RESEARCH PAPER

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TITLE : ***Projective Geometry—((ARC-RATIO)).—S.B's Theorems.***

KEYWORDS:

Projectivity, perspectivity, cross-ratio, complete quadrangle , SB's theorem, Division ring , Two extension theorems , Collineation , ARC-Ratios and related theorem, some relations, example .

ABSTRACT :

In Projective geometry we never measure anything as in Euclidean geometry ; instead , we relate one set of points to another by a projectivity. Generally , Projective geometry is a part of education in general mathematics like Differential equation , Operation research and Statistics , Dynamics , Statics etc .Furthermore , projective geometry is very much prerequisite for algebraic geometry , one of modern day's most vigorous , interesting and exciting branches of mathematics on which a lots of research works are due , to be completed .

In this paper I have tried to express some new ideas in lucid manner which are logically logical with diagrams and easy for the student audiences.

This paper has two sections. Section-I. contains mainly three theorems (SB's Theorem.1.and two extension theorems of two famous theorems) and some discussions .

In section.II. , I have introduced a totally new idea ARC – RATIO.

ARC – RATIO of four distinct concyclic points A, B ,C, D is denoted by ***((AB ,CD))*** *. All these ideas come to me in course of teaching .*

Hope , this enthusiastic research work will create some interests in the student's mind .

INTRODUCTION : ……………………………

Recall:

1) On exactly one line , any two distinct points are incident .
2) Any two distinct lines (whether parallel or not) are incident with at least one point.
3) Every line is incident with at least three distinct points .
4) There exist three non-coplanar points .
5) Three non-collinear points must lie on a unique plane .
6) ∃ a point and a plane that are not incident .
7) Any two distinct planes have at least a line in common or at least two common points .
8) There exist four non-coplanar points , no three of which are collinear.
9) If P, Q, R and S are four points such that PQ meets RS , then PR meets QS. (PQ $\neq$ RS)[1] .

10) The three diagonal points of a complete quadrangle are never collinear. (In fig. 1.) . Here PQRS be a complete quadrangle whose three diagonal points are A , B and V which are not collinear. Also C is the harmonic conjugate of D w.r.t. A and B. Here A , B , C , D form a harmonic set denoted by H(AB , CD) .
11) Four collinear points A , B , C , D are said to be a harmonic set if there exists a complete quadrangle PQRS (In fig. 1.) such that two of the points (here A and B) are diagonal points of the complete quadrangle and the other two points (here C and D) are on the opposite sides determined by the third diagonal point (here V).
12) A diagonal point of a complete quadrangle (here PQRS) is a point incident with opposite sides of the quadrangle. By fig.1.diagonal point A is incident on sides PQ and SR , similarly point B on QR and PS , point V on PR and QS .
13) If two complete quadrangles have the same set of diagonal points , then their eight vertices must lie on a conic .
14) A projectivity is a perspectivity iff their common element is self-corresponding .
15) The cross-ratio of four points remains invariant under projection .
16) The collinearity of points is an invariant under a collineation .
17) The inverse transformation of a collineation is a collineation .
18) The set of all collineations of a space to itself forms a group , called collineation group.
19) An involution is a projectivity of period two .
20) If a be a unit in R with unity , then a is not a divisor of zero and vice versa.
21) The set of all units in a ring R with unity forms a group w.r.t. multiplication.
22) Every non-zero element in a ring R having no divisor of zero is a unit .
23) A division ring contains no divisor of zero.
24) The set of all non-zero elements of a division ring forms a group under multiplication.
25) A division ring has all the properties of real numbers except for commutative of multiplication , ordering and completeness .

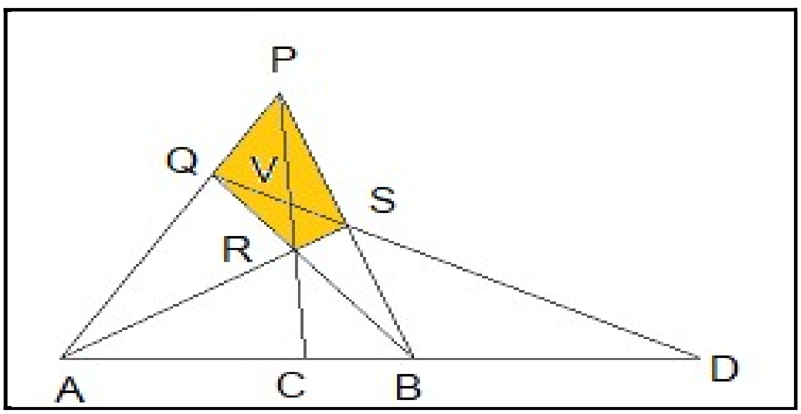


Fig.1.

HISTORICAL BASIS OF PROJECTIVE GEOMETRY :

Realising the facts from various sources and from the writings of eminent scholars , we have come to know that without having the profound knowledge of modern Projective Geometry , the artists of Renaissance periods did their works so intelligently and delicately that motivation for this kind of Geometry came from their fine arts.Thus the fundamental ideas that form the nucleus of Projective Geometry started from the works of artists and sculptures during the period of Renaissance , though , nowadays we know that some distinctive isolated works from ancient times are being recognized as a part of the same area.

The Italian School of Artists was the first to develop the idea of vertical perspectivity. Parallel lines were used in the scene to represent and create the sense of depth in the centre of the picture. The modern system of focused perspective was discovered by Brunelleschi , an Italian sculptor and architect and was consolidated into treatise , years later by Leone Battista Alverti , an eminent painter and architect.

The earliest truly , projection theorems were discovered by Pappus of Alexandria in the third century AD and Jean-Victor Ponchelet first proved such theorems by purely projective meaning. Italian artist and scientist Leonardo da Vinci and his counterpart Michelangelo and Albrecht Dürer perfected the method through their numerous paintings,sculptures and writings.

Examples of projective geometry as a Technique in real life (during 1558 – 1570).

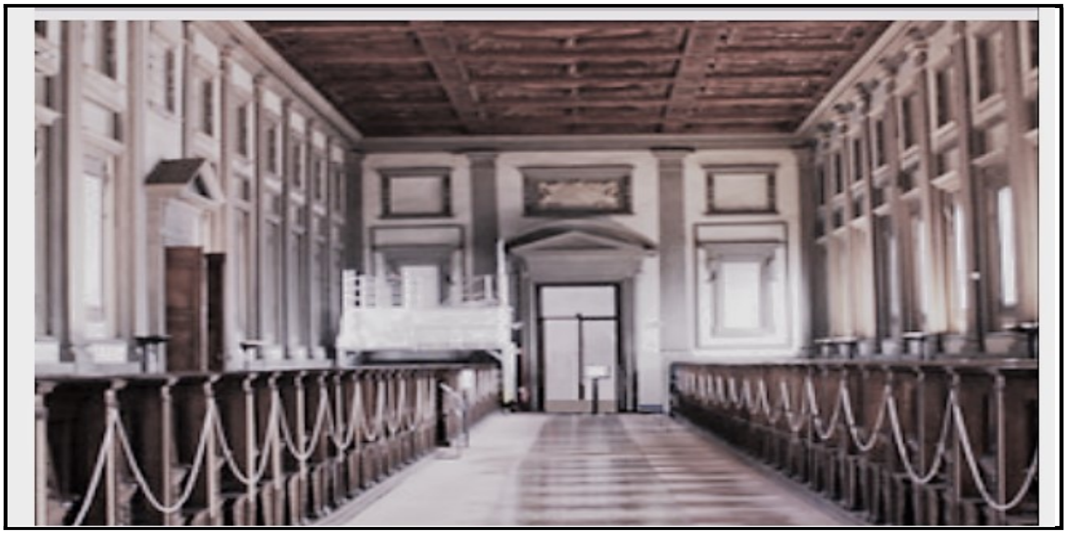


Fig.2. Laurention Library , Florence. ITALY

German astronomer Johann Kepler was first to declare ‘parabola’ as a geometrical figure which has two foci , one of which is at infinite distance in both of two opposite directions and that any point on the curve is joined to this focus by a line parallel to the axis. From that time we started to design Projective geometry so as to deal with **‘** Points at Infinity **’** and regular points (in n-dimensional space) in a uniform way, without making any proper difference among them .

The foundation of Projective Geometry , laid mainly by G.Desargues , B.Pascal and G.Monge , was of great values for the further development of Projective Geometry.

The French architect and army officer Girard Desargues declared that two or more parallel lines have a common point at an infinite distance (nowadays , considered to be in projective spaces) and if no point of a line is at a finite distance , the line itself is at an infinite distance.

The possibility of making a replacement of a statement about points and lines by a dual statement about lines and points(i.e. the pole and polar lines associated with conics) is known as “Principal of Duality” by Ponchelet , a pupil of Monge.

German Mathematician Karl George Christian von Staudt**’s** first work attracts special attention because it helped to determine the orbit of a comet. By his work **“** *Geometric der Lage* **”** ,
he tried to free Projective Geometry completely from any metrical basis.

Historically, we came to know about the axiomatic approach in the works of Hilbert ,Veblen and Young , Coxeter , Emil Artin , ……. so many . Technically, we can nowadays define Projective Geometry axiomatically or by depending upon linear algebra. One of the great virtues of Projective geometry is that it gives us a very clear presentation of curves and surfaces (rational).

In very recent years , new , modern and important practical applications have been discovered . The ideas and structures of classical projective geometry have been proved to be the ideal tools for modern error coding theory and cryptography.Also, Projective geometry is in the curriculam of engineering , architechture ,geography and so many streams .

Examples of projective geometry as a *technique in real life . [Figure. 3 – 5]*

Fig.3. Calcutta High Court . INDIA . (1862) .

Fig.4. The Lake Pontchartrain Causway , Louisians . USA . (1956) .

Fig.5. Sky – Walk , Kolkata . INDIA (2018)

SECTION - I

1.1. Projectivity between two pencils **:** A one-to-one correspondence between the lines of two pencils (O) and (O′) such that the cross-ratio of any four lines of one pencil is equal to the cross-ratio of the corresponding four lines of the other pencil is said to be a projectivity between the lines of the two pencils.In such a case the pencils are said to be projective.(Fig.6.).

1.2. Projectivity between two rows **:** A one-to-one correspondence between the points of two ranges or rows (*l*) and *(l ′)* such that the cross-ratio of any four points of one row is equal to that of the corresponding four points of the other row is said to be a projectivity between the points of the two rows. In such a case the two rows are said to be projective .(Fig.7.).

If a projectivity leaves invariant each of three points on a row **,** it leaves invariant every point on the row.

