Primary decomposition of graded secondary modules graded over finitely generated abelian groups

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Abstract
Let R be a commutative ring which is graded by a finitely generated Abelian group $\mathcal{G}$. Here, we initiate a primary decomposition of $\mathcal{G}$-graded secondary R-modules. And also, show the annihilator of $\mathcal{G}$ -graded secondary module by graded prime ideals.
Keywords: Graded rings, and modules, rings and modules of finite generation.
2000 Mathematics Subject Classification:16W50, 13E15

Introduction

The graded prime ideals are the prime concern of study in graded commutative ring theory. To acquire a outright delineation of all structure of graded ideals in a graded commutative ring may be impossible summation, i.e. One limit recognition to the structure of graded prime ideals which can be frequently made finer advancement, however receiving fine details on the same whole structure of ring. For instance if the size and structure of graded commutative ring can be computed by creating sequence of graded prime ideals and determining the prolonged chain. This is also known as Krull dimension of the graded ring which is essential in graded commutative ring.The number of theories like unique factorization, technique of localization and algebraic geometry are closely contained the concept of graded prime ideals. These subjects guide to provide the supplementary application in topics of graduate algebra and modern research. The number of authors had done a lot of work on the theory of secondary module (see [1], [3], [7], and [9]). In this article we are doing study on representation of grade secondary modules over a $\mathcal{G}$-grade ring and graded primary submodule of grade secondary module. The prime objective of this paper to study the primary decomposition over graded secondary modules and show the representation of graded secondary module is also graded. This paper is comprised among three section. In first section, we state the basic definition and terminology. Section two of research paper show some inferences which are widely used to conclude the results. Section three contained the results which establish the relations between the primary decomposition over the graded modules and graded secondary modules. Through out the paper R is $\mathcal{G}$ graded commutative ring, $\mathcal{A}$ is finitely generated $\mathcal{G}$ graded module over R. "f.g." means finitely generated and $\mathfrak{M}$ denotes annihilator of $\mathcal{A}$.

Definition 1.1 *[4] A$\mathcal{G}$-graded R-module $\mathcal{A}$ is called to be $\mathcal{G}$-secondary if $\mathcal{A} \neq 0$ and if, for each $r \in h(R)$ the endomorphism $\lambda_{r,\mathcal{A}}$ is either surjective or nilpotent.*

Definition 1.2 *Suppose $\mathcal{A}$ be a f.g $\mathcal{G}$- graded secondary R-module.*

- An ideal$Ass^{\mathcal{G}}\mathbb{U} \subset R$ iff $\mathbb{U}$ is $\mathcal{G}$-prime and $\mathbb{U} = Ann(t)$ for a few component $t \in \mathcal{A}$.
- $Ass^{\mathcal{G}}\mathcal{A}$represent the set of all $\mathcal{G}$-Associated ideals of $\mathcal{A}$.
- $\mathcal{A}$is$\mathcal{G}$-secondary coprimary if $Ass^{\mathcal{G}}\mathcal{A} = \{q\}$ for a few $\mathcal{G}$-Prime ideal q.
- A $\mathcal{G}$-graded secondary submodule $\mathcal{B}$ of $\mathcal{A}$ is called a $\mathcal{G}$-primary if the quotient module $\mathcal{A}/\mathcal{B}$ is $\mathcal{G}$-secondary coprimary.
- Assume that $\mathcal{B}$ is a $\mathcal{G}$-graded secondary submodule of $\mathcal{A}$, then there is articulated $\mathcal{B} = \cap_{i \in \mathbb{U}} \mathbb{V}_i$ a $\mathcal{G}$-primary decomposition of $\mathcal{B}$ in $\mathcal{A}$ if and only if the $\mathbb{V}_i$ are $\mathbb{G}$-Primary secondary submodules of $\mathcal{A}$ with $Ass^{\mathcal{G}}\mathcal{A}/\mathbb{V}_i = \{q_i\}$ and q_i are $\mathcal{G}$-associated to $\mathcal{A}/\mathcal{B}$.
- A $\mathcal{G}$-primary decomposition $\mathcal{B} = \cap_{i \in \mathbb{U}} \mathbb{V}_i$ of $\mathcal{B}$ in $\mathcal{A}$ is said to be diminished if all the q_i are well defined and there is no existence of $i \in \mathbb{U}$ such that $\cap_{j \neq i} \mathbb{V}_j \subset \mathbb{V}_i$.

