Optimal Harvesting of Three Species Dynamic Model with Bionomic Equilibrium

Dr.M.Gunasekaran[a], A.M.Sarravanaprabhu [b]

[a] Assistant Professor, PG Department of Mathematics, Sri Subramaniya Swami Government Arts College, Tiruttani – 631 209, India.

[b] Assistant Professor, Department of Mathematics, St. ThomasCollege of Arts and Science, Koyambedu, Chennai – 600 044, India and

[b] Research Scholar, PG Department of Mathematics, Sri Subramaniya Swami Government Arts College,Tiruttani – 631 209, India. saravanaprabum@gmail.com

Abstract: This paper presents three species, ecological models with optimal harvesting effect. The proposed model of species, termed as Prey (P), Middle-level Predator (Q) and Top-level Predator (R) is formed by a set of first order non-linear differential equation. This system initially examines all the possible equilibrium points for this planned model, subsequently investigates the stability analysis, optimal harvesting and quantitative bionomic equilibrium aspect of the prey-predator species and their continued existence in the ecological system. Finally, local and global stability were studied by using Routh Hurwitz criteria and Lyapunov function respectively under positive equilibrium state.

Keywords: Prey,Middle-level Predator, Top-level Predator, Equilibrium point, Stability Analysis, Optimal Harvesting, Quantitative Bionomic Equilibrium, Routh Hurwitz Criterion, Lyapunov Function,

1. Introduction

Mathematical modelling is a dominant and resourceful tool in a biological system to describe the experimental population dynamics and predict the future circumstance of the ecosystem. Earlier ecological models have commonly assumed that the densities of species are spatially homogeneous or the environment is well mixed. Any species live in a community and interacts with some other species along with the surroundings. Its relationship consists of the interactions among species and their subsequent effects on one another are focused. The system deals with population interaction with the environment to develop their growth exponentially in the dynamic space.

The prey-predator contacts were well studied by Lotka [5] and Volterra [19]. The extensive studies are carried out in various ecological models by quite a lot of ecologists and mathematicians, namely Freedman. H. I [1], Kapur J.N [4], and Murray J.D. [7]. Prasad B.S.R.V., *et.al*[9] focused on the coexistence of the predator with mutual interference whereas very little prey exists in the dynamic environment. Saifuddina Md., *et.al*[11] has explained weak Allee effect and various competition coefficients in the eco-epidemiological model. Various parameters to examine the prey-predator model such as Modified Transmission function, hydra effect, mutually help, and Stochastic Model. Satyathi GAL, *et.al*[12] mentioned the harvesting effort of prey in a dynamic model. Suresh kumar Y., *et.al* [15] express the stochastic process among the three species. Gupta R.P., *et.al* [2] discussed the Leslie-Gower model under the bifurcation analysis and prey harvesting by michaelis-menten method. Kalyan Das, *et.al*[3] has considered a fishery resource system consisting of two predators competing for the same prey to survive and also investigated the bionomical and biological equilibrium of the model. Motuma S.T. [6] described the interaction of the population species with harvesting function and several types of functional responses. Pal D., *et.al*[8] taken study about the prey-predator harvesting model underneath impreciseness and introduce the biological parametric form with interval. Sachin Kumar, *et.al*[10] focused on the stability and bifurcation analysis of the prey-predator system under the occurrence of group defense and prey harvesting. Srinivas M.N., *et.al*[13] has things to see how the commensal and host species are harvested optimally and analyzed the stability by stochastically to find the variance of the species. Sujatha.K.,*et.al*[14] projected an optimal control problem on SIR epidemic model with disease population specified in prey. Tiwari V., *et.al* [16] has analyzed the predator harvesting with Crowley martin functional response and discussed a diffusive predator-prey system. Vidyananth T., *et.al*[17,18] has explained the detailed study about the predator and two preying species which mutually help each other in the ecological model and the analytical results of the bionomic equilibrium of threes species model with optimal harvesting for prey species.

2. Mathematical Model

The proposed ecological model describes the three species are specifically as P, Q and R are the population density of the Prey, Middle-level Predator and Top-level Predator respectively. The Middle-level predator is like omnivores that compete with prey and mutualise with the Top-level predators. The species R communicates natural interaction with prey. These species are live in the same environment and utilize the resources among them by interacting and mutualising with each other. In addition to that death rate and optimal harvesting were calculated for each species. The eight equilibrium points were acknowledged and derived. The system discusses optimal harvesting and the bio-economic equilibrium of the three species and also their coexistence in a bio-network. Finally, the local and global stability of all the three species in existence state are also depicted.

The first order non-linear differential equation for the given prescribed model is as follows,

$$\frac{dP}{dt} = (a_1 - d_1 - q_1E_1)P - \alpha_1P^2 - \beta_1PQ - \gamma_1PR$$
$$\frac{dQ}{dt} = (a_2 - d_2 - q_2E_2)Q - \alpha_2Q^2 - \beta_2PQ + \gamma_2QR$$
$$\frac{dR}{dt} = (a_3 - d_3 - q_3E_3)R - \alpha_3R^2 + \beta_3PR + \gamma_3QR$$

2.1. Parameters

P, Q, R – Population density for Prey (P), Middle-level Predator(Q), Top-Level Predator(R) respectively.

$a_i's$ – Growth rate of the species P, Q, R respectively (where i=1,2,3).

$\alpha_i's$ – Rate of intra specific competition of the species P, Q, R respectively (where i=1,2,3).

β_1, β_2 – Rate of inter specific competition between the species P and Q.

γ_1, β_3 – Rate of the predation between the species P and R.

γ_2, γ_3 – Increase rate of the species Q and R under mutualism.

$d_i's$ – Death rate of species P, Q, R respectively (where i=1, 2, 3).

$q_iE_i's$ – Optimal harvesting rate for catch ability coefficient and effort applied to the species P, Qand R respectively (where i=1,2,3).

Further the Population density variable P, Q, R are non-negative and the parameters of the system $a_i's$,$\alpha_i's$, β_1, β_2,β_3, γ_1, γ_2, γ_3, $d_i's, q_iE_i's$are supposed to be non-negative constants.