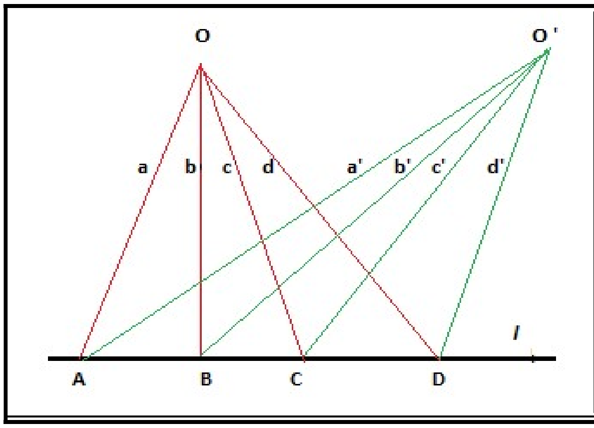


Fig.6 .

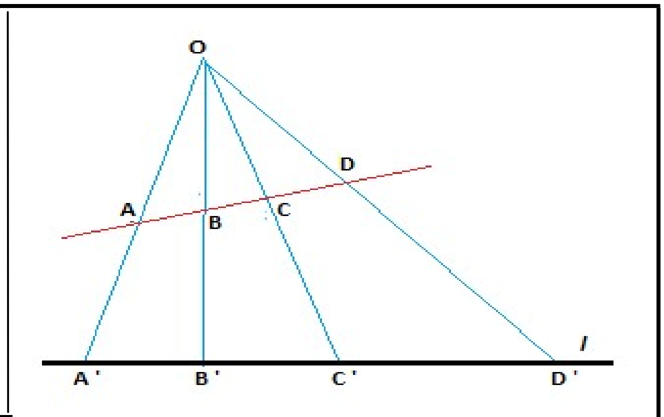


Fig.7 .

1.3. Perspectivity between two points **:** A projectivity between two points (O) and (O′) is said to be perspectivity if the points of intersection of the corresponding rays of (O) and (O′) are collinear . In such a case the pencils are said to be perspective . The line of collinearily is said to be the axis of perspectivity . (Fig.8.)

1.4. Perspectivity between two rows **:** A projectivity between two rows (*l*) and *(l′*) is said to be perspectivity if the lines joining the corresponding points of (*l*) and (*l′*) are concurrent. In such a case the rows are said to be perspective and the point of concurrence (S) is said to be the centre of perspectivity .(Fig.9.)

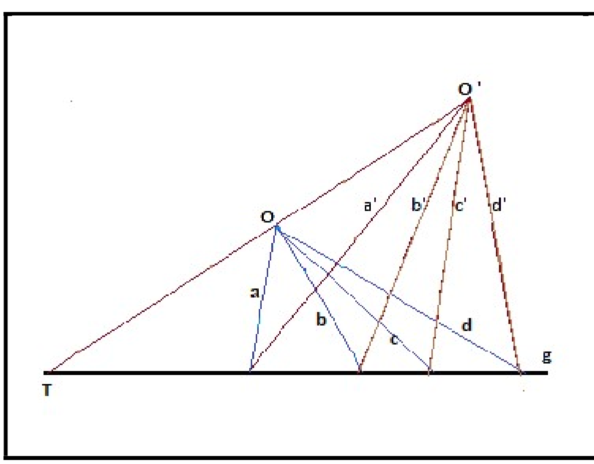


Fig .8 .

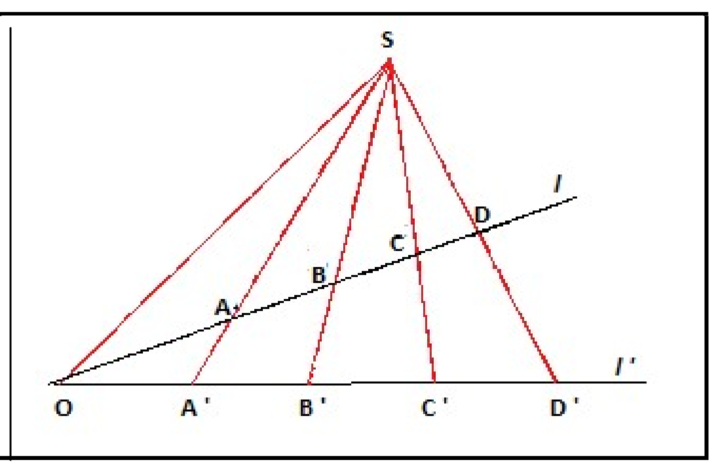


Fig . 9 .

A perspectivity between two pencils of points A , B , C , D and A′, B′,C ′, D′ with axis ***l*** and ***l ′*** is denoted by $l \overset{o}{\barwedge} l'$ or , ABC $\overset{o}{\barwedge}$ A′ B′ C ′ , where O is the centre of perspectivity. Similarly a perspectivity between two pencils of lines is denoted by O $\overset{g}{\barwedge}$ O′ , where g is the axis of perspectivity .

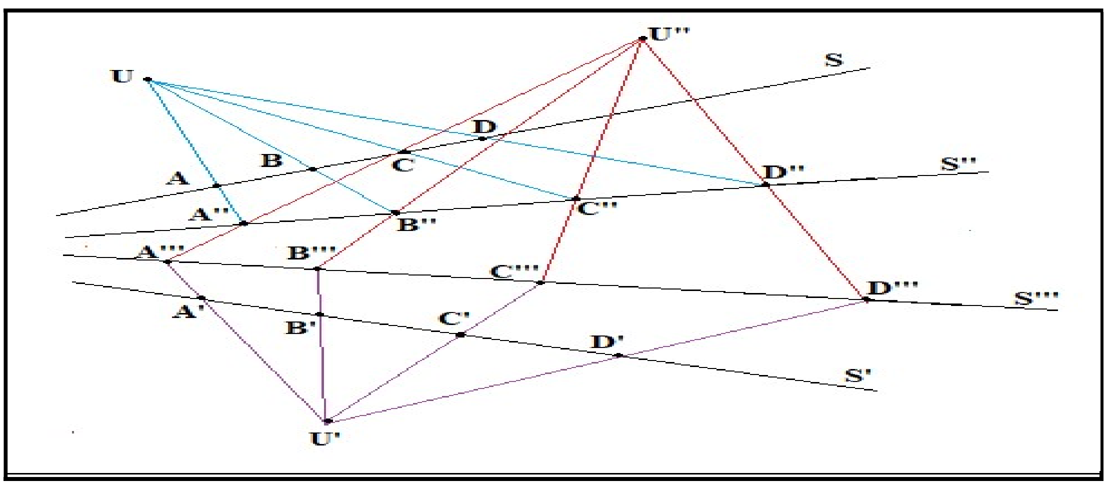
1.5. <u>Note , that a projectivity is a one-to-one correspondence or mapping between two pencils of points (rows) if the correspondence is a</u> <u>composition of finitely many perspectivities.</u> With the help of this idea we will prove Fundamental theorem (which is stated below) .

<u>**Fundamental theorem of projective geometry**</u> **:** A projectivity is uniquely determined when three pairs of corresponding elements are given .

<u>**Proof**</u> **:** **New method**

<u>In case of two rows:</u> <u>Composition of finite perspectivities (Fig.10.)</u>

Let two rows be (A B C…..) and (A′ B′ C′…..) on two distinct lines s and s′ respectively and let A′,B′,C′ be the corresponding points of A,B,C .We have to determine the projectivity between the rows s and s′ .Let us take three fixed points U ,U ′and U ″ which will be centres of three perspectives . According to the Fig.10. , two pencils of points A ,B ,C on s and A″,B″, C″ on s″ are perspective as the lines incident with the corresponding points of the two pencils are concurrent at U . Here U is the centre of perspectivity .



<u>Fig.10.</u>

. Again , two pencils of points A″, B″,C″ on s″ and A‴, B‴,C‴ on s‴ are perspective as the lines incident with the corresponding points of the two pencils are concurrent at U ″.

Here U ″ is the centre of perspectivity.

Further , two pencils of points A‴, B‴, C‴ on s‴ and A′, B′, C′ on s′ are perspective as the lines incident with the corresponding points of two pencils are concurrent at U ′ . Here U ′ is the centre of perspectivity.

Then the rows (A B C....) and (A″ B″ C″….) are sections of same pencil through U. Similarly (A″ B″ C″….) and (A‴ B‴ C‴….) are sections of the same pencil through U ″. Also (A‴ B‴ C‴….) and (A′ B′ C′…..) are sections of the same pencil through U′.Consequently , the rows (A B C....) and (A′ B′ C′…..) are projective to one another (by the composition of perspectivities) . Thus we have

To find the point D′ of s′ which corresponds to a point D of s , we join U and D to meet s″ at D″ and then we join U ″ and D″ to meet s‴ at D‴ and now D′ is the point of intersection of line U ′ D‴ with s′ . If D be the point at infinity on s , then to find its projection on s′, we draw UD″ meeting s″ at D″ (at infinity) such that UD″ meets s at a point D at infinity .We join U ″ D″ meeting s‴ at D‴ (at infinity) . Now we join U ′ D‴ meeting s′ at a point D′ (at infinity) which corresponds to the point D at infinity on s .

Hence , the projectivity is uniquely determined when three pairs (at least) of corresponding points are given .

In the case of two pencils : Composition of finite perspectivities(Fig.11.)

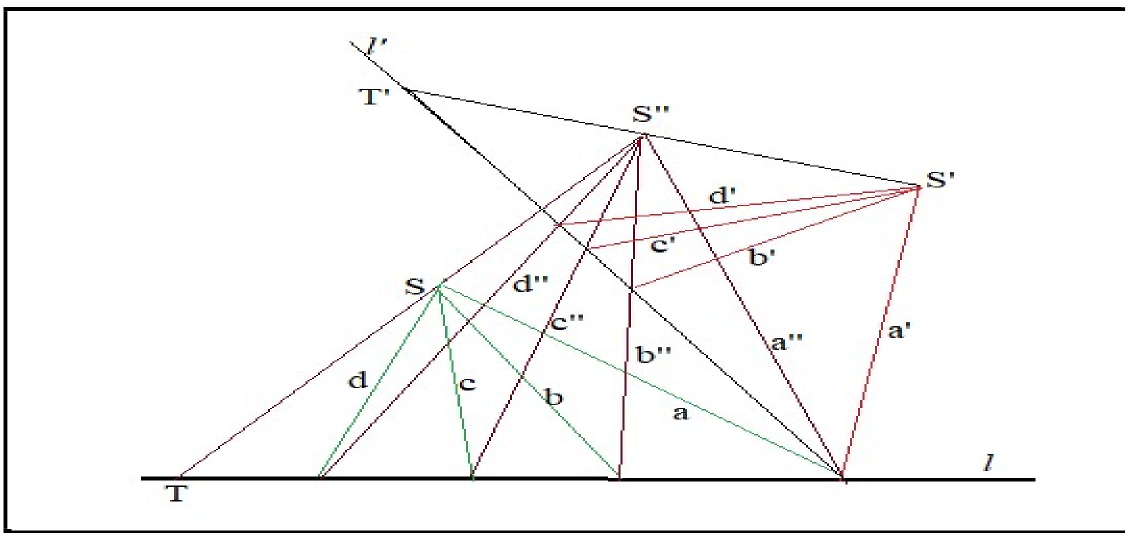


Fig.11.