Section-2

Assume $\mathcal{G}$ is f. g. Abelian group and R a $\mathcal{G}$-graded ring and $\mathcal{A}$a $\mathcal{G}$-graded secondary R-modules are considered to be f.g., throughout the paper. Generally, for $\mathcal{G}$-prime ideals, it is noted that, analogously there is the existence of usual prime ideals and greatest $\mathcal{G}$- prime ideals as a result of Zorn's lemma. There are following facts:

Proposition 1.3 *For every graded prime ideal q of a ring R and every graded secondary sub module $\mathcal{A} \neq 0$of R/q, $Ass(\mathcal{A}) = \{q\}$.*

Proof. Let R be a commutative graded ring and if it is quotient by graded prime ideal q then R/q become integral domain, the annihilator of an element of $R/q \neq 0$is q.

Proposition 1.4 *Suppose $\mathcal{A}$ is a graded secondary module over a graded ring R. Each greatest element of the set of ideals $Ann(x)$of R, where x runs through the set of elements $\neq 0$ of $\mathcal{A}$belongs to $Ass(\mathcal{A})$.*

Proof. Assume that $a' = Annx(x \in \mathcal{A}, x \neq 0)$ is a greatest element which is sufficient to show that a' is prime. As $x \neq 0, a' \neq R$. Let e', f' be two elements of R such that $e'f' \in a' \Rightarrow (e'f')^n x = 0$ for some $n > 0$. If $f \notin a \Rightarrow \lambda_{f\prime,x}$ is surjective, that is $f'^n x = x$. Then $e'^n x = e'^n(f'^n x) = (e'f')^n x = 0 \Rightarrow a$ is nilpotent.then $(x \neq 0, e \in Ann(f'x))$ and $a' \subset Ann(f'x)$. As a' is greatest element, $Ann(f'x) = a'$hence $e' \in a'$, so that a' is prime.

Proposition 1.5 *Assume $\mathcal{A}$ is graded secondary R-module and $\mathcal{B}$ a graded secondary submodule of $\mathcal{A}$. Then:*

$$Ass(\mathcal{B}) \subset Ass(\mathcal{A}) \subset Ass(\mathcal{B}) \cup Ass(\mathcal{A}/\mathcal{B})$$

.

Proof. The inclusion $Ass(\mathcal{B}) \subset Ass(\mathcal{A})$ is obvious. Let $q \in Ass(\mathcal{A})$, $\mathcal{E}$ graded secondary submodule of $\mathcal{A}$ is isomorphic to R/q, and $\mathbb{F} = \mathbb{E} \cap \mathbb{B}$. If $\mathbb{F} = 0$, $\mathbb{E}$ is isomorphic to a graded secondary submodule of $\mathcal{A}/\mathcal{B}$, whence $q \in Ass(\mathcal{A}/\mathcal{B})$. If $\mathbb{F} \neq 0$, $q = Ann(\mathbb{F})$ where each element $\neq 0$ (by preposition 1.3) $q \in Ass(\mathbb{F}) \subset Ass(\mathcal{B})$.

Corollary 1.6 *If a graded secondary R-module $\mathcal{A}$ is the direct sum of the family $(\mathcal{A})_{i\in\mathbb{U}}$ of submodule then $Ass^{\mathcal{G}}\mathcal{A} = \cup_{i\in\mathbb{U}} Ass\mathcal{A}_i$.*

Corollary 1.7 *If $\mathcal{A}$ is a graded secondary R-module and $(\mathbb{V}_i)_{i\in\mathbb{U}}$ a defined family of a graded secondary sub module of $\mathcal{A}$. If $\cap_{i\in\mathbb{U}} \mathbb{V}_i = 0$, then $Ass(\mathcal{A}) \subset \cup_{i\in\mathbb{U}} Ass(\mathcal{A}/\mathcal{V})$.*

Proposition 1.8 *Assume that $\mathcal{A}$ is a $\mathcal{G}$-graded secondary module, then*

(1) $Ann(l)$will be a homogeneous ideal If $l \in \mathcal{A}$ is homogeneous
(2) $\mathfrak{M}$is a homogeneous ideal.