3. Equilibrium states

In this part, we investigate the stability performance of the system at eight equilibrium states which have given below

ES_1: All the three species are washout state.

$$\overline{P} = 0, \quad \overline{Q} = 0, \quad \overline{R} = 0$$

ES_2: Prey exist equilibrium state

$$\overline{P} = \left(\frac{a_1 - d_1 - q_1E_1}{\alpha_1}\right), \quad \overline{Q} = 0, \quad \overline{R} = 0$$

ES_3: Middle-level predator exists equilibrium state

$$\overline{P} = 0, \quad \overline{Q} = \left(\frac{a_2 - d_2 - q_2E_2}{\alpha_2}\right), \quad \overline{R} = 0$$

ES_4: Top-level predator exist equilibrium state

$$\overline{P} = 0, \quad \overline{Q} = 0, \quad \overline{R} = \left(\frac{a_3 - d_3 - q_3E_3}{\alpha_3}\right)$$

ES_5: The existence of Prey and Middle- level predator equilibrium state.

$$\overline{P} = \frac{(a_1 - d_1 - q_1E_1)\alpha_2 - (a_2 - d_2 - q_2E_2)\beta_1}{\alpha_1\alpha_2 - \beta_1\beta_2},$$

$$\overline{Q} = \frac{(a_2 - d_2 - q_2E_2)\alpha_1 - (a_1 - d_1 - q_1E_1)\beta_2}{\alpha_1\alpha_2 - \beta_1\beta_2}, \quad \overline{R} = 0$$

ES_6: The existence of Prey and Top-level predator equilibrium state

$$\overline{P} = \frac{(a_1 - d_1 - q_1E_1)\alpha_3 - (a_3 - d_3 - q_3E_3)\gamma_1}{\alpha_1\alpha_3 + \gamma_1\beta_3},$$

$$\overline{Q} = 0, \quad \overline{R} = \frac{(a_3 - d_3 - q_3E_3)\alpha_1 + (a_1 - d_1 - q_1E_1)\beta_3}{\alpha_1\alpha_3 + \gamma_1\beta_3}$$

ES_7: The existence of Middle-level and Top-level predator equilibrium state

$$\overline{P} = 0, \quad \overline{Q} = \frac{(a_2 - d_2 - q_2E_2)\alpha_3 + (a_3 - d_3 - q_3E_3)\gamma_2}{\alpha_2\alpha_3 - \gamma_2\gamma_3}$$

$$\overline{R} = \frac{(a_3 - d_3 - q_3E_3)\alpha_2 + (a_2 - d_2 - q_2E_2)\gamma_3}{\alpha_2\alpha_3 - \gamma_2\gamma_3},$$

ES_8: All the three species are existing

$$\overline{P} = \frac{(a_1 - d_1 - q_1E_1)(\alpha_2\alpha_3 - \gamma_2\gamma_3) - (a_2 - d_2 - q_2E_2)(\beta_1\alpha_3 + \gamma_1\gamma_3) - (a_3 - d_3 - q_3E_3)(\beta_1\gamma_2 - \alpha_2\gamma_1)}{\alpha_1(\alpha_2\alpha_3 - \gamma_2\gamma_3) + \beta_1(\beta_3\gamma_2 - \alpha_3\beta_2) + \gamma_1(\alpha_2\beta_3 - \beta_2\gamma_3)},$$

$$\overline{Q} = \frac{(a_1 - d_1 - q_1E_1)(\beta_3\gamma_2 - \alpha_3\beta_2) + (a_2 - d_2 - q_2E_2)(\alpha_1\alpha_3 + \beta_3\gamma_1) + (a_3 - d_3 - q_3E_3)(\alpha_1\gamma_2 - \beta_2\gamma_1)}{\alpha_1(\alpha_2\alpha_3 - \gamma_2\gamma_3) + \beta_1(\beta_3\gamma_2 - \beta_2\alpha_3) + \gamma_1(\alpha_2\beta_3 - \beta_2\gamma_3)},$$

$$\overline{R} = \frac{(a_1 - d_1 - q_1E_1)(\alpha_2\beta_3 - \beta_2\gamma_3) + (a_2 - d_2 - q_2E_2)(\alpha_1\gamma_3 - \beta_1\beta_3) + (a_3 - d_3 - q_3E_3)(\alpha_1\alpha_2 - \beta_1\beta_2)}{\alpha_1(\alpha_2\alpha_3 - \gamma_2\gamma_3) + \beta_1(\beta_3\gamma_2 - \beta_2\alpha_3) + \gamma_1(\alpha_2\beta_3 - \beta_2\gamma_3)}$$

4. Analysis of Steady state

In this section, we analyse the dynamic behaviour of the three species for all possible equilibrium states. The jacobian matrix (A) of the system under study at any arbitrary point is given by

$$A = \begin{bmatrix} (a_1 - d_1 - q_1E_1) - 2\alpha_1P - \beta_1Q - \gamma_1R & -\beta_1P & -\gamma_1P \\ -\beta_2Q & (a_2 - d_2 - q_2E_2) - 2\alpha_2Q - \beta_2P + \gamma_2R & \gamma_2Q \\ \beta_3R & \gamma_3R & (a_3 - d_3 - q_3E_3) - \alpha_3R + \beta_3P + \gamma_3Q \end{bmatrix}$$

The result of computing the variation matrixes corresponding to each equilibrium state is as follows.

4.1 Dynamic behaviour of Equilibrium state ES_1 at (0,0,0)

$$\text{Let } A(ES_1) = \begin{bmatrix} (a_1 - d_1 - q_1E_1) & 0 & 0 \\ 0 & (a_2 - d_2 - q_2E_2) & 0 \\ 0 & 0 & (a_3 - d_3 - q_3E_3) \end{bmatrix}$$

The Eigen values of the above matrix is

$\lambda_1 = (a_1 - d_1 - q_1E_1),\quad \lambda_2 = (a_2 - d_2 - q_2E_2)$ and $\lambda_3 = (a_3 - d_3 - q_3E_3)$

Here birth rate of $a_i's$ is always greater than death and harvested rate.

So, the Eigen values of $\lambda_1, \lambda_2, \lambda_3$are positive.

Hencethe system of equilibrium pointsA(ES_1) is unstable.

4.2 Dynamic behaviour of Equilibrium state ES_2 at $\left(\frac{(a_1-d_1-q_1E_1)}{\alpha_1},\ 0,\ 0\right)$

Let $A(ES_2)$

$$= \begin{bmatrix} -(a_1 - d_1 - q_1E_1) & \frac{-\beta_1(a_1 - d_1 - q_1E_1)}{\alpha_1} & \frac{-\gamma_1(a_1 - d_1 - q_1E_1)}{\alpha_1} \\ 0 & (a_2 - d_2 - q_2E_2) - \frac{\beta_2(a_1 - d_1 - q_1E_1)}{\alpha_1} & 0 \\ 0 & 0 & (a_3 - d_3 - q_3E_3) + \frac{\beta_3(a_1 - d_1 - q_1E_1)}{\alpha_1} \end{bmatrix}$$

The Eigen values of $A(ES_2)$ are

$$\lambda_1 = -(a_1 - d_1 - q_1E_1),$$

$$\lambda_2 = (a_2 - d_2 - q_2E_2) - \frac{\beta_2(a_1 - d_1 - q_1E_1)}{\alpha_1} \text{ and}$$

$$\lambda_3 = (a_3 - d_3 - q_3E_3) + \frac{\beta_3(a_1 - d_1 - q_1E_1)}{\alpha_1}$$

Where based on the three eigen values, the stability of equilibrium point exists for following two cases

Case (i): since $\lambda_1 < 0$ (always) $\lambda_2 < 0,\ \lambda_3 > 0$ the system of equilibrium point is unstable.