Let the two pencils be S(abc….) and S′(a′b′c′….) where S and S′ are their centres. Let us draw two lines *l* and *l* ′ through the point of intersection of any pair of corresponding lines, say a and a′ as in Fig.7 . Let us denote the lines joining Sb and S′b′ by b″ ; Sc and S′c′ by c″ . Let the lines b″ and c″ meet at S″. We denote the line joining S″ and aa′ by a″. Let the line S′S″ meets the line *l* ′ at T ′ . Here S(abc….) and S(a″ b″ c″….) are any two perspective pencils and *l* be the axis of perspectivity. Let their Common element (i.e. the common line) SS″ meets *l* in T.
Now, since any two corresponding lines of the pencils intersect on *l* , ST and S″ T are corresponding lines .Since S , S″ , T are collinear , SS″ is self-corresponding. Hence , the pencils S and S″ are projective.

Similarly ,we can prove that the pencils S″(a″b″c″….) and S′(a′b′c′….) are two perspective pencils and *l*′ be the axis of perspectivity. Hence ,it follows that the pencils S (abc….) and S′(a′b′c′….) are projective. (by the composition of perspectivities).

i.e. $(a,b,c) \overset{l}{\barwedge} (a'',b'',c'') \overset{l'}{\barwedge} (a',b',c') \longrightarrow (a,b,c) \barwedge (a',b',c')$

To find the corresponding ray of d , we can join Sd and S″ by d″ and then again join d″ with S′ by d′ and hence d′ will be the required corresponding ray .

Hence , the projectivity is uniquely determined when any three pairs of corresponding lines are given .

Hence the theorem is proved .

.1.6. S.B's Theorem[2] on Projective Geometry : (First)

Statement :

Two complete quadrangles are in perspective axially iff they are in perspective centrally .

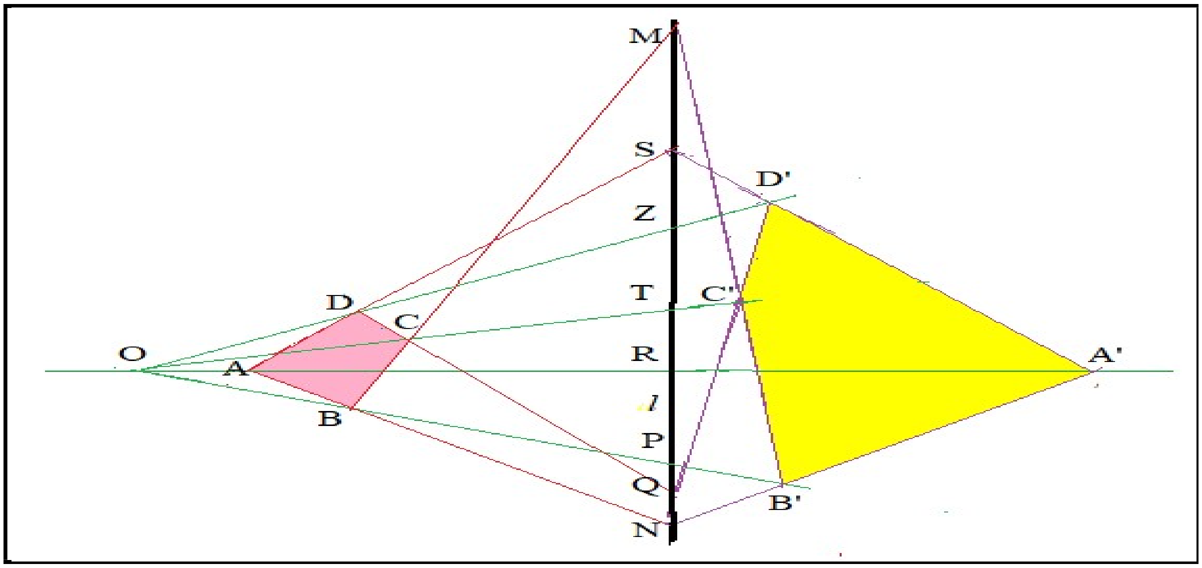


Fig.12.

Proof :

At first , we take two centrally perspective complete quadrangles ABCD and A′B′C′D′ . Let the lines joining the corresponding vertices of the two complete quadrangle ABCD and A′B′C′D′ intersect at O. Let BC×B′C′= M, CD×C′D′= Q , DA×D′A′= S and AB×A′B′= N. We are to show that M , S, Q, N are collinear.(Fig.12.)

Let (a_i),(b_i),(c_i),(d_i) and (a'_i),(b'_i),(c'_i),(d'_i) be respectively the coordinates of A,B,C,D and A′,B′,C′, D′ .

Now O being the the point. of concurrence of AA′ , BB′ , CC ′, DD′ the coordinates of O can be written as ……

$$X_i = \alpha a_i + \alpha' a'_i = \beta b_i + \beta' b'_i = \gamma c_i + \gamma' c'_i = \delta d_i + \delta' d'_i .$$

$$\text{or , } \alpha a_i - \beta b_i = \beta' b'_i - \alpha' a'_i \ ; \ \beta b_i - \gamma c_i = \gamma' c'_i - \beta' b'_i \ ; \ \gamma c_i - \delta d_i = \delta' d'_i - \gamma' c'_i \ ; \ \delta d_i - \alpha a_i = \alpha' a'_i - \delta' d'_i \ .$$

From these four equations , we can write ----------

$AB \times A'B' \equiv (\alpha a_i - \beta b_i)$ or $\equiv (\beta' b'_i - \alpha' a'_i)$

$BC \times B'C' \equiv (\beta b_i - \gamma c_i)$ or $\equiv (\gamma' c'_i - \beta' b'_i)$

$CD \times C'D' \equiv (\gamma c_i - \delta d_i)$ or $\equiv (\delta' d'_i - \gamma' c'_i)$

$DA \times D'A' \equiv (\delta d_i - \alpha a_i)$ or $\equiv (\alpha' a'_i - \delta' d'_i)$

Adding above four relations we get ---------

$$(\alpha a_i - \beta b_i) + (\beta b_i - \gamma c_i) + (\gamma c_i - \delta d_i) + (\delta d_i - \alpha a_i) = 0$$

$$\text{or , } (\beta' b'_i - \alpha' a'_i) + (\gamma' c'_i - \beta' b'_i) + (\delta' d'_i - \gamma' c'_i) + (\alpha' a'_i - \delta' d'_i) = 0$$

i.e. $AB \times A'B' + BC \times B'C' + CD \times C'D' + DA \times D'A = 0$.

From these we can conclude that AB × A′B′, BC × B′C ′, CD × C ′ D′, DA × D′A′ are linearly dependent and hence they are collinear. ***l*** is the line of collinearity .

Now, conversely let , two complete quadrangles ABCD and A′B′C ′D′ be perspective from a line ***l*** i.e. ***l*** contains four points N, M , Q , S such that N = AB × A′B′ , M = BC × B′C ′, Q = CD × C ′D′ , S = DA × D′C ′ . We want prove that AA′, BB′, CC ′, DD′ all pass through one common point O as in Fig.12.

Case I : Three dimensional diagram (In projective space) :

According to Recall point.(9)[1]. of this article , Since AB meets A′B′ , so AA′ meets BB′ , again since BC×B′C′ , so BB′ × CC′ and by the same reason CC′ meets DD′ ⇒ DD′ meets AA′ . Thus the four lines AA′ , BB′ , CC′, DD′ are concurrent at the point O(say).If the planes ABCD and A′B′C′D′ are distinct, the four lines must be concurrent , if not , they would form a quadrangle (or a triangle) which would lie in both the planes .

Case II : Two dimensional diagram (In projective plane) :

If ABCD and A′B′C′ D′ are in one plane , we draw , in another plane through O , four non-concurrent lines through N_I , M_I, Q_I, S_I. respectively , so as to form quadrangle $A_IB_IC_ID_I$ with A_IB_I through N , B_IC_I through M , C_ID_I through Q , $D_I A_I$ through S . This quadrangle is perspective from O with both ABCD and A′B′C′ D′. By the result of non coplanarity the lines AA_I , BB_I ,CC_I , DD_I pass through one point E & three lines $A'A_I$, $B'B_I$, $C'C_I$, $D'D_I$, pass through another point E′. (points E & E′ are distinct) . Since A_I lies on both AE and A′ E′ , (from Recall pt.(11) of this article) we can say that EE′ meets AA′. Similarly EE′ meets BB′ , CC′ and DD′ . Hence , four lines AA′ , BB′ ,CC′, DD′, all pass through the point O = ABCD × EE′. Thus the theorem is proved .

NOTE : SB's theorem[2] , above , is true for the real projective planes and for any projective space in which Pappus's theorem holds , in particular , theorem holds for any plane that lies in a projective 3-space . The projective plane P^2 (R) over a division ring R always satisfies the above theorem[2].

Again, it is clear from the above theorem that the above configuration requires 13 points, namely one vertex , two complete quadrangles and four points of intersection of corresponding sides. So , above SB's theorem[2] is meaningless in the Fano plane which contains seven points only.

Remarks : I was inspired and encouraged for thinking and authoring such above theorem by Desargues theorem on perspectivities of a triangle axially or centrally , which has been considered as a pioneer in this area .

1.7. **Division ring** and Projective planes / Projective spaces :

Definition :

Units : In a non-trivial ring R with unity I , an element x in R is said to be a unit if ∃ an element y in R such that x·y = y·x = I , I being the unity in R , y is said to be a multiplicative inverse of x .

Division ring / Skew field : A non-trivial ring R with unity is said to be a Division ring if every non-zero element of R is a unit .i.e.a division ring is a ring such that (R – {0} , · , **1**) is a group .

Example :1) The ring of real quaternions H = $\left\{ \begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix} : a , b , c , d \in \mathbb{R} \right\}$ is a division ring under matrix addition and matrix multiplication. (where $\mathbb{R}$ is a set of real numbers) .