Proof. (1) Suppose that $a' = \sum_{g\in\mathcal{G}} a'_g \in Ann(l)$and $a'.l = 0$. As l is homogeneous, hence the degrees of each term $a'_g.l$ is pairwise well defined., and so $a'_g.l$ should be disappear for all $g \in \mathcal{G}$ and $a'_g \in Ann(l)$ for each $g \in \mathcal{G}$.
(2) Similarly, for each element $a' \in \mathfrak{M}$ we can say $\mathfrak{M}$ is a homogeneous ideal. .

Proposition 1.9 *Suppose $\mathcal{A}$ is a $\mathcal{G}$-graded secondary module, then*

(1) Each $Ass^{\mathcal{G}}\mathcal{A}$ is the annihilator of $\{h(l) | l \in \mathcal{A}\}$
(2) The set of all homogeneous greatest annihilator ideals belong to $Ass^{\mathcal{G}}\mathcal{A}$;
(3) $Ass^{\mathcal{G}}\mathcal{A} = \phi$ iff $\mathcal{A} = 0$:
(4) $h(\mathcal{A}) = \bigcup_{(q\in Ass^{\mathcal{G}}\mathcal{A})} q^{\mathcal{G}}$, where $q^{\mathcal{G}}$ represent the set of homogeneous elements in q that contains exactly 0 and all homogeneous proper zero divisors of $\mathcal{A}$.

Proof. (1) Suppose q is a $\mathcal{G}$-prime ideal linked along $\mathcal{A}$. Then $q = Ann(l)$ for a few $l \in \mathcal{A}$. Take l as a defined sum $\sum_{g\in\mathcal{G}} l_g$, where $l_g \in \mathcal{A}_g$ for each $g \in \mathcal{G}$. Then for each homogeneous component $b' \in q$, there is $b'_l = \sum_{g\in\mathcal{G}} b'l_g = 0$ for every $g \in \mathcal{G}$. Therefore $b'l_g$ has pairwise different degree and this should be disappear independently, i.e$b'l_g = 0$ for each $g \in \mathcal{G}$. For $a', b' \in h(R)$, let $a'b' \in \mathfrak{M} \Rightarrow (a'b')^n\mathcal{A} = 0$ for some $n > 0$. If $b' \notin \mathfrak{M} \Rightarrow \lambda_{b',\mathcal{A}}$ is surjective, that is $b'^n\mathcal{A} = \mathcal{A}$. Then $a'^n\mathcal{A} = a'^n(b'^n\mathcal{A}) = (a'b')^n\mathcal{A} = 0 \Rightarrow a$ is nilpotent. Thus $a' \in q \subset Ann(l_g)$ for every $g \in \mathcal{G}$ and $a' \in \mathfrak{M}$); besides $\bigcap_{g\in\mathcal{G}} Ann(l_g) \subset q$, and as the $Ann(l_g)$ is homogeneous and q is $\mathcal{G}$-prime ideal, here is the existence of a few $h \in \mathcal{G}$ s.t. $Ann(l_g) \subset q$, consequently $Ann(l_h) = q$.