Case (ii): since $\lambda_1 < 0$ (always) $\lambda_2 > 0,\ \lambda_3 > 0$ then the first eigen value are negative and the remaining two eigen values are positive. The system of equilibrium points A(ES_2) is unstable and saddle points exist.

4.3 Dynamic behaviour of equilibrium state ES_3at $\left(0,\ \frac{(a_2-d_2-q_2E_2)}{\alpha_2},\ 0\right)$

Let $A(ES_3)$

$$= \begin{bmatrix} (a_1 - d_1 - q_1E_1) - \frac{\beta_1(a_2 - d_2 - q_2E_2)}{\alpha_2} & 0 & 0 \\ -\frac{\beta_2(a_2 - d_2 - q_2E_2)}{\alpha_2} & -(a_2 - d_2 - q_2E_2) & \frac{\gamma_2(a_2 - d_2 - q_2E_2)}{\alpha_2} \\ 0 & 0 & (a_3 - d_3 - q_3E_3) + \frac{\gamma_3(a_2 - d_2 - q_2E_2)}{\alpha_2} \end{bmatrix}$$

The eigen values of $A(ES_3)$ are

$$\lambda_1 = (a_1 - d_1 - q_1E_1) - \frac{\beta_1(a_2 - d_2 - q_2E_2)}{\alpha_2},$$

$$\lambda_2 = -(a_2 - d_2 - q_2E_2)\text{and}$$

$$\lambda_3 = (a_3 - d_3 - q_3E_3) + \frac{\gamma_3(a_2 - d_2 - q_2E_2)}{\alpha_2}$$

Where based on the three Eigen values the stability of equilibrium point exist for following two cases

Case (i):Since $\lambda_2 < 0$ (always)$\lambda_1 < 0,\ \lambda_3 > 0$.The system of equilibrium point $A(ES_3)$ is unstable.

Case (ii): Since $\lambda_2 < 0$ (always) $\lambda_1 > 0,\ \lambda_3 > 0$. The system of equilibrium point $A(ES_3)$ is unstable and saddle points exist when second eigen value is negative and the remaining two eigen values are positive.

4.4 Dynamic behaviour of equilibrium state ES_4 at $\left(0,\ 0,\ \frac{(a_3-d_3-q_3E_3)}{\alpha_3}\right)$

Let $A(ES_4)$

$$=\begin{bmatrix} (a_1-d_1-q_1E_1)-\dfrac{\gamma_1(a_3-d_3-q_3E_3)}{\alpha_3} & 0 & 0 \\ 0 & (a_2-d_2-q_2E_2)+\dfrac{\gamma_2(a_3-d_3-q_3E_3)}{\alpha_3} & 0 \\ \dfrac{\beta_3(a_3-d_3-q_3E_3)}{\alpha_3} & \dfrac{\gamma_3(a_3-d_3-q_3E_3)}{\alpha_3} & -(a_3-d_3-q_3E_3) \end{bmatrix}$$

The eigen values of $A(ES_4)$ are

$$\lambda_1 = (a_1-d_1-q_1E_1)-\frac{\gamma_1(a_3-d_3-q_3E_3)}{\alpha_3},$$

$$\lambda_2 = (a_2-d_2-q_2E_2)+\frac{\gamma_2(a_3-d_3-q_3E_3)}{\alpha_3} \quad and$$

$$\lambda_3 = -(a_3-d_3-q_3E_3)$$

The stability of equilibrium points exist for following two cases based on the obtained eigen values.

Case(i) : since $\lambda_3 < 0$ (always) $\lambda_1 < 0,\ \lambda_2 > 0$ the system of equilibrium point $A(ES_4)$ is unstable.

Case(ii): since $\lambda_3 < 0$ (always) $\lambda_1 > 0,\ \lambda_2 > 0.$ The system of equilibrium point $A(ES_4)$ is unstable and saddle points exist whereasthe third eigen values is negative and the remaining two eigen values are positive.

4.5 Dynamic behaviour of equilibrium state ES_5 at W_1

$$W_1 = \left(\frac{(a_1-d_1-q_1E_1)\alpha_2-\beta_1(a_2-d_2-q_2E_2)}{\alpha_1\alpha_2-\beta_1\beta_2}, \frac{(a_2-d_2-q_2E_2)\alpha_1-\beta_2(a_1-d_1-q_1E_1)}{\alpha_1\alpha_2-\beta_1\beta_2}, 0\right)$$

$$\text{Let } A(ES_5) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$a_{11} = \frac{-(a_1-d_1-q_1E_1)\alpha_1\alpha_2+\alpha_1\beta_1(a_2-d_2-q_2E_2)}{\alpha_1\alpha_2-\beta_1\beta_2}$$

$$a_{12} = \frac{-(a_1-d_1-q_1E_1)\alpha_2\beta_1+\beta_1^2(a_2-d_2-q_2E_2)}{\alpha_1\alpha_2-\beta_1\beta_2}$$

$$a_{13} = \frac{-(a_1-d_1-q_1E_1)\alpha_2\gamma_1+\beta_1\gamma_1(a_2-d_2-q_2E_2)}{\alpha_1\alpha_2-\beta_1\beta_2}$$

$$a_{21} = \frac{-(a_2-d_2-q_2E_2)\alpha_1\beta_2+\beta_2^2(a_1-d_1-q_1E_1)}{\alpha_1\alpha_2-\beta_1\beta_2}$$

$$a_{22} = \frac{-(a_2-d_2-q_2E_2)\alpha_1\alpha_2+\alpha_2\beta_2(a_1-d_1-q_1E_1)}{\alpha_1\alpha_2-\beta_1\beta_2}$$

$$a_{23} = \frac{(a_2-d_2-q_2E_2)\alpha_1\gamma_2-\beta_2\gamma_2(a_1-d_1-q_1E_1)}{\alpha_1\alpha_2-\beta_1\beta_2}$$

$$a_{31} = 0,\ a_{32} = 0,$$

$$a_{33} = \frac{(a_3-d_3-q_3E_3)(\alpha_1\alpha_2-\beta_1\beta_2)+(a_1-d_1-q_1E_1)(\alpha_2\beta_3-\beta_2\gamma_3)+(a_2-d_2-q_2E_2)(\alpha_1\gamma_3-\beta_1\beta_3)}{\alpha_1\alpha_2-\beta_1\beta_2}$$