2) The ring of rational quaternions G = $\left\{ \begin{pmatrix} u+iv & w+ix \\ -w+ix & u-iv \end{pmatrix} : u , v , w , x \in \mathbb{Q} \right\}$ is also a division ring under same condition. (where $\mathbb{Q}$ is the set of rational numbers) .

Projective planes and Projective spaces : The totality of all quadruplate of $(x_1 , x_2 , x_3, x_4) \neq (0, 0 , 0 , 0)$ of all real numbers x_i's is said to constitute a projective space of three dimension over the field of real numbers.This space will be denoted by P^3. Every quadruplate $(x_1 , x_2 , x_3, x_4) \neq (0, 0 , 0 , 0)$ will represent a point of P^3 and two points will be considered as identical if the corresponding quadruplates are proportional . Thus (x_1 , x_2 , x_3, x_4) and

$\alpha(x_1, x_2, x_3, x_4)$, $\alpha \neq 0$ represent the same point . A point (x_1, x_2, x_3, x_4) of P^3 will be denoted by (x_i) .

The locus of a point (x_1, x_2, x_3, x_4) satisfies a linear homogeneous equation ,

$w_1 x_1 + w_2 x_2 + w_3 x_3 + w_4 x_4 = 0$, where $(w_1, w_2, w_3, w_4) \neq (0, 0, 0, 0)$, is said to be a plane (P^2) of P^3.

Two planes $\sum_{i=1}^{4} \boldsymbol{w}_i x_i = 0$ and $\sum_{i=1}^{4} \boldsymbol{v}_i x_i = 0$ will said to be identical iff $\boldsymbol{w}_i$ and $\boldsymbol{v}_i$ are proportional i.e. $w_i = \alpha v_i$, where $\alpha \neq 0$ $\forall$ i .Thus $\alpha(w_1, w_2, w_3, w_4)$, $\alpha \neq 0$ may be taken to represent the plane $\sum \boldsymbol{w}_i x_i = 0$.

Let us consider a plane $u_1 x_1 + u_2 x_2 + u_3 x_3 + u_4 x_4 = 0$ --(1)

The maximum number of independent solution of "(1)" is 3 .So maximum number of independent points is 3 .

The set of all planes dependent of two distinct planes is said to form a pencil of planes which may be represented by $\lambda u_i + \mu v_i$; $(\lambda, \mu) \neq (0, 0)$ where u_i and v_i are two planes .

1.8 <u>Theorem</u> : . The projective plane $P^2(R)$ over a division ring R always satisfies the above S.B's theorem[2].

<u>Proof</u> : One can define projective 3-space $P^3(R)$ by taking points to be equivalence classes (x_1, x_2, x_3, x_4) , $x_i \in R$ not all zero and where this is equivalent to $(x_1\lambda, x_2\lambda, x_3\lambda, x_4\lambda)$. Planes are defined by linear equations , $\sum_{i=1}^{4} \boldsymbol{c}_i x_i = 0$ and lines as intersections of planes.

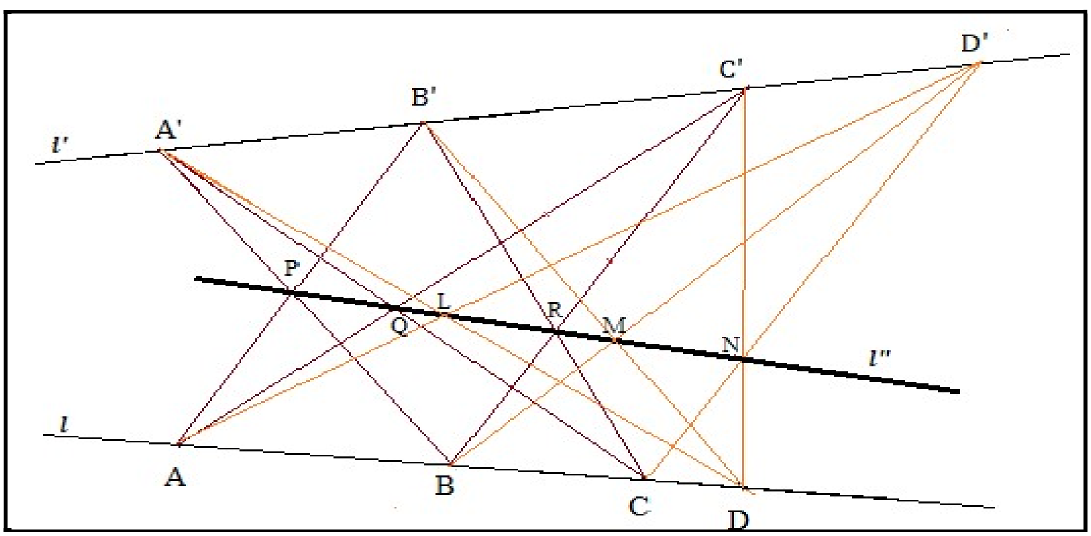
Now , $P^3(R)$ is a projective space as it satisfies all prerequisite conditions to be a projective space .

Now , $P^2(R)$ can be embedded as the plane $x_4 = 0$ in this projective 3-space and since S.B's theorem[2] holds for any plane that lies in a projective 3-space , $P^2(R)$ over a division ring R always satisfies the above theorem .

1.9 <u>Extension theorem (1)</u> <u>of **Pappus'** theorem</u> : (Fig.13).

If A,B,**C** be three distinct points on a line *l* and A′, B′, C′ be three distinct ponts on a different line *l′*, different from X and if we define P = AB′×A′B , Q = AC′×A′C , R = BC′×B′C, then P ,Q and R are collinear. *l″* is the line of collinearity (by **Pappus'** Theorem) . Now , for any distinct point D on *l*, different from X , we can determine an unique point D′ on ***l′***, different from X , such that the points L (AD×A′D) , M (BD′×B′D) and N (CD′×C′D) are collinear with the points P,Q,R. The line of collinearity is ***l″***.

<u>Proof</u> : Proof of the above theorem is obvious from the Fig.13. given below .



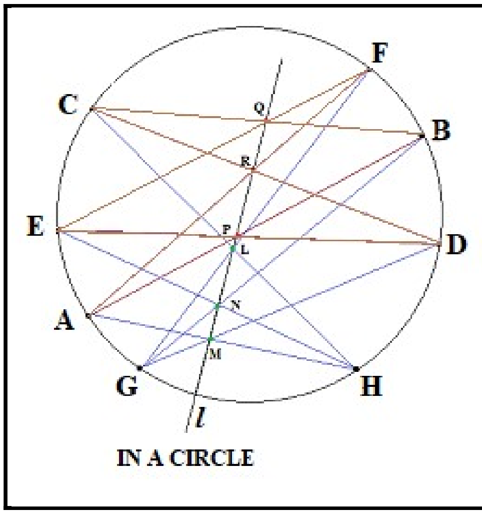
<u>Fig.13.</u>

1.10.<u>Extension theorem (2)</u> of <u>Pascal's</u> <u>theorem :</u> (Fig.14. and Fig.15.)

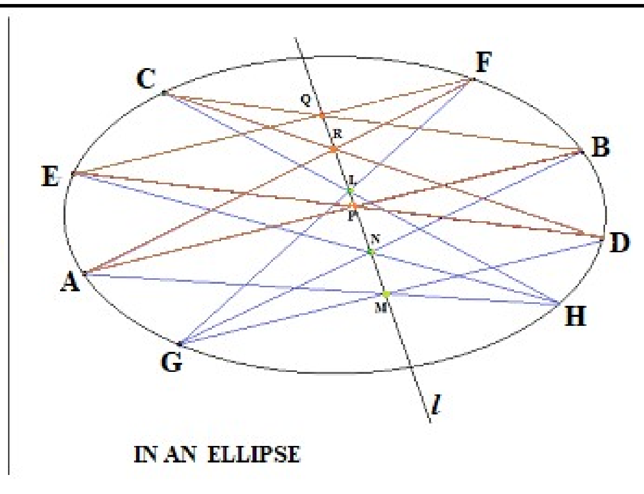
If the six vertices of a hexagon ABCDEF lie on a conic **,**the points of intersection of the three pairs of opposite sides P = AB × DE **;** Q = BC × EF **;** R = CD × FA are collinear on the Pascal line (***l***) (by Pascal's theorem) **,** then for any arbitrary point G on that conic we can find an unique point H on that conic such that the points of intersection of the three pairs of opposite sides L= (GF × CH)**,**M = (GD × AH) **,** N = (GB × EH) are collinear with P**,**Q**,** and R on the same Pascal line (*l*) . Here ABCDEFGH is an <u>octagon</u> inscribed in the conic (which may be a circle or an ellipse **,** parabola or hyperbola).

<u>Proof</u> : Proof of the above theorem is obvious from the Fig.14.and Fig.15. given below.

Here according to the figures 14 and 15 **,** ABCDEF is a hexagon inscribed in the conic section **,** where points of intersection of the three pairs of opposite sides P = AB×DE **;** Q = BC×EF **;** R = CD×FA are collinear on the Pascal line (***l***).Again **,** for any point G we can obviously find an unique point H on the same conic such that points of intersection of the six pairs of opposite sides P = (AB×DE) **;** Q = (BC × EF) **;** R = (CD × FA) **;** L = (GF × CH) **;** M = (GD × AH) **;** N = (GB × EH) are collinear on the same Pascal line (***l***) and thus we have proved the theorem . Here ABCDEFGH forms an octagon inscribed in the same conic section.



<u>Fig.14.</u>



<u>Fig.15.</u>

<u>Note</u> : Above two Extension theorems can be proved easily for any pair of points (5**,**6 or 7 and so on …)

1.11. C<u>ollineation-----------Transformation----------Classification</u> :

In projective geometry **,** a collineation is a bijective mapping from a projective space to another projective space or from a projective space into itself **,** such that the relation of incidence is always preserved .Thus it transforms points into points **,** ranges into ranges **,** pencils into pencils **,** quadrangles into quadrangles and so on . So **,** collineation is said to be an isomorphism between two projective spaces or an automorphism from a projective space into itself . Clearly **,** it is a self-dual concept **,** the inverse of a collineation is a collineation . Such a mapping may be obtained by matrix multiplication by a 3×3 matrix ***U*** **.** We can transform every point p into a point p*´* such that p*´* = **U**p . In the case of transforming a line ***l*** into a line ***l*´,** we note that collinearity must be preserved i.e**.** if a point p lies on the line ***l*,** then the corresponding point p´ must lie on the corresponding line ***l*´** therefore -------

$$p^{U} \boldsymbol{l} = (\mathbf{U}^{-1} p')^{U} \boldsymbol{l} = (p')^{U} (\mathbf{U}^{-U} \boldsymbol{l}) \qquad \text{which indicates that } \boldsymbol{l}' = \mathbf{U}^{-U} \boldsymbol{l} .$$

We can use the transformation and collineation interchangeably **.**

Regarding transformation **,** we can say that -------------------

Euclidean $\subset$ **Similarity** $\subset$ **Affine** $\subset$ **Projective**.