(2) Suppose $\mathbb{U} = Ann(l)$ is a greatest annihilator ideal, where $0 \neq l$ and $a', b' \in R$ homogeneous s.t. $a'b' \in \mathcal{U} \Rightarrow (a'b')^n\mathcal{A} = 0$, $n > 0$. If $b' \notin \mathbb{U} \Rightarrow \lambda_{b',\mathcal{A}}$ is surjective, that is $b'^n\mathcal{A}$=$\mathcal{A}$. Then $a'^n\mathcal{A} = a'^n(b'^n\mathcal{A}) = (a'b')^n\mathcal{A} = 0 \Rightarrow a'$ is nilpotent. Thus, $a' \in$

$Ann(b'_l)$moreover $\mathbb{U} \subset Ann(b'_l)$. Because $\mathbb{U}$ is greatest and $b'_l \neq 0$, it come after that $\mathbb{U} = Ann(b'_l)$ and $a' \in \mathbb{U}$. Whence $\mathbb{U}$ is $\mathcal{G}$-prime.

(3) If $\mathcal{A} = 0$, understandably $Ass^{\mathcal{G}}\mathcal{A} = \phi$. Otherwise, if $\mathcal{A} \neq 0$, by (2) there stands a greatest ideal of the form $Ann(l)$ for a few $l \in \mathcal{A}$, that relates to $Ass^{\mathcal{G}}\mathcal{A}$.

(4) It is similarly as mentioned above part(2).

Proposition 1.10 *Assume a decomposition $\mathcal{G} = \mathcal{P} \times \mathcal{Q}$, where $\mathcal{P}, \mathcal{Q}$ are free and finite, and the projection of $\mathcal{G}$ on to the corresponding factors induces the $\mathcal{P}$ and $\mathcal{Q}$ grading of $\mathcal{A}$. Then:*

(1) Each one $\mathcal{G}$-prime ideal is $\mathcal{Q}$-prime,
(2) $Ass^{\mathcal{P}}\mathcal{A} = Ass\mathcal{A}$and $Ass^{\mathcal{G}}\mathcal{A} = Ass^{\mathcal{Q}}\mathcal{A}$.

Proof. (1) The proof is stated forward.

(2) The first averment follows from the second, where $mathcalQ$ is trivial by(1) it is sufficient to show that $Ass^{\mathcal{Q}}\mathcal{A} \subset Ass^{\mathcal{G}}\mathcal{A}$. Let $q \in Ass^{\mathcal{Q}}\mathcal{A}, l \in \mathcal{A}$ homogeneous element of degree t s.t. $q = Ann(l)$, $x \in q$ homogeneous of $\mathcal{Q}$ degree s. It can be written $x = \sum_{f \in \mathcal{P}} x_{f,s}$and $L = \sum_{g \in \mathcal{P}} l_{g,t}$. Here, we can take linear map $\pi: \mathcal{P} \to \mathbb{Z}$ concerning the non zero degrees of $x$and l. Consider q is a prime ideal associated with $\mathcal{A}$and $Ann(x) \in \mathcal{A}$. And $(x_i)_{i \in \Delta}$ is the family of homogeneous element of x; let $i(1) < i(2) \ldots < i(r)$. Consider $a \in q$ by the induction on the number of indices i such that $a_i \neq 0$. We prove that $a \in q$ the induction hypothesis applied to $a - a_i$ will give the conclusion. Now using the fact that the homogeneous element of degree $s + t$ of $ax = 0$ we obtain $\sum_{i \in \Delta} a_{s-1}x_{t+1} = 0$ then we conclude that $a_s x_t$ is a linear map combination of x_i of indices $i > t$. In particular, $a_s x_i(r) = 0$ whence descending induction on $n < r a_s^{r-n+1} x_(n) = 0$and $a_s^r x = 0$. Whence $a_s^r \in q$ and, as q is prime $a \in q$. And $x^s m = 0$ hence $x^s \in Ann(m) = q$ which prove that $Ass^{\mathcal{G}} \subset q$ as $\mathcal{A} \neq \phi$and $Ass^{\mathcal{G}}\mathcal{A} = q$.