The characteristic equation of the above matrix $A(ES_5)$ is

$$\lambda^3 + X_1\lambda^2 + X_2\lambda + X_3 = 0$$

The eigen values of the above equation is

$$\lambda_1 = \frac{1}{2}\left[-(a_{11}+a_{22}) + \sqrt{(a_{11}+a_{22})^2 - 4(a_{11}a_{22}-a_{12}a_{21})}\right] < 0,$$

$$\lambda_2 = \frac{1}{2}\left[-(a_{11}+a_{22}) - \sqrt{(a_{11}+a_{22})^2 - 4(a_{11}a_{22}-a_{12}a_{21})}\right] < 0 \text{ and } \lambda_3 = a_{33}.$$

Where $[(a_{11}+a_{22})^2 - 4(a_{11}a_{22}-a_{12}a_{21})] > 0$, $-(a_{11}+a_{22}) < 0$ and

$$\sqrt{(a_{11}+a_{22})^2 - 4(a_{11}a_{22}-a_{12}a_{21})} < (a_{11}+a_{22}).$$

The steady state of the equilibrium points exist for subsequent cases support on the obtained eigen values.

Case(i) : Since $\lambda_1 < 0,\ \lambda_2 < 0$(always) and $\lambda_3 > 0$ the system of equilibrium point $A(ES_5)$ is unstable.

Case(ii): Since $\lambda_1 < 0,\ \lambda_2 < 0$(always) and $\lambda_3 < 0$are negative real number thenthe stability of equilibrium point $A(ES_5)$ is stable.

Hence the steady state of the system$A(ES_5)$ is locally asymptotically stable.

4.6 Dynamic behaviour of equilibrium state ES_6 at W_2

$$W_2 = \left(\frac{(a_1-d_1-q_1E_1)\alpha_3 - (a_3-d_3-q_3E_3)\gamma_1}{\alpha_1\alpha_3+\gamma_1\beta_3}, 0, \frac{(a_3-d_3-q_3E_3)\alpha_1 + (a_1-d_1-q_1E_1)\beta_3}{\alpha_1\alpha_3+\gamma_1\beta_3}\right)$$

$$\text{Let } A(ES_6) = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$b_{11} = \frac{-(a_1-d_1-q_1E_1)\alpha_1\alpha_3 + \alpha_1\gamma_1(a_3-d_3-q_3E_3)}{\alpha_1\alpha_3+\gamma_1\beta_3}$$

$$b_{12} = \frac{-(a_1-d_1-q_1E_1)\alpha_3\beta_1 + \beta_1\gamma_1(a_3-d_3-q_3E_3)}{\alpha_1\alpha_3+\gamma_1\beta_3}$$

$$b_{13} = \frac{-(a_1-d_1-q_1E_1)\alpha_3\gamma_1 + \gamma_1^2(a_3-d_3-q_3E_3)}{\alpha_1\alpha_3+\gamma_1\beta_3}$$

$$b_{21} = 0, \qquad b_{23} = 0$$

$$b_{22} = \frac{(a_2-d_2-q_2E_2)(\alpha_1\alpha_3+\gamma_1\beta_3) + (a_1-d_1-q_1E_1)(\beta_3\gamma_2-\alpha_3\beta_2) + (a_3-d_3-q_3E_3)(\alpha_1\gamma_2+\beta_2\gamma_1)}{\alpha_1\alpha_3+\gamma_1\beta_3}$$

$$b_{31} = \frac{(a_3-d_3-q_3E_3)\alpha_1\beta_3 + \beta_3^2(a_1-d_1-q_1E_1)}{\alpha_1\alpha_3+\gamma_1\beta_3}$$

$$b_{32} = \frac{(a_3-d_3-q_3E_3)\alpha_1\gamma_3 + \beta_3\gamma_3(a_1-d_1-q_1E_1)}{\alpha_1\alpha_3+\gamma_1\beta_3}$$

$$b_{33} = \frac{-(a_3-d_3-q_3E_3)\alpha_1\alpha_3 - \alpha_3\beta_3(a_1-d_1-q_1E_1)}{\alpha_1\alpha_3+\gamma_1\beta_3}$$

The characteristic equation of the above matrix $A(ES_6)$ is

$$\lambda^3 + Y_1\lambda^2 + Y_2\lambda + Y_3 = 0$$

The eigen values of the above characteristic equation is

$$\lambda_1 = \frac{1}{2}\left[-(b_{11}+b_{33}) + \sqrt{(b_{11}+b_{33})^2 - 4(b_{11}b_{33} - b_{13}b_{31})}\right] < 0,\ \lambda_2 = b_{22} \text{ and}$$

$$\lambda_3 = \frac{1}{2}\left[-(b_{11}+b_{33}) - \sqrt{(b_{11}+b_{33})^2 - 4(b_{11}b_{33} - b_{13}b_{31})}\right] < 0$$

Where $[(b_{11}+b_{33})^2 - 4(b_{11}b_{33} - b_{13}b_{31})] > 0$, $-(b_{11}+b_{33}) < 0$ and

$$\sqrt{(b_{11}+b_{33})^2 - 4(b_{11}b_{33} - b_{13}b_{31})} < (b_{11}+b_{33}).$$

The system of the equilibrium points exist for the successive cases stand on the eigen values.

Case(i) : Since $\lambda_1 < 0,\ \lambda_3 < 0$(always) and $\lambda_2 > 0$ then the system of equilibrium point $A(ES_6)$ is unstable.

Case(ii): Since $\lambda_1 < 0,\ \lambda_3 < 0$ (always) and $\lambda_2 < 0$are negative real number then the system of the equilibrium point $A(ES_6)$ is stable.

Hence the steady state of the system$A(ES_6)$ is locally asymptotically stable.

4.7 Dynamic behaviour of equilibrium state (ES_7) at W_3

$$W_3 = \left(0, \frac{(a_2 - d_2 - q_2E_2)\alpha_3 + (a_3 - d_3 - q_3E_3)\gamma_2}{\alpha_2\alpha_3 - \gamma_2\gamma_3}, \frac{(a_3 - d_3 - q_3E_3)\alpha_2 + (a_2 - d_2 - q_2E_2)\gamma_3}{\alpha_2\alpha_3 - \gamma_2\gamma_3}\right)$$

$$\text{Let } A(ES_7) = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

$$c_{11} = \frac{(a_1 - d_1 - q_1E_1)(\alpha_2\alpha_3 - \gamma_2\gamma_3) - (a_2 - d_2 - q_2E_2)(\alpha_3\beta_1 + \gamma_1\gamma_3) - (a_3 - d_3 - q_3E_3)(\beta_1\gamma_2 + \alpha_2\gamma_1)}{\alpha_2\alpha_3 - \gamma_2\gamma_3}$$