A property which remains invariant under the group of collineations is called projective properties . Collinearity of points **,**concurrency of lines **,** cross-ratio **,** principle of duality etc ***.*** are projective properties . A property which remains invariant under the group of affinities **,** but not under the projective group **,** is called an affine property in the strict sense . Parallelism of lines is an affine property.

Unlike affine transformations **,** similarity transformations preserve angles **,** areas and ratio of lengths .

Unlike projective **,** affine or similarity transformations **,** scale is important in Euclidean transformations .

A linear transformation of the type $\rho x_i' = \sum u_{ij} x_j$ **,** $|u_{ij}| \neq 0$, ……………………………………….(2)

where i = 1,2,3,…..n. is called a projective transformation or Collineation of P_n .If we impose the conditions… $u_{o1} = u_{o2} = \dots\dots\dots = u_{on} = 0$ and $u_{oo} \neq 0$. Then “(2)” reduces to

$$\left.\begin{aligned}
\rho x_o' &= u_{oo} x_o . \\
\rho x_1' &= u_{1o} x_o + u_{11} x_1 \ \dots\dots\dots\dots\ u_{1n} x_n \\
&\text{- -} \\
\rho x_n' &= u_{no} x_o + u_{n1} x_1 \ \dots\dots\dots\dots\ u_{nn} x_n
\end{aligned}\right\} \text{- - - - - - - - - - - - - - - - - - - -} (3)$$

$$\text{with} \quad u_{oo} \begin{vmatrix} u_{11} & \text{- - - - - - - - - -} & u_{1n} \\ u_{21} & \text{- - - - - - - - - -} & u_{2n} \\ \text{- - - -} & \text{- - - - - - - - - -} & \text{- - - -} \\ u_{n1} & \text{- - - - - - - - - -} & u_{nn} \end{vmatrix} \neq 0$$

All such transformations are called affinities of P_n .Thus the affinities are achieved from collineation by imposing a particular condition .

Since $u_{oo} \neq 0$ **,** we can take $u_{oo} = 1$, without loss of generality .Using non-homogeneous coordinates…………

$x = x_1 / x_o$, $y = x_2 / x_o$,…………………., $w = x_n / x_o$, we can write equation “(3)” in the form……………

$$\begin{aligned}
x' &= u_{11} x + u_{12} y + \ \dots\dots\dots\dots\ u_{1n} w + u_{1o} \\
y' &= u_{21} x + u_{22} y + \dots\dots\dots\dots\dots\dots\dots\dots\dots u_{1n} w + u_{2o} \\
&\text{- -} \\
w' &= u_{n1} x + u_{n2} y + \dots\dots\dots\dots\dots\dots\dots\dots .. \ u_{nn} w + u_{no}
\end{aligned}$$

Here $\Delta = \begin{vmatrix} u_{11} & \cdots & u_{1n} \\ u_{21} & \cdots & u_{2n} \\ \cdots & \cdots & \cdots \\ u_{n1} & \cdots & u_{nn} \end{vmatrix} \neq 0$

If we impose the conditions $\sum_{k=1}^{n} u_{ik} \, . u_{jk} = \begin{cases} v^2 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$ i , j = 1 , 2 , 3 ,………n ;

(where v is a non-zero constant) on the affinities the transformations are called similarities .
In this case $\Delta^2 = v^{2n} \Rightarrow \Delta = \pm v^n$.
Furthermore , if we impose the condition $v^2 = 1$, we will get orthogonal transformations. For these transformations $\Delta = \pm 1$. The orthogonal transformations for which $\Delta = 1$, are called rigid motions (Euclidean transformations) and for which $\Delta = -1$, are called symmetries .The symmetries do not form a group .

Corollary (1.11 . 1) : The cross-ratio of any four distinct collinear points remains invariant under a collineation .

Proof : Let P, Q , R , S be four collinear points and let us assume that P′,Q′,R′,S′ be their transforms under the collineation given by the relation---- $\alpha P' = PA$, $\alpha \neq 0$ and det$A \neq 0$.

Then we take, $\alpha_1 P' = PA$, $\alpha_2 Q' = QA$, $\alpha_3 R' = RA$, $\alpha_4 S' = SA$ where $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, $\alpha_3 \neq 0$, $\alpha_4 \neq 0$ and det$A \neq 0$. Since , R is collinear with P , Q , we express R as…$R = \lambda P + \mu Q$, so , $\alpha_3 R' = RA = (\lambda P + \mu Q)A = \lambda PA + \mu QA = \lambda \alpha_1 P' + \mu \alpha_2 Q'$

so , $R' = \frac{\lambda \alpha 1}{\alpha 3} P' + \frac{\mu \alpha 2}{\alpha 3} Q' = \lambda' P' + \mu' Q'$ .where $\lambda' = \frac{\lambda \alpha 1}{\alpha 3}$ and $\mu' = \frac{\mu \alpha 2}{\alpha 3}$.

Similarly , if $S = \gamma P + \delta Q$ we get S′ as $S' = \gamma' P' + \delta' Q'$ where $\gamma' = \frac{\gamma \alpha 1}{\alpha 4}$ and $\delta' = \frac{\delta \alpha 2}{\alpha 4}$.

Cross-ratio (PQ , RS) = $\frac{\mu}{\lambda} \Big/ \frac{\delta}{\gamma} = \frac{\mu\gamma}{\lambda\delta}$ --(4)

Cross-ratio (P′Q′ , R′S′) = $\frac{\mu'\gamma'}{\lambda'\delta'} = \frac{\frac{\mu \alpha 2}{\alpha 3}}{\frac{\lambda \alpha 1}{\alpha 3}} \times \frac{\frac{\gamma \alpha 1}{\alpha 4}}{\frac{\delta \alpha 2}{\alpha 4}} = \frac{\mu\gamma}{\lambda\delta}$ which is same as --"(4)".

Hence , we have attained the desired result .

Corollary (1.11 . 2): The collinearity of points is an invariant under a Collineation .

Proof : Let x , y , z be three distinct collinear points and let us assume that x , y , z be their transformations under a collineation given by the relation …………

$\rho x' = xA$, $\rho \neq 0$, det$A \neq 0$

Then we take $\rho_1 x' = xA$, $\rho_2 y' = yA$, $\rho_3 z' = zA$

where $\rho_1 \neq 0$, $\rho_2 \neq 0$, $\rho_3 \neq 0$, det$A \neq 0$. Since , z is collinear with x , y , we may express z as $z = \lambda x + \mu y$. So $\rho_3 z' = zA = (\lambda x + \mu y)A = \lambda \rho_1 x' + \mu \rho_2 y'$.

therefore , $z' = (\lambda \rho_1 / \rho_3)\, x' + (\mu \rho_2 / \rho_3)\, y' = \lambda' x' + \mu' y'$.

Thus , x' , y' , z' are collinear i.e. the collinearity of points under a collineation remains invariant .

Corollary (1.11 . 3):If x_1 , x_2 , x_3 , x_4 are the distances of four distinct points P_1 , P_2 , P_3 , P_4 from any fixed origin lying on the straight line on which the given four points lie , then find the conditions when the cross-ratios are (A) $(P_1P_2 , P_3P_4) = -1$. (B) $(P_1P_2 , P_3P_4) = 1$.

Solutions :

(A) We have ------------ $(P_1P_2 , P_3P_4) = -1$

or , $(x_1 - x_3)(x_2 - x_4) \;/\; (x_2 - x_3)(x_1 - x_4) = -1$

or , $(x_1 - x_3)(x_2 - x_4) + (x_2 - x_3)(x_1 - x_4) = 0$

or , $x_1x_2 - x_1x_4 - x_3x_2 + x_3x_4 + x_1x_2 - x_2x_4 - x_3x_1 + x_3x_4 = 0$

or , $2(x_1x_2 + x_3x_4) = x_1x_4 + x_3x_2 + x_2x_4 + x_3x_1 = (x_1 + x_2)(x_3 + x_4)$.

Therefore , the desired condition is $(x_1 + x_2)(x_3 + x_4) = 2(x_1x_2 + x_3x_4)$.

(B) Here , $(P_1P_2 , P_3P_4) = 1$ i.e. $(x_1 - x_3)(x_2 - x_4) \;/\; (x_2 - x_3)(x_1 - x_4) = 1$

or , $(x_1 - x_3)(x_2 - x_4) = (x_2 - x_3)(x_1 - x_4)$.

or , $x_1x_2 - x_1x_4 - x_3x_2 + x_3x_4 = x_1x_2 - x_2x_4 - x_3x_1 + x_3x_4$.

or , $(x_1 - x_2)(x_3 - x_4) = 0$ i.e. either $x_1 = x_2$ or , $x_3 = x_4$

i.e. either P_1 coincides with P_2 or , P_3 coincides with P_4 , but which contradicts our assumption (P_1 , P_2, P_3 , P_4 are four distinct points).

Corollary(1.11 . 4) :If l_1 and l_2 be two intersecting straight lines and p and q are the internal and external bisectors of the angles between them , then find the cross ratio $(l_1 l_2 , pq)$.

Solution :

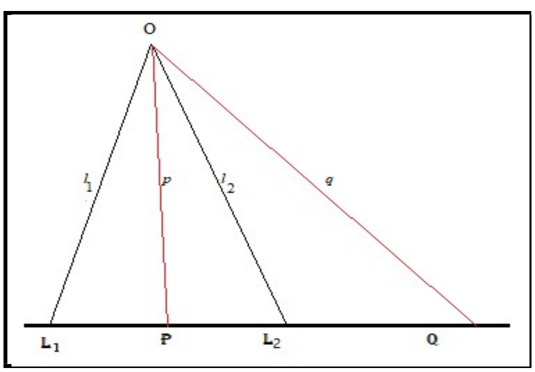


Fig.16.

$(l_1 l_2 , pq) = (L_1 L_2 , PQ)$

$= (PL_1 / PL_2) / (QL_1 / QL_2)$

$= [\sin(p , l_1) / \sin(p, l_2)] / [\sin(q , l_1) / \sin(q, l_2)] = 1$.

[since , $\sin(p , l_1) = \sin(p, l_2)$ and $\sin(q , l_1) = \sin(q, l_2)$]

N.B . : 1.11.1) By a Collineation , a polarity is transformed to a polarity and a null system to a null system .