Proposition 1.11 *Assume $\mathcal{A}$ is a $\mathcal{G}$- graded secondary module.*
(1) Assume $\mathcal{B} \subset \mathcal{A}$. Then $Ass^{\mathcal{G}}\mathcal{A} \subset Ass^{\mathcal{G}}\mathcal{B} \cup Ass^{\mathcal{G}}\mathcal{A}/\mathcal{B}$
(2) Assume $\{\mathbb{V}_i\}_{i \in \mathbb{U}}$ is defined family of $\mathcal{G}$-submodules of $\mathcal{A}$s.t $\cap_{i \in \mathbb{U}} \mathbb{V}_i = 0$. Then $Ass^{\mathcal{G}}\mathcal{A} \subset \cup_{i \in \mathbb{U}} Ass^{\mathcal{G}}\mathcal{A}/\mathbb{V}_i$.
(3) Let $\mathbb{U} \subset Ass^{\mathcal{G}}\mathcal{A}$. Then there exist a submodule $\mathcal{B}$ of $\mathcal{A}$ such that $Ass^{\mathcal{G}}\mathcal{B} = Ass^{\mathcal{G}}\mathcal{A}/\mathbb{U}$and $Ass^{\mathcal{G}}\mathcal{A}/\mathcal{B} = \mathbb{U}$.

Proof. Suppose that $\{\mathcal{A}_j\}_j \in \mathbb{K}$ is a of graded secondary submodules of $\mathcal{A}$, then $\cup_{j \in \mathbb{K}} \mathcal{A}_j$ is a graded secondary module too. If it characterize along $\mathcal{A}$, then $\mathcal{A}$ is graded secondary itself and $Ass^{\mathcal{G}}\mathcal{A} = \cup_{j \in \mathbb{K}} Ass^{\mathcal{G}}\mathcal{A}_j$.
(1) Now, it is clear that $Ass^{\mathcal{G}}\mathcal{B} \subset Ass^{\mathcal{G}}\mathcal{A}$. Let $q \in Ass^{\mathcal{G}}\mathcal{A}$ there is the existence of a

submodule $\mathcal{Q}$ of $\mathcal{A}$ isomorphic to R/q. Here, we take $\mathcal{P} = \mathcal{Q} \cap \mathcal{B}$ which is secondary graded. $Ass^{\mathcal{G}}\mathcal{P} \subset Ass^{\mathcal{G}}\mathcal{B}$. If $\mathbb{P} = 0$, then $Ass^{\mathbb{G}}\mathbb{P} = \phi$ by proposition 1.9(3). But then $\mathcal{Q} \cong \mathcal{A}\ \ \mathcal{B}$ and thus $q \in Ass^{\mathcal{G}}\mathcal{A}/\mathcal{B}$. If $\mathcal{P} \neq 0$, then $Ass^{\mathcal{G}}\mathcal{P} = q$ (see prop 1.3 and 1.5).
(2) We do the same in direct sum $\bigoplus_{i\in\mathbb{U}} \mathcal{P}/\mathbb{V}_i$ (see corollaries 1.6 and 1.7 to prop 1.5).
(3) Let $\mathbb{E}$ be the set of graded secondary submodule $\mathbb{P}$ of $\mathcal{A}$ such that $Ass^{\mathcal{G}}\mathbb{P} \subset Ass^{\mathcal{G}}\mathcal{A}/\mathbb{U}$ by using $Ass^{\mathcal{G}}\mathcal{A} = \cup_{j\in\mathbb{K}} Ass^{\mathcal{G}}\mathcal{A}_j$. The set is nonempty and $\{0\} \in \mathbb{E}$and $\mathbb{E} \neq \mathbb{U}$. Let $\mathcal{B}$ be a greatest element of $\mathbb{E}$. Then $Ass^{\mathcal{G}}\mathcal{B} \subset Ass^{\mathcal{G}}\mathcal{A}/\mathbb{U}$. We shall see that $Ass^{\mathbb{G}}\mathbb{A}\ \ \mathbb{B} \subset \mathbb{U}$. Let $q \in Ass^{\mathcal{G}}\mathcal{A}/\mathcal{B}$, then $\mathcal{A}/\mathcal{B}$ contain a submodule $\mathcal{P}/\mathcal{B} \cong R/q$ by (1), $Ass^{\mathcal{G}}\mathcal{P} \subset Ass^{\mathcal{G}}\mathcal{B} \cup \{q\}$. But $\mathcal{Q} \notin \mathbb{E}$ as $\mathcal{B}$ is greatest element. Thus $q \in \mathbb{U}$ (see chap.- 4,prop.-4 in [2]).