$$c_{12} = 0, \qquad c_{13} = 0$$

$$c_{21} = \frac{-(a_2 - d_2 - q_2E_2)\alpha_3\beta_2 - (a_3 - d_3 - q_3E_3)\beta_2\gamma_2}{\alpha_2\alpha_3 - \gamma_2\gamma_3}$$

$$c_{22} = \frac{-(a_2 - d_2 - q_2E_2)\alpha_2\alpha_3 - (a_3 - d_3 - q_3E_3)\alpha_2\gamma_2}{\alpha_2\alpha_3 - \gamma_2\gamma_3}$$

$$c_{23} = \frac{(a_2 - d_2 - q_2E_2)\alpha_3\gamma_3 + (a_3 - d_3 - q_3E_3)\gamma_2\gamma_3}{\alpha_2\alpha_3 - \gamma_2\gamma_3}$$

$$c_{31} = \frac{(a_3 - d_3 - q_3E_3)\alpha_2\beta_3 + (a_2 - d_2 - q_2E_2)\beta_3\gamma_3}{\alpha_2\alpha_3 - \gamma_2\gamma_3}$$

$$c_{32} = \frac{(a_3 - d_3 - q_3E_3)\alpha_2\gamma_3 + (a_2 - d_2 - q_2E_2)\gamma_3^2}{\alpha_2\alpha_3 - \gamma_2\gamma_3}$$

$$c_{33} = \frac{-(a_3 - d_3 - q_3E_3)\alpha_2\alpha_3 - (a_2 - d_2 - q_2E_2)\alpha_3\gamma_3}{\alpha_2\alpha_3 - \gamma_2\gamma_3}$$

The characteristic equation of the above matrix $A(ES_7)$ is

$$\lambda^3 + Z_1\lambda^2 + Z_2\lambda + Z_3 = 0$$

The eigen values of the above equation is

$$\lambda_1 = c_{11}, \qquad \lambda_2 = \frac{1}{2}\left[-(c_{22}+c_{33}) + \sqrt{(c_{22}+c_{33})^2 - 4(c_{22}c_{33} - c_{23}c_{32})}\right] < 0 \text{ and}$$

$$\lambda_3 = \frac{1}{2}\left[-(c_{22}+c_{33}) - \sqrt{(c_{22}+c_{33})^2 - 4(c_{22}c_{33} - c_{23}c_{32})}\right] < 0$$

Where $[(c_{22}+c_{33})^2 - 4(c_{22}c_{33} - c_{23}c_{32})] > 0$ and $-(c_{22}+c_{33}) < 0$

$\sqrt{(c_{22}+c_{33})^2 - 4(c_{22}c_{33} - c_{23}c_{32})} < (c_{22}+c_{33})$.

The system of the equilibrium points exist for the successive two cases based on the eigen values.

Case(i) : Since $\lambda_2 < 0,\ \lambda_3 < 0$(always) and $\lambda_1 > 0$ then the stability of the system $A(ES_7)$ is unstable.

Case(ii): Since $\lambda_2 < 0,\ \lambda_3 < 0$ (always) and $\lambda_1 < 0$ are negative real number then the stability of the system $A(ES_7)$ is stable.

Hence the steady state of the system$A(ES_7)$ is locally asymptotically stable.

4.8 Dynamic behaviour of equilibrium state (ES_8) at $(\overline{P}, \overline{Q}, \overline{R})$

Local Stability of the positive interior steady state

Herewe discuss the positive equilibrium state (ES_8) of the three species is locally asymptotically stable.

Let $P = \overline{P} + u_1,\ Q = \overline{Q} + u_2,\ R = \overline{R} + u_3$ where $U = (u_1, u_2, u_3)^T$the perturbation is over the equilibrium state.

The scaled equations are linearized to obtain the equation for the perturbed state.

Which gives $\frac{dU}{dt} = AU$

$$A(ES_8) = \begin{bmatrix} (a_1 - d_1 - q_1E_1) - 2\alpha_1P - \beta_1Q - \gamma_1R & -\beta_1P & -\gamma_1P \\ -\beta_2Q & (a_2 - d_2 - q_2E_2) - 2\alpha_2Q - \beta_2P + \gamma_2R & \gamma_2Q \\ \beta_3R & \gamma_3R & (a_3 - d_3 - q_3E_3) - \alpha_3R + \beta_3P + \gamma_3Q \end{bmatrix}$$

The characteristic equation of the matrix is $|A - \lambda I| = 0$

The steady state is present only if the solutions of the characteristic equation are negative parts of real or complex number.

The linearized equation as follows,

$$\frac{du_1}{dt} = -\alpha_1\overline{P}\,u_1 - \beta_1\overline{Q}\,u_1 - \gamma_1\overline{R}\,u_1$$

$$\frac{du_2}{dt} = -\beta_2\overline{P}u_2 - \alpha_2\overline{Q}u_2 + \gamma_2\overline{R}u_2$$

$$\frac{du_3}{dt} = \beta_3\overline{P}u_3 + \gamma_3\overline{Q}\,u_3 - \alpha_3\overline{R}u_3$$

Then we form the matrix of the above equation

$$\frac{dU}{dt} = \begin{bmatrix} -\alpha_1\overline{P} & -\beta_1\overline{Q} & -\gamma_1\overline{R} \\ -\beta_2\overline{P} & -\alpha_2\overline{Q} & \gamma_2\overline{R} \\ \beta_3\overline{P} & \gamma_3\overline{Q} & -\alpha_3\overline{R} \end{bmatrix}$$

The Characteristic polynomial of the above matrix is $\lambda^3 + h_1\lambda^2 + h_2\lambda + h_3 = 0$

$\lambda^3 + (\alpha_1\overline{P} + \alpha_2\overline{Q} + \alpha_3\overline{R})\lambda^2$

$+[(\alpha_2\,\alpha_3 - \gamma_2\gamma_3)\overline{QR} + (\alpha_1\alpha_3 + \beta_3\gamma_1)\overline{PR} + (\alpha_1\alpha_2 - \beta_1\beta_2)\overline{PQ}]\lambda$

$+[\alpha_1(\alpha_2\,\alpha_3 - \gamma_2\gamma_3) + \beta_1(\beta_3\gamma_2 - \alpha_3\beta_2) + \gamma_1(\alpha_2\,\beta_3 - \beta_2\gamma_3)]\overline{PQR} = 0$

Where $h_1 = \alpha_1\overline{P} + \alpha_2\overline{Q} + \alpha_3\overline{R}$

$h_2 = (\alpha_2\,\alpha_3 - \gamma_2\gamma_3)\overline{QR} + (\alpha_1\alpha_3 + \beta_3\gamma_1)\overline{PR} + (\alpha_1\alpha_2 - \beta_1\beta_2)\overline{PQ}$

$$h_3 = [\alpha_1(\alpha_2\,\alpha_3 - \gamma_2\gamma_3) + \beta_1(\beta_3\gamma_2 - \alpha_3\beta_2) + \gamma_1(\alpha_2\,\beta_3 - \beta_2\gamma_3)]\overline{\mathrm{PQR}}$$

Here $\alpha_2\,\alpha_3 - \gamma_2\gamma_3 > 0,\ \alpha_1\alpha_2 - \beta_1\beta_2 > 0,\ \beta_3\gamma_2 - \alpha_3\beta_2 > 0,\ \alpha_2\,\beta_3 - \beta_2\gamma_3 > 0.$

By using Routh-Hurwitz criterion, the system is stable

if $h_1 > 0,\ h_3 > 0, (h_1h_2 - h_3) > 0,\ h_3(h_1h_2 - h_3) > 0.$

Hence the coexistence of the steady state A(ES_8) is locally asymptotically stable.