1.11.2) For a null system every point is self-conjugate.

1.11.3) The product of two correlations is a Collineation and the same is true for the product of an even number of correlations .

SECTI*ON* -- II

Let us pause for a while to watch what we have done and where we are going .Till now **,** we have been studying some new ideas **,** theorems and corollaries **,** based on the works of respected great mathematicians on the subject of Projective Geometry.

Now **,** we are going to enter into a totally new topic **,** named --- **Arc-Ratios** .

CONCEPT OF ARC-RATIO

2.1. Ratio in which a point divides two other points on a circle :

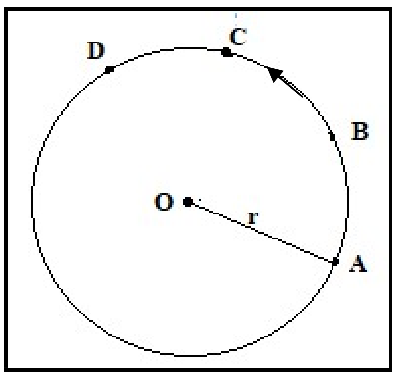


Fig.17.

Let A **,** B **,** C be any three distinct points on a circle (considered as being contained in the real projective plane or we may consider the circle as the intersection of a sphere in space by a plane) . The ratio in which C divides arc.AB is given by $arc.AC/arc.BC$.

Eventually **,** $arc.AC/arc.BC = arc.CA/arc.CB$. (Fig.17.)

2.2. Arc-Ratio[3] of four concyclic points : (Definition) . (Fig.17.)

If **A , B , C , D** be four distinct concyclic points **,** then their Arc-Ratio will be denoted by ((AB**,**CD)) and is defined by the quotient ---------

((AB,**C**D)) = $\frac{(ratio\ in\ which\ C\ divides\ arc.AB)}{(ratio\ in\ which\ D\ divides\ arc.AB)} = \frac{arc.CA/arc.CB}{arc.DA/arc.DB} = \frac{arc.CA}{arc.CB} \times \frac{arc.DB}{arc.DA}$.

2.3. Properties[4] of Arc-Ratio : (All by Fig.17.)

Let A**,**B**,**C**,**D be four distinct concyclic points **,** then -----------

(**2.3.1**). The interchange of two pair of elements does not change in Arc-Ratio i.e. ((AB**,** CD)) = ((CD **,** AB)) .

Proof : ((CD **,** AB)) = $\frac{arc.AC/arc.AD}{arc.BC/arc.BD} = \frac{arc.CA/arc.DA}{arc.CB/arc.DB} = \frac{arc.CA/arc.CB}{arc.DA/arc.DB}$ = ((AB **,** CD)) .

(**2.3.2**). The reversal of order in both pairs does not change the Arc-Ratio i.e. ((AB **,** CD)) = ((BA **,** DC)) .

Proof : ((BA **,** DC)) = $\frac{arc.DB/arc.DA}{arc.CB/arc.CA} = \frac{arc.CA/arc.DA}{arc.CB/arc.DB} = \frac{arc.CA/arc.CB}{arc.DA/arc.DB}$ = ((AB **,** CD))

(**2.3.3**). The reversal of order in one pair changes the Arc-Ratio into its reciprocal i.e., ((AB , DC)) = $\frac{1}{((AB,CD))}$ and ((BA , CD)) = $\frac{1}{((AB,CD))}$

Proof : ((AB , DC)) = $\frac{arc.DA/arc.DB}{arc.CA/arc.CB} = \frac{1}{\frac{arc.CA/arc.CB}{arc.DA/arc.DB}} = \frac{1}{((AB,CD))}$.

Similarly , the other result.

(**2.3.4**). ((AB , CD)) + ((AC , BD)) = 1

Proof : L.H.S. = ((AB , CD)) + ((AC , BD))

$$= \frac{arc.CA \times arc.DB}{arc.CB \times arcDA} + \frac{arc.BA \times arc.DC}{arc.BC \times arcDA} = \frac{arc.CA \times arc.DB - arc.BA \times arc.DC}{arc.CB \times arc.DA} \quad (\text{ as , } arc.BC = -arc.CB)$$

$$= \frac{(arc.CB + arc.BA)\ (arc.DA - arc.BA) - arc.BA\,(arc.DA - arc.CB - arc.BA)}{arc.CB \times arc.DA}$$

$$= \frac{arc.CB \times arc.DA}{arc.CB \times arcDA} = 1 = \text{R.H.S.} \quad \text{Proved .}$$

$$\left\{ \begin{array}{l} \text{Since , } \because \textbf{arc.CA = arc.CB + arc.BA , arc. DB = arc. DA − arc. BA} \\ \textbf{arc.DC = arc. DA − arc.CA = arc.DA − arc.CB − arc.BA} \end{array} \right\}$$

2.4. Arc-Ratios in terms of central angles :

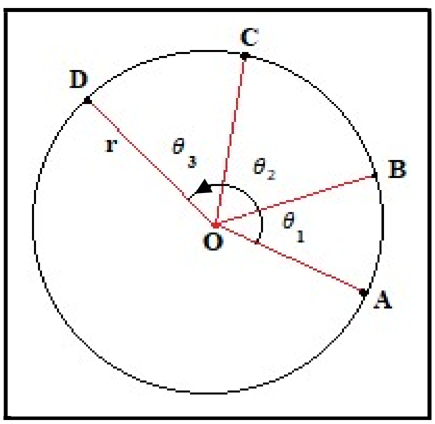


Fig.18.

Let **A , B, C , D** be four distinct concyclic points on a circle with centre at O and with radius r . Let the angles at the centre by the arc.AB , arc.BC and arc.CD are $\theta_1, \theta_2, \theta_3$ respectively . (Fig.18).

Here all three angles are positive .

Then arc.AB = r θ_1, arc.B**C** = r θ_2, arc.AD = r ($\theta_1 + \theta_2 + \theta_3$) , arc.AC = r ($\theta_1 + \theta_2$) ,

arc.BD = r ($\theta_2 + \theta_3$) , arc.CD = r θ_3 -------

Then the Arc-Ratio ((AB , CD)) can be denoted by ((AB ,CD)) = $\frac{arc.CA \times arc.DB}{arc.CB \times arcDA} = \frac{\theta 1 + \theta 2}{\theta 2} \times \frac{\theta 2 + \theta 3}{\theta 1 + \theta 2 + \theta 3}$

2.5. Some relations of Arc-Ratios and central angles or Properties :(By Fig.18.)

If **A,B,C,D** be any four concyclic points on a circle with the centre at O with radius r , then………

(2.5.1) Relation : When ((AB,CD)) = − 1 then ,

$$\frac{(\theta 1 + \theta 2)}{\theta 2} \times \frac{(\theta 2 + \theta 3)}{(\theta 1 + \theta 2 + \theta 3)} = -1$$

or , $\theta_1\,\theta_3 + \theta_1\,\theta_2 + \theta_2\,\theta_3 + \theta_2^2 + \theta_1\,\theta_2 + \theta_2^2 + \theta_2\,\theta_3 = 0$

or, $\theta_1\theta_3 + 2\theta_1\theta_2 + 2\theta_2\theta_3 + 2\theta_2^2 = 0$

or, $\theta_1(\theta_3+2\theta_2)+\theta_2(\theta_3+2\theta_2)+\theta_2\theta_3=0$

or, $(\theta_3+2\theta_2)(\theta_1+\theta_2)+\theta_2\theta_3=0$

which is not possible **,** because all the angles $\boldsymbol{\theta_1}$**,** $\boldsymbol{\theta_2}$**,** $\boldsymbol{\theta_3}$ are positive (not equal to zero) and so we can say that ((AB **,** CD)) $\neq -1$.

<u>(2.5.2)Relation</u> : When ((AB , CD)) **<** -1 then……

$$\frac{(\boldsymbol{\theta 1}+\boldsymbol{\theta 2})}{\boldsymbol{\theta 2}}\times\frac{(\boldsymbol{\theta 2}+\boldsymbol{\theta 3})}{(\boldsymbol{\theta 1}+\boldsymbol{\theta 2}+\boldsymbol{\theta 3})} < -1$$

or , $\theta_1\theta_3+\theta_1\theta_2+\theta_2\theta_3+\theta_2^2+\theta_1\theta_2+\theta_2^2+\theta_2\theta_3 < 0$

which is not possible **,** as in the case (2.5.1) .

So **,** we conclude that ((AB **,** CD)) can never be negative (if all angles are measured in positive direction) .

(2.5.3) <u>Relation</u> : When ((AB **,**CD)) = 1 then **,**

$$\frac{\theta 1+\theta 2}{\theta 2}\times\frac{(\boldsymbol{\theta 2}+\boldsymbol{\theta 3})}{(\boldsymbol{\theta 1}+\boldsymbol{\theta 2}+\boldsymbol{\theta 3})} = 1$$

or , $(\theta_1+\theta_2)(\theta_3+\theta_2)-\theta_2(\theta_1+\theta_2+\theta_3) = 0$

or , $\theta_1\theta_3+\theta_1\theta_2+\theta_2\theta_3+\theta_2^2-\theta_1\theta_2-\theta_2^2-\theta_2\theta_3=0$

or , $\theta_1\theta_3=0$.

So **,** either $\theta_1=0$ or **,** $\theta_3=0$ i.e. either A coincides with B or C coincides with D but which is not possible because A**,**B**,**C**,**D are four distinct points and so both the angles are positive.

So we can say that ((AB**,**CD)) $\neq$ 1 .

(2.5.4)<u>Relation</u> : When ((AB**,**CD)) = 0

then , $(\theta_1+\theta_2)(\theta_3+\theta_2)=0$ **,**

i.e. either $\theta_1+\theta_2=0$ or $\theta_3+\theta_2=0$ **,**

i.e. either A coincides with C or B coincides with D , which contradicts the assumption that A , B , C , D are distinct points and so $\theta_1+\theta_2>0$, $\theta_3+\theta_2>0$.

So **,** we can say that ((AB**,**CD)) $\neq$ 0

<u>(2.5.5)Relation</u> : When 0 $<$ ((AB**,**CD)) $<$ 1 and ((AB **,** CD)) $>$ 1

The relation is possible and the deduction is so simple that it is left for the readers.

Finally **,** we can conclude that ((AB**,**CD)) >0 **,** but ((AB**,**CD)) $\neq 1$.