Proposition 1.12 *Suppose that $\mathcal{A}$ is a $\mathcal{G}$- graded secondary module and $\mathbb{H} \subset R$ is a multiplicatively closed subset s.t.* $\mathbb{H}' = \cup_{g\in\mathcal{G}} \mathbb{H} \cap R_g$.
(1) Then $Ass^{\mathcal{G}}_{\mathbb{H}^{-1}R}\mathbb{H}^{-1}\mathcal{A} = \{\mathbb{H}^{-1}q | q \in Ass^{\mathcal{G}}_R\}$and $q \cap \mathbb{H} = \phi$.
(2) Assume the set of the elements $\mathbb{U} \subset Ass^{\mathcal{G}}\mathcal{A}$ do not converge to $\mathbb{H}$. Then the kernel $\mathcal{B}$ of the canonical mapping $\mathcal{A} \to \mathbb{H}^{-1}\mathcal{A}$ is the unique submodule of $\mathcal{A}$ such that $Ass^{\mathbb{G}}\mathcal{B} = Ass^{\mathcal{G}}\mathcal{A}/\mathbb{U}$and $Ass^{\mathcal{G}}\mathcal{A}/\mathcal{B} = \mathbb{U}$.

Proof. Compare [1,ch 4.article 1,No. 4,proposition 5 and 6 in [2]]

Proposition 1.13 *Suppose that $\mathcal{A}$ is $\mathcal{G}$-graded secondary module*
(1) There exist a formation sequence $\mathcal{A}_k \subset \mathcal{A}_{k-1} \subset \ldots\ldots \subset \mathcal{A}_0 = \mathcal{A}$
s.t. the $\mathcal{A}_i$ are $\mathcal{G}$-graded and for $0 < i < k-1$, $\mathcal{A}_i/\mathcal{A}_{i+1} \cong R/q_i$, here q_i is a $\mathcal{G}$-prime ideal of R.
(2) A formation of sequence as earlier, $Ass^{\mathcal{G}}\mathcal{A} \subset \{q_0, \ldots, q_{k-1}\}$.
(3) $Ass^{\mathcal{G}}\mathcal{A}$is finite.

Proof. To get the proof of this theorem we need to follow the concept of [$\oint$ 1,Chap.$-$ 4, article 1 No$-$ 2, Thms. -1 and 2 ,and Coro.-1in [2]].

Lemma 1.14 *Suppose that $\mathcal{A}$ is a $\mathcal{G}$ coprimary secondary module with $Ass^{\mathcal{G}}\mathcal{A} = \{q_1\}$. Then there is the existence of a few $q \in Ass\mathcal{A}$s.t $q_1 = q'$.*

Proof. Assume q is any prime ideal and associated with $\mathcal{A}$. If there exists another prime ideal q then by the proposition (1.9(4)) the set of homogeneous element of $q^{\mathcal{G}} = q'^{\mathcal{G}}$ is contained in ${q_1}^{\mathcal{G}}$, hence $q' \subset q_1$. Otherwise, the zero divisors of $\mathcal{A}$ is contained in $q_1$and $q_1 \subset \bigcup_{q\in Ass\mathcal{A}} q$. So there exists some $q \in Ass\mathcal{A}$ such that $q_1 \subset q$. Now, we can say $q_1 \subset q' \subset q$and the result follows.

Proposition 1.15 *Assume $\mathcal{A}$ is a f.g. $\mathcal{G}$-graded secondary module, Then* $Ass^{\mathcal{G}}\mathcal{A} \subset \{q' | q \in Ass\mathcal{A}\}$.