5. Global stability analysis

In this phase, we examine the equilibrium state of ES_5 to ES_8 of the three species is globally asymptotically stable.

Let us take the system of the steady state by applying the Lyapunov function.

5.1 Global stability of the Equilibrium state (ES_5)

$$V(\overline{\mathrm{P}},\overline{\mathrm{Q}}) = \left\{P - \overline{\mathrm{P}} - \overline{\mathrm{P}}\ln\left(\frac{\mathrm{P}}{\overline{\mathrm{P}}}\right)\right\} + \left\{Q - \overline{\mathrm{Q}} - \overline{\mathrm{Q}}\ln\left(\frac{\mathrm{Q}}{\overline{\mathrm{Q}}}\right)\right\}$$

$$\frac{dV}{dt} = \frac{dP}{dt}\left[1 - \frac{\overline{\mathrm{P}}}{P}\right] + \frac{dQ}{dt}\left[1 - \frac{\overline{\mathrm{Q}}}{Q}\right]$$

$$= [P - \overline{\mathrm{P}}][(\mathrm{a}_1 - \mathrm{d}_1 - \mathrm{q}_1\mathrm{E}_1) - \alpha_1\mathrm{P} - \beta_1\mathrm{Q}] + [Q - \overline{\mathrm{Q}}][(\mathrm{a}_2 - \mathrm{d}_2 - \mathrm{q}_2\mathrm{E}_2) - \alpha_2\mathrm{Q} - \beta_2\mathrm{P}]$$

Substitute

$$(\mathrm{a}_1 - \mathrm{d}_1 - \mathrm{q}_1\mathrm{E}_1) = \alpha_1\overline{\mathrm{P}} + \beta_1\overline{\mathrm{Q}} \text{ and } (\mathrm{a}_2 - \mathrm{d}_2 - \mathrm{q}_2\mathrm{E}_2) = \alpha_2\overline{\mathrm{Q}} + \beta_2\overline{\mathrm{P}}$$

$$\frac{dV}{dt} = -\left(\alpha_1 + \frac{\beta_1 + \beta_2}{2}\right)[P - \overline{\mathrm{P}}]^2 - \left(\alpha_2 + \frac{\beta_1 + \beta_2}{2}\right)[Q - \overline{\mathrm{Q}}]^2$$

$$\frac{dV}{dt} < 0$$

Therefore, the equilibrium state ES_5 is globally asymptotically stable.

5.2 Global stability of the Equilibrium state (ES_6)

$$V(\overline{\mathrm{P}},\overline{\mathrm{R}}) = \left\{P - \overline{\mathrm{P}} - \overline{\mathrm{P}}\ln\left(\frac{\mathrm{P}}{\overline{\mathrm{P}}}\right)\right\} + \left\{R - \overline{\mathrm{R}} - \overline{\mathrm{R}}\ln\left(\frac{\mathrm{R}}{\overline{\mathrm{R}}}\right)\right\}$$

$$\frac{dV}{dt} = \frac{dP}{dt}\left[1 - \frac{\overline{\mathrm{P}}}{P}\right] + \frac{dR}{dt}\left[1 - \frac{\overline{\mathrm{R}}}{R}\right]$$

$$= [P - \overline{\mathrm{P}}][(\mathrm{a}_1 - \mathrm{d}_1 - \mathrm{q}_1\mathrm{E}_1) - \alpha_1\mathrm{P} - \gamma_1\mathrm{R}] + [R - \overline{\mathrm{R}}][(\mathrm{a}_3 - \mathrm{d}_3 - \mathrm{q}_3\mathrm{E}_3) - \alpha_3\mathrm{R} + \beta_3\mathrm{P}]$$

Substitute

$$(\mathrm{a}_1 - \mathrm{d}_1 - \mathrm{q}_1\mathrm{E}_1) = \alpha_1\overline{P} + \gamma_1\overline{\mathrm{R}} \text{ and } (\mathrm{a}_3 - \mathrm{d}_3 - \mathrm{q}_3\mathrm{E}_3) = \alpha_3\overline{\mathrm{R}} - \beta_3\overline{\mathrm{P}}$$

$$\frac{dV}{dt} = -\left(\alpha_1 + \frac{\gamma_1 - \beta_3}{2}\right)[P - \overline{\mathrm{P}}]^2 - \left(\alpha_3 + \frac{\gamma_1 - \beta_3}{2}\right)[R - \overline{\mathrm{R}}]^2$$

$$\frac{dV}{dt} < 0$$

Therefore, the equilibrium state ES_6 is globally asymptotically stable.

5.3 Global stability of the Equilibrium state (ES_7)

$$V(\overline{\mathrm{Q}},\overline{\mathrm{R}}) = \left\{Q - \overline{\mathrm{Q}} - \overline{\mathrm{Q}}\ln\left(\frac{\mathrm{Q}}{\overline{\mathrm{Q}}}\right)\right\} + \left\{R - \overline{\mathrm{R}} - \overline{\mathrm{R}}\ln\left(\frac{\mathrm{R}}{\overline{\mathrm{R}}}\right)\right\}$$

$$\frac{dV}{dt} = \frac{dQ}{dt}\left[1 - \frac{\overline{Q}}{Q}\right] + \frac{dR}{dt}\left[1 - \frac{\overline{R}}{R}\right]$$

$$= \left[Q - \overline{Q}\right]\left[(a_2 - d_2 - q_2E_2) - \alpha_2\overline{Q} + \gamma_2\overline{R}\right] + \left[R - \overline{R}\right]\left[(a_3 - d_3 - q_3E_3) - \alpha_3\overline{R} + \gamma_3\overline{Q}\right]$$

Substitute

$$(a_2 - d_2 - q_2E_2) = \alpha_2\overline{Q} - \gamma_2\overline{R} \text{ and } (a_3 - d_3 - q_3E_3) = \alpha_3\overline{R} - \gamma_3\overline{Q}$$

$$\frac{dV}{dt} = -\left(\alpha_2 - \left(\frac{\gamma_2 + \gamma_3}{2}\right)\right)\left[Q - \overline{Q}\right]^2 - \left(\alpha_3 - \left(\frac{\gamma_2 + \gamma_3}{2}\right)\right)\left[R - \overline{R}\right]^2$$

$$\frac{dV}{dt} < 0$$

Therefore, the equilibrium state ES_7 is globally asymptotically stable.