2.6. <u>Arc-Ratio of four distinct concyclic points in terms of a single parameter</u> :

From four distinct concyclic points we can obtain 4! = 24 Arc-Ratios **,** accordingly to order in which the points are taken . But these 24 Arc-Ratios are not all different because of the above properties .In fact they can be arranged in six rows **,** the values of the Arc-Ratios in a row being equal.

If we assume ((AB ,CD)) = λ **,** then we have-----

(2.6.1) ((AB , CD)) = ((BA , DC)) = ((DC , BA)) = ((CD , AB)) $= \lambda$.

(2.6.2) ((AB , DC)) = ((BA , CD)) = ((CD , BA)) = ((DC , AB)) $= \frac{1}{\lambda}$.

(2.6.3) ((AC , BD)) = ((CA , DB)) = ((DB , CA)) = ((BD , AC)) $= 1 - \lambda$.

(2.6.4) ((AC , DB)) = ((CA , BD)) = ((BD , CA)) = ((DB , AC)) $= \frac{1}{1-\lambda}$.

(2.6.5) ((AD , BC)) = ((BC , AD)) = ((CB , DA)) = ((DA , CB)) $= 1 - \frac{1}{\lambda}$.

(2.6.6) ((AD , CB)) = ((BC , DA)) = ((DA , BC)) = ((CB , AD)) $= \frac{\lambda}{\lambda-1}$.

Above six relations can be easily proved by definition of Arc – Ratio[3] with the help of the properties[4] of Arc-Ratio given above **,** so left for the readers .

2.7. S.B's Theorem[5] on Projective Geometry :(second)

Statement : Arc-Ratio remains invariant under projection. (Fig.19)

Proof :

Let MABCDN be a circle with centre O and with radius r (considered as being cut off from a sphere **,** contained in any projective space **,** by a plane) . A source of light is at S such that SO is perpendicular to the plane of the circle . From S some rays of light fall on the circle (i.e. on the object sphere) and make its shadow on a plane (*l*) parallel to the circle MABCDN and the formed shadow is again a circle M′A′B′C ′D′N′ i.e. the points **A′, B′, C ′, D′** are the projections of the points **A , B , C , D** i.e. central projection. It is very clear from two diagrams of Fig.18. that-----

$\angle$AOB $= \angle$ A′O′B′ $= \theta_1$, $\angle$ BOC $= \angle$ B′O′C′$= \theta_2$, (as the angle is formed by two intersecting planes SBB′O′ and SCC′O′ at the line of intersection SOO′ and similarly for other angles also), and the angle $\angle$COD $= \angle$ C ′O′D′$= \theta_3$ where S ,O ,O′ are collinear and perpendicular to the planes of the circles ($\angle$SOC =$\angle$SO′C′ $= \angle$SOB $= \angle$SO′B′ $= \angle$SOA $= \angle$SOD $= \angle$SO′A′.........$= 90^o$)

Now we have to prove that Arc-Ratio ((AB ,CD)) = ((A′B′ , C′D′)) .

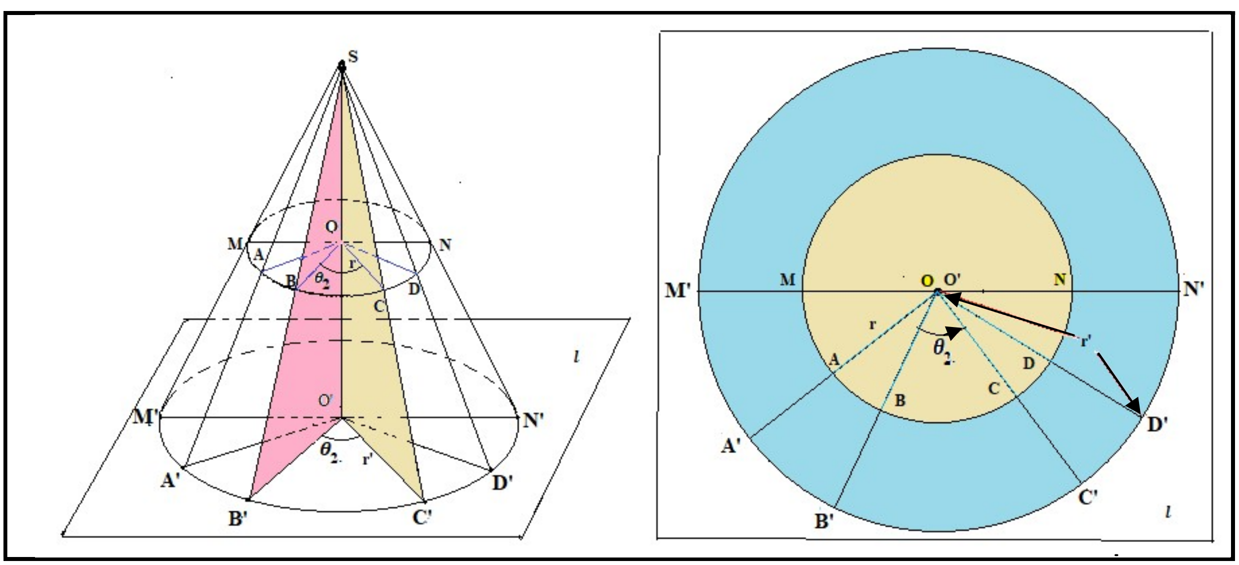


Fig.19.

According to the Fig.19.---------

In the small circle ***MABCDN*** (object) with radius **r , ((AB ,CD)) =** $\frac{\theta 1+\theta 2}{\theta 2} \times . \frac{\theta 2+\theta 3}{\theta 1+\theta 2+\theta 3}$

Again in the bigger circle (shadow) $M'A'B'C'D'N'$ with radius r ((A′B′ , C′D′)) = $\frac{\theta 1+\theta 2}{\theta 2} \times . \frac{\theta 2+\theta 3}{\theta 1+\theta 2+\theta 3}$.

So **,** Arc-Ratio ((AB **,**CD)) = ((A′B′ **,**C′D′)) .We have come to a conclusion that Arc-Ratio remains invariant under projection.

Note : It is a projection of a projective plane(i.e. the object is a circle) on another plane (*l*) **,**

from a point S outside the planes .

PRACTICAL EXAMPLE OF THE ABOVE THEOREM [5] :

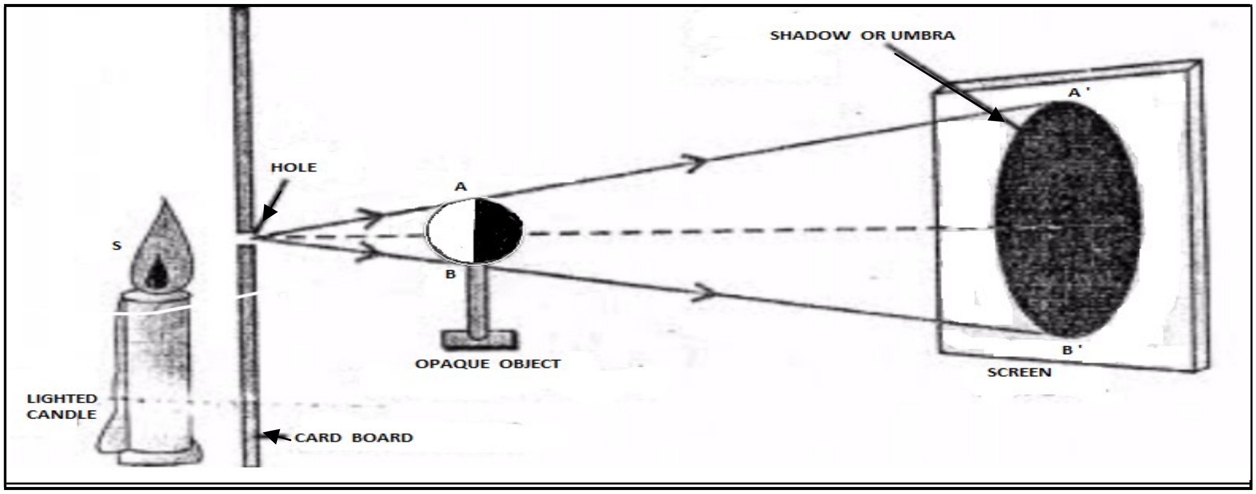


Figure.20.

Some corollaries of Theorem[5] :

Corollary (2.7.1):

Again **,** there is a one-to-one correspondence between the points A**,**B**,**C**,**D of one circle and the points A′**,**B′**,**C′**,**D′ of another circle such that their respective Arc-Ratios are equal .This one-to-one correspondence is said to be a projectivity between the points of two circles . Here the lines joining the corresponding points AA′**,** BB′**,** CC′**,** DD′ are concurrent at S .In such a case the two circles are said to be perspective and the point of concurrence (S) is the centre of perspectivity.

Corollary (2.7.2):

According to Fig.19. we can conclude that there are infinite numbers of circles **,** perspective with the circle M′A′B′C ′D′N′ **,** with the centres lying on SO′**,** with different radius . In all cases the Arc-Ratios remain invariant .

Corollary (2.7.3) **:**

(Here source and object circle both are moved so that shadows fall on the same circle.)

Note that a projectivity is a one-to-one mapping between two sets of concyclic points if the mapping is a composition of finitely many perspectivities .

According to the Fig.21. **,** circle MABCDN with centre O and circle UA′′B′′C′′D′′V with centre O′′ are perspective.

So **,** ((AB **,**CD)) = ((A′′B′′**,**C′′D′′)) and the lines joining the corresponding points AA′′**,** BB′′ **,** CC′′**,** DD′′ are concurrent at S and the point of concurrence (S) is the centre of perspectivity..

Again , the circle UA′′B′′C′′D′′V with centre O′′ and circle PA′B′C′D′Q with centre O′ are perspective .So , ((A′B′ ,C′D′)) = ((A′′B′′ ,C′′D′′)) and the lines joining the corresponding points A′A′′, B′B′′ , C′C′′, D′D′′ are concurrent at S′ and the point of concurrence (S′) is the centre of perspectivity.. So , ((AB ,CD)) = ((A′B′ ,C′D′)) .

Therefore , the points A,B,C,D and the points A′,B′,C′,D′ are projective .