Proof. The concept of induction is used on the cardinality of $Ass^{\mathcal{G}}\mathcal{A}$. If $\mathcal{A}$ is secondary coprimary, it follows the disclosure of lemma 1.14. Now we take $q \in Ass^{\mathcal{G}}\mathcal{A}$ such that $Ass^{\mathcal{G}}\mathcal{A}/\{q_1\}$ is non void. Consider $\mathbb{H}$ is the set of the homogeneous element of R and q does not contain it. The kernel $\mathcal{B}$ of the natural mapping $\mathcal{A} \to \mathbb{H}^{-1}\mathcal{A}$ is $\mathcal{G}$-graded secondary. We know that $Ass^{\mathcal{G}}\mathcal{B} = Ass^{\mathcal{G}}\mathcal{A}/\{q\}$and $Ass^{\mathcal{G}}\mathcal{A}\ \ \mathcal{B} = \{q_1\}$, by proposition 1.12-(2). Further more, by [$\oint$ 1, No.2, proposition 6 in [2]] we have $Ass\mathcal{A}/\mathcal{B} \subset Ass\mathcal{A}$and $Ass\mathcal{A}/\mathcal{B} = \{q \in Ass\mathcal{A} | q \cap \mathbb{H} = \phi\} = \{q \in Ass\mathcal{A} | q' \cap \mathbb{H} = \phi\}$. By the lemma 1.14 $q_1 = q'$ for some $q \in Ass\mathcal{A}/\mathcal{B} \subset Ass\mathcal{A}$.

Section-3
Here, $\mathbb{R}$ is considered G-graded commutative Noetherian ring.

Proposition 1.16 *Suppose $\mathcal{A}$ is a f.g. $\mathcal{G}$-graded module over $\mathbb{R}$ such that $Ass^{\mathcal{G}}\mathcal{A} = q$, where q is $\mathcal{G}$ prime. And q^l annihilates $\mathcal{A}$where $l > 0$.*

Proof*.* Suppose p is a smallest graded $\mathcal{G}$ -prime ideal over $\mathfrak{M}$, where $\mathcal{A}$ is $\mathcal{G}$-graded secondary module over $\mathbb{R}$. The component in $\mathbb{R}/p$ is determined homogeneous by (prop.-3.9,[6])and q is considered without loss of generality a greatest graded $\mathcal{G}$- prime ideal of $\mathbb{R}$. If p is containing $\mathfrak{M}$, then it is associated with $\mathcal{A}$ and correspond along q which is smallest over $\mathfrak{M}$. At the moment this is stated that for any $y \in q$ and for a few $l > 0$ s.t. $y^l \in \mathfrak{M}$. Let there is a few $x \in q$ s.t. $x^l \notin \mathfrak{M}, \forall l > 0$. Now we contemplate few ideal $p \subset q$ that is greatest with respect to the characteristic of $\mathfrak{M}$ and not contained in power of x. Suppose $x_1, x_2 \in \mathbb{R}/p$ are homogeneous. we know that p is greatest then there is existence of $j > 0$ s.t. $x^j \in (p+< x_1 >) \cap (p+< x_2 >)$ such that $x^j = a + kx - 1 = b + nx_2$ for any $a, b \in p$ and$k, n \in \mathbb{R}$ this comes after that $x^2 j \in p$ that is discrepancy. If $x_1 x_2 \in p$ and for few component $x \in q$. Whence $\mathbb{R}$ is graded Noetherian $\exists l > 0$s.t $q^l \subset \mathfrak{M}$.

Theorem 1.17 *Assume $\mathcal{A}$ is G-graded secondary module over $\mathbb{R}$ and $\mathcal{B} \subset \mathcal{A}$ and $\mathcal{B} = \bigcap_{i \in \mathbb{U}} \mathbb{Y}_i$ a Primary decomposition of $\mathcal{B}$in $\mathcal{A}$. Then:*
(1) Suppose $\mathbb{Y}'_i$ is the biggest $\mathcal{G}$-graded submodule of $\mathcal{A}$ consist of $\mathbb{Y}'_i$
(2) There stands a$\mathbb{K} \subset \mathbb{U}$ s.t. $\mathcal{B} = \bigcap_{i \in j} \mathbb{Y}'_i$ is decreased $\mathcal{G}$-Primary decomposition.
(3) If a few $\mathbb{Y}_i$ match to a $\mathcal{G}$-prime ideal q_i that is a smallest element of $Ass^{\mathcal{G}}\mathcal{A}/\mathcal{B}$, then $\mathbb{Y}_i$ is $\mathcal{G}$-graded.