5.4 Global stability of the Equilibrium state (ES_8)

Here, we discuss the positive equilibrium state (ES_8) of the three species as globally asymptotically stable.

By applying the Lyapunov Function for the positive interior stability state

$$V(\overline{P}, \overline{Q}, \overline{R}) = \left\{P - \overline{P} - \overline{P}\ln\left(\frac{P}{\overline{P}}\right)\right\} + \left\{Q - \overline{Q} - \overline{Q}\ln\left(\frac{Q}{\overline{Q}}\right)\right\} + \left\{R - \overline{R} - \overline{R}\ln\left(\frac{R}{\overline{R}}\right)\right\}$$

Determine $\frac{dV}{dt}$

$$\frac{dV}{dt} = \frac{dP}{dt}\left[1 - \frac{\overline{P}}{P}\right] + \frac{dQ}{dt}\left[1 - \frac{\overline{Q}}{Q}\right] + \frac{dR}{dt}\left[1 - \frac{\overline{R}}{R}\right]$$

$$= \left[P - \overline{P}\right]\left[(a_1 - d_1 - q_1E_1) - \alpha_1P - \beta_1Q - \gamma_1R\right]$$

$$+\left[Q - \overline{Q}\right]\left[(a_2 - d_2 - q_2E_2) - \alpha_2Q - \beta_2P + \gamma_2R\right]$$

$$+\left[R - \overline{R}\right]\left[(a_3 - d_3 - q_3E_3) - \alpha_3R + \beta_3P + \gamma_3Q\right]$$

Substitute

$$(a_1 - d_1 - q_1E_1) = \alpha_1\overline{P} + \beta_1\overline{Q} + \gamma_1\overline{R},$$

$$(a_2 - d_2 - q_2E_2) = \beta_2\overline{P} + \alpha_2\overline{Q} - \gamma_2\overline{R},$$

$$(a_3 - d_3 - q_3E_3) = -\beta_3\overline{P} - \gamma_3\overline{Q} + \alpha_3\overline{R}$$

We get

$$\frac{dV}{dt} \le -\alpha_1\left[P - \overline{P}\right]^2 - \alpha_2\left[Q - \overline{Q}\right]^2 - \alpha_3\left[R - \overline{R}\right]^2$$

$$-\beta_1\left[P - \overline{P}\right]\left[Q - \overline{Q}\right] - \beta_2\left[P - \overline{P}\right]\left[Q - \overline{Q}\right] + \beta_3\left[P - \overline{P}\right]\left[R - \overline{R}\right]$$

$$-\gamma_1\left[P - \overline{P}\right]\left[R - \overline{R}\right] + \gamma_2\left[Q - \overline{Q}\right]\left[R - \overline{R}\right] + \gamma_3\left[Q - \overline{Q}\right]\left[R - \overline{R}\right]$$

$$\frac{dV}{dt} \le -\left(\alpha_1 + \frac{\beta_1 + \beta_2 - \beta_3 + \gamma_1}{2}\right)\left[P - \overline{P}\right]^2 - \left(\alpha_2 + \frac{\beta_1 + \beta_2 - \gamma_2 - \gamma_3}{2}\right)\left[Q - \overline{Q}\right]^2$$

$$-\left(\alpha_3 + \left(\frac{\gamma_1 - \gamma_2 - \gamma_3 - \beta_3}{2}\right)\right)\left[R - \overline{R}\right]^2$$

$$\frac{dV}{dt} < 0$$

Therefore, the equilibrium state ES_8 is globally asymptotically stable.

6. Quantitative Bionomic Equilibrium

It is generally linked to the study of the dynamic of living by economic equilibrium. The bionomic equilibrium is supposed to be achieved when the entire revenue acquired by selling the harvested biomass equals the entire cost consume in harvesting. The biological equilibrium is specified by

$$\frac{dP}{dt} = 0, \qquad \frac{dQ}{dt} = 0, \qquad \frac{dR}{dt} = 0,$$

Let us assume that,

C_1 be the harvesting cost for each unit effort of prey species (P), C_2 be the harvesting cost for each unit effort of Middle-Level Predator species (Q) and C_3 be the harvesting cost for each unit effort of Top-Level Predator species (R). M_1 be the price for each unit biomass of the prey, M_2 be the price for each unit biomass of the Middle-Level Predator and M_3 be the price for each unit biomass of the Top-Level Predator.

Thus economic rent or Net revenue at any time known by $B = B_1 + B_2 + B_3$.

Where $B_1 = (M_1 q_1 P - C_1)E_1$, $B_2 = (M_2 q_2 Q - C_2)E_2$, $B_3 = (M_3 q_3 R - C_3)E_3$

Here B_1, B_2, B_3 represent Net revenue for Prey, Middle-Level Predator and Top-Level Predator.

The bionomic equilibrium$((P)_\infty (Q)_\infty (R)_\infty (E_1)_\infty (E_2)_\infty (E_3)_\infty)$ is given by the subsequent equations.

$$(a_1 - d_1 - q_1 E_1)P - \alpha_1 P^2 - \beta_1 PQ - \gamma_1 PR = 0 \quad \text{---(1)}$$

$$(a_2 - d_2 - q_2 E_2)Q - \alpha_2 Q^2 - \beta_2 PQ + \gamma_2 QR = 0 \quad \text{---(2)}$$

$$(a_3 - d_3 - q_3 E_3)R - \alpha_3 R^2 + \beta_3 PR + \gamma_3 QR = 0 \quad \text{---(3)}$$

$$B = (M_1 q_1 P - C_1)E_1 + (M_2 q_2 Q - C_2)E_2 + (M_3 q_3 R - C_3)E_3 = 0 \quad \text{---(4)}$$

In order to determine the Bionomic equilibrium, we come across the following cases

Case (i) : if $C_1 > M_1 q_1 P$, $C_2 > M_2 q_2 Q$, $C_3 > M_3 q_3 R$ then the cost is bigger than revenue for three species and the whole system will be closed.