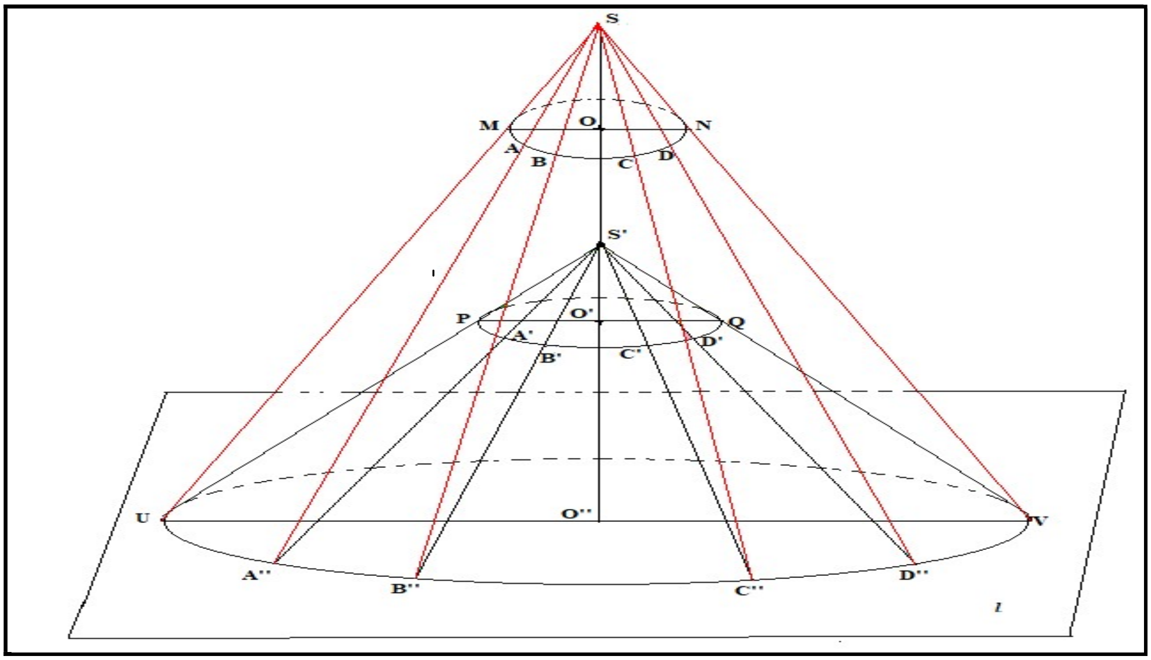


Fig . 21.

Corollary (2 .7 . 4) :

In Fig.22(a) ,the object is moved but the source of light (S) and the plane (*l*) on which shadows are fallen , are remained in the same position. By the above theorem , here also………. Arc-Ratio ((AB ,CD)) = ((A′′ B ′′ , C′′D′′)) = ((PQ, RT)) = ((A′ B′, C′ D′)) . Therefore , the points A,B,C,D and the points A′,B′,C′,D′ are projective (as a composition of perspectivities).

In Fig.22(b) , the source of light (S′) is moved to (S) but the object and the plane (*l*) on which shadow falls , are remained in the same position. By the above theorem , here also………. Arc–Ratio ((AB, CD)) = ((A′ B′, C′ D′)) = ((A′′B′′,C′′D′′)) . Therefore , the points A,B,C,D and the points A′′, B′′,C′′, D′′ are projective(as a composition of perspectivities).

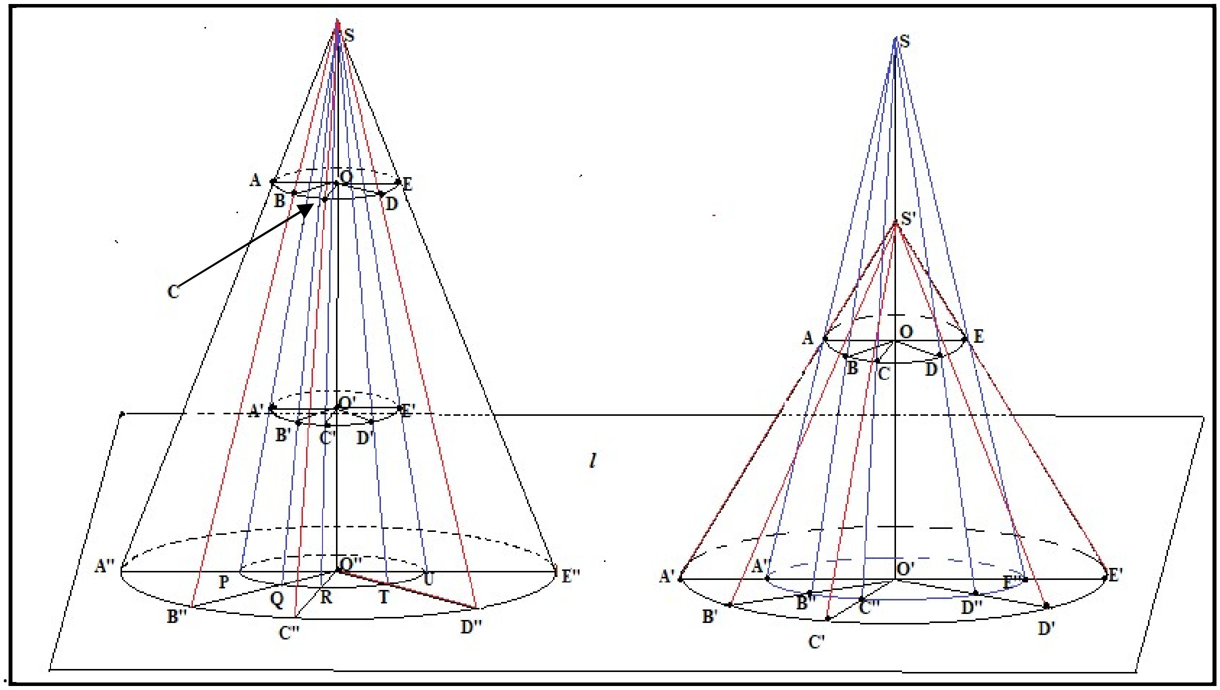


Fig.22(a). *Fig.22(b).*

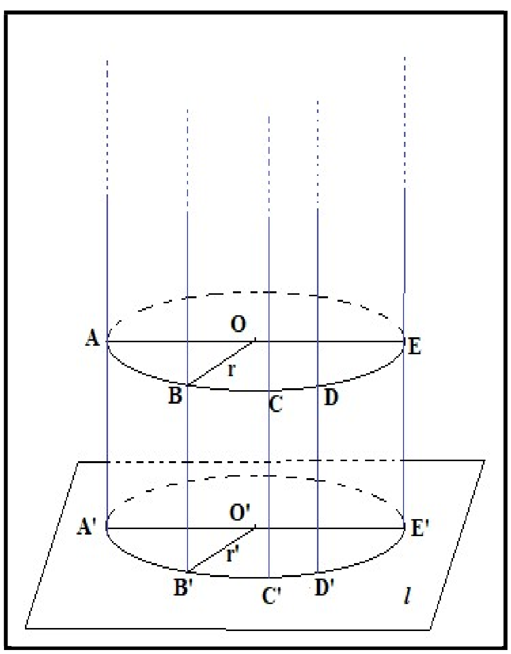


Fig.23.

Corollary (2. 7. 5):(Fig.23.)

If the object circle ABCDE with radius r and the plane (*l*) on which shadow

A′B′C′D′E′ with radius r′ falls **,** are remained in the same position and we assume that the point (S) source of light is

at infinity then the rays SAA′**,** SBB′**,** SCC′**,** SDD′ ,SEE′ seem to be parallel (perspective) and in this case the circles ABCDE and A′B′C ′D′E′ are of same radius i.e. r = r′ (Fig.22.).

So**,** Arc-Ratio ((AB ,CD)) = ((A′B′,C ′D′)) **,** the points A**,**B**,**C**,**D and points A′**,**B′**,**C′**,**D′ are projective.

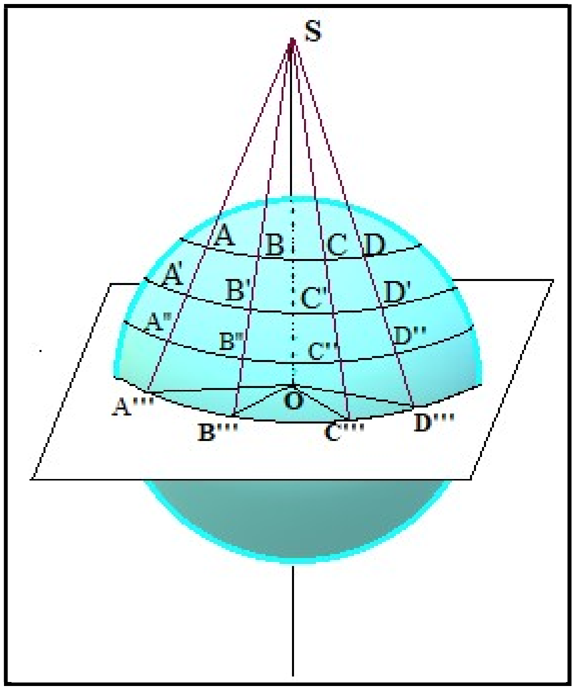


Fig.24 .

Corollary (2.7. 6) : Fig.24.

If we cut a sphere by different parallel planes (assumed to intersect each other at infinity) , we will get different circles including the great circle. We take A,B,C,D ; A′,B′,C′,D′ ; A′′,B′′,C′′,D′′ and A′′′,B′′′,C ′′′,D′′′ which are four sets of points on four such circles(according to Fig. .) formed by a central projection from a point S(source of light) outside the sphere. O is centre of the sphere .It is clear from the figure that the Arc – Ratios are equal, i.e. ((AB,CD)) = ((A′B′,C′D′)) = ((A′′B′′,C ′′D′′)) = ((A′′′B′′′,C ′′′D′′′)) , as the four sets of points are projective.

RESULT : Thus we have got the result that Arc-Ratio is a projective property or we can say Arc-Ratio remains constant under projection.

FINAL CONCLUSION : Hope , the new theorems and some new approaches in section-I will grow interests in junior researchers and scientists for thinking in a different way.

I think , the new topic **"Arc-Ratio "** in section-II will help the scientists to determine the relative positions of celestial bodies which are revolving in circular orbits . Definitely this is a matter of further research .

References **……………**

[1] *Foundations of Projective Geometry .   By Robin Hartshorne . (1967) .*

[2] *Projective Geometry: From Foundations to Applications. By Albrecht Beutelspacher and Utc Rosenbaum. Cambridge University Press. ISBN:0521482771.*

[3] *Introduction To Projective Geometry .By C .R . WYLIE , JR .ISBN:048646895X .*

[4] *Projective Geometry .By H .S M. Coxeter.(2nd edition) ISBN:0387965327.*

[5] *Projective Geometry. By T .EWAN FAULKNER. ISBN:048645326X .*

[6] *Projective Geometry . An introduction. By Louis R.A.Casse , Rey Casse. Oxford University Press ,ISBN : 9780199298860.*