Proof. Suppose that $\mathcal{B} = \bigcap_{i \in \mathbb{U}} \mathbb{Y}_i$ is the Primary decomposition of $\mathcal{B}$in $\mathcal{A}$.
(1) The $\mathbb{Y}_i$ is $\mathcal{G}$-graded primary with respect to $\mathcal{A}$ and furthermore, $\mathcal{B} \subset \mathbb{Y}'_i \subset \mathbb{Y}_i$, so that $\mathcal{B} = \bigcap_{i \in \mathbb{U}} \mathbb{Y}'_i$ is a decomposition of $\mathcal{B}$ by $\mathcal{G}$-Primary graded secondary modules By proposition 3.3 from [6]
(2) We have $Ass^{\mathcal{G}}\mathcal{A}\ \ \mathcal{B} \subset \{q' | q \in Ass\mathcal{A}\ \ \mathcal{B}\}$ through the proposition 1.15. Here $q_i \in Ass^{\mathcal{G}}\mathcal{A}/\mathcal{B}$ and the $\mathcal{G}$-prime ideal matching a part of $i \in \mathbb{U}$. Then we primed $\mathcal{B} = \bigcap_{q_i \in Ass^{\mathcal{G}}\mathcal{A}\ \mathcal{B}} \mathbb{Y}_i/\mathcal{B}$. It follows that $Ass^{\mathcal{G}}\mathcal{B} = \phi$ and thus $\mathcal{B} = 0$. Whence $\bigcap_{q_i \in Ass^{\mathcal{G}}\mathcal{A}\ \mathcal{B}} \mathbb{Y}_i$ is a $\mathcal{G}$-Primary decomposition of $\mathcal{B}$ in$\mathcal{A}$. By the proposition 1.11(2) it pursue that no one of the q_i can be excepted and so the $\mathcal{G}$-Primary decomposition is curtailed.

(3) It pursue that the biggest $\mathcal{G}$-graded secondary sub module $\mathbb{Y}'_i$ of $\mathbb{Y}_i$ is q_i primary, as $q'_i = q_i$, and so $\mathbb{Y}_i = \mathbb{Y}'_i$, as q_i is a smallest element of $Ass^{\mathcal{G}}\mathcal{A}/\mathcal{B}$ by the concept of proposition 3.3 from [6].

References

(1) Atani E. (2002). On secondary modules over pullback rings. *Communication in Algebra*, 30(6), 2675-2685,
(2) Bourbaki N.(2011). Commutative Algebra.*Springer-verlag*, (1989), *Communications in Algebra*, 39.
(3) Macdonald I. G.(1973). Secondary representation of modules over a commutative ring.*Symposia Mathematica, 11*, 23-43.
(4)Mishra R. K., Pratibha, Kumar S. D. and Srinivas B. Group graded secondary representation of the graded modules. *Appearing*.
(5) Nastasescu C., Oystaeyen V. F. (2004). Methods of graded rings, *Lecture Notes in Mathematics, Springer-Verlag*.
(6) Perling M. and Kumar S. D. (2007). Primary decomposition over rings graded by finitely generated abelian groups. *Journal of Algebra*, 318, 553–561.
(7) Pratibha, Mishra R. K. and Mohan R. (2017). A report on graded rings and graded modules, *Global Journal of Pure and Applied Mathematics*. 13(9) , 6827–6853.
(8) Refai M., Al-Zoubi K. (2004). On Graded Primary Ideals.*Turkish Journal of Mathematics*, 28, 217-229.
(9) Sharp Y. R.(1986). Asymptotic behavior of certain sets of attached prime ideals. *Journal of the London Mathematical Society,* 34 , 212-218.