Case (ii): if $C_1 < M_1 q_1 P$, $C_2 < M_2 q_2 Q$, $C_3 < M_3 q_3 R$ then the cost is smaller than revenue for all the three species being positive, then the whole system will be in operation.

$$P = \frac{C_1}{M_1 q_1}, \qquad Q = \frac{C_2}{M_2 q_2}, \qquad R = \frac{C_3}{M_3 q_3}$$

Now substitute $(P)_\infty (Q)_\infty (R)_\infty$ in equation (1) – (4), then we get

$$(E_1)_\infty = \frac{1}{q_1}\left[a_1 - d_1 - \frac{\alpha_1 C_1}{M_1 q_1} - \frac{\beta_1 C_2}{M_2 q_2} - \frac{\gamma_1 C_3}{M_3 q_3}\right]$$

$$(E_2)_\infty = \frac{1}{q_2}\left[a_2 - d_2 - \frac{\alpha_2 C_2}{M_2 q_2} - \frac{\beta_2 C_1}{M_1 q_1} + \frac{\gamma_2 C_3}{M_3 q_3}\right]$$

$$(E_3)_\infty = \frac{1}{q_3}\left[a_3 - d_3 - \frac{\alpha_3 C_3}{M_3 q_3} + \frac{\beta_3 C_1}{M_1 q_1} + \frac{\gamma_3 C_2}{M_2 q_2}\right]$$

Now

$$(E_1)_\infty > 0 \; if \; a_1 > d_1 + \frac{\alpha_1 C_1}{M_1 q_1} + \frac{\beta_1 C_2}{M_2 q_2} + \frac{\gamma_1 C_3}{M_3 q_3} \quad \text{---(5)}$$

$$(E_2)_\infty > 0 \; \text{if} \; a_2 + \frac{\gamma_2 C_3}{M_3 q_3} > d_2 + \frac{\alpha_2 C_2}{M_2 q_2} + \frac{\beta_2 C_1}{M_1 q_1} \quad \text{---(6)}$$

$$(E_3)_\infty > 0 \; \text{if} \; a_3 + \frac{\beta_3 C_1}{M_1 q_1} + \frac{\gamma_3 C_2}{M_2 q_2} > d_3 + \frac{\alpha_3 C_3}{M_3 q_3} \quad \text{---(7)}$$

Hence the non-trivial bionomic equilibrium point $((P)_\infty (Q)_\infty (R)_\infty (E_1)_\infty (E_2)_\infty (E_3)_\infty)$ exist if conditions (5)-(7) hold.

7. Conclusion

This study considered two predators and one prey with continuous optimal harvest. The catch per unit criterion was measured for optimal harvesting. All three species are harvested continuously subject to minimum economic revenue. The analysis results exposed the fittest survey of all the species in the environment. The positive equilibrium point can play important role in locally and globally that provides the result of the proposed model is asymptotically stable and the system inspect the possibilities of the continued existence of the species in the state of bionomic equilibrium. The benefit of the study communicates to society and the species are preserved in the bio-network.

References

[1]. Freedman H.I., deterministic Mathematical models in population ecology, Marcel Decker, New York (1980).

[2]. Gupta R.P., Peeyush Chandra "Bifurcation Analysis Of Modified Leslie-Gowerpredator-Prey Model With Michaelis-Menten Type Prey Harvesting" Elsevier-Journal of Mathematical Analysis and Applications. 398 (2013), pp.278-295.

[3]. Kalyan Das, Srinivas M.N., Srinivas M.A.S., Gazi N.H. "Chaotic Dynamic Of A Three Species Prey Predator Competition Model With Bionomic Harvesting Due To Delayed Environmental Noise As External Driving Force" , Elsevier-Comptes Rendus Biologies 335(2012) pp.503-513.

[4]. Kapur J.N., Mathematical Model in Biology and Medicine, Affiliate East West (1985).

[5]. Lotka A.J., elements of Physical biology, Williams and Wilkins, Baltimore (1925).

[6]. Motuma S.T. "Mathematical Model of Population Interactions with Functional Responses and Harvesting Function", International Journal of Scientific Research in Mathematical and Statistcal Sciences, volume-7, Isssue-3, pp.33-38, June(2020).

[7]. Murray J. D., Mathematical biology. Biomathematics, vol. 19, Springer, Berlin, Germany, 1989.

[8]. Pal D., G.S.Mahaptra, Samanta G.P. " Optimal Harvesting Of prey- predator withinterval biological parameters: A Bioeconomic Model" Elesevier - Mathematical Biosciences 241(2013),181-187.

[9]. Prasad B.S.R.V., Malay Banerjee , Srinivasu P.D.N. "Dynamics of additional food provided predator–prey system with mutually interfering predators" Elsevier–Mathematical Biosciences 246 (2013) 176–190.

[10]. Sachin Kumar, Harsha Kharbanda " Chaotic Behavior of Predator-Prey Model with Group Defense and Non-Linear Harvesting in Prey", Elsevier-Chaos, Solitons and Fractals 119 (2019) pp.19-28.

[11]. Saifuddina Md., Santanu Biswas , Sudip Samanta , Susmita Sarkar, Joydev Chattopadhyay "Complex dynamics of an eco-epidemiological model with different competition coefficients and weak Allee in the predator" Elsevier – Chaos, Solitons and Fractals 91 (2016) 270–285.

[12]. Satyavathi G A L, Paparao, A.V, Sobhan Babu K., Dynamics of Strong Prey and weak Predator with Harvesting of Prey, International Journal of Recent echnology and Engineering, vol 7, Issue 6S2, April 2019.

[13]. Srinivas M.N., shiva Reddy K., Sabarmathi A. "Optimal Harvesting Strategy and Stochastic Analysis for a Two Species Commensaling System" Engineering Physics and Mathematics- Elsevier B.V on behalf of Ain Shams Engineering Journal, October 2013.

[14]. Sujatha.,K and Gunasekaran.M., 2019, "Optimal control of disease of an Eco- Epidemiological Prey-Predator System", International Journal of Research and Analytical Reviews, E- ISSN: 2348-1269, P- ISSN: 2349-5138, Volume-6, Issue 1 (2019).

[15]. Suresh kumar Y., Seshagiri Rao N., AppaRao B.V "A Stochastic Model for Three Species" International Journal of Engineering & Technology 7 (4.10), pp 497–503, 2018.

[16]. Tiwari V., Tripathi J.P., Abbas S., Wang J.S., Sun G.Q. "Qualitative Analysis Of Diffusive Crowley-Martin Predator Prey Mode; The Role Of Nonlinear Predator Harvesting" Springer Nature B.V. September 2019.

[17]. Vidyanath T., Lakshmi Narayan K., Shahnaz Bathul "A Three Species Ecological Model with a Predator and Two preying Species" SciPress, Switzerland, Vol.9, pp 26–32, 2016.

[18]. Vidyanath T., Lakshmi Narayan K., Shahnaz Bathul "Bionomic Equilibrium of a Three Species Model with Optimal Harvesting for the First Prey", International Journal of Pure and Applied Mathematics, Vol 113, No.9 pp.220 –233, 2017.

[19]. Volterra V., Leconnssen la theorice mathematique de la leitte pou lavie, gautheir villars, paris(1931).