	1.41321	1.39471
	MSE	0.01900	0.00073	0.00006	0.00018	0.00005
100	Mean	1.40048	1.41803	1.40888	1.41154	1.40247
	MSE	0.00981	0.00033	0.00008	0.00013	0.00001

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