A Study on Zermelo Navigation Problem

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Abstract
In this paper, mathematical formulation of navigation problem using Riemannian &Finsler geometry is discussed.Special situations with complete mathematical solutions of the problemare mentioned, namelyFinsler space with constant flag curvature and other. Modeling of the aircraft motion is introduced.
Keywords: Finsler Space, Riemannian Geometry, Geodesics

Introduction

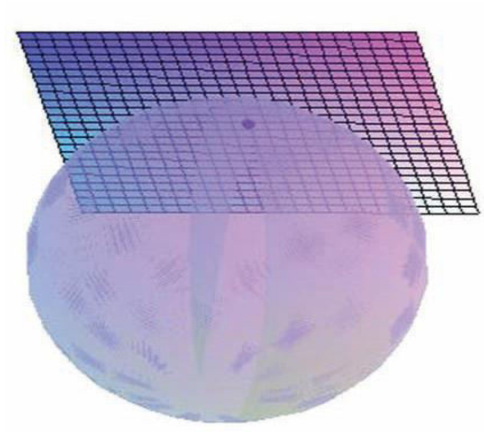
It is well known that the straight line is the shortest path in the Euclidean space $E_n$with the standard norm, which induces the distance function. On the sphere Sn, again with the natural distancefunction induced by the norm defined on tangent vectors andinherited from the standard norm on the ambient space E_{n+1},locally, the shortest path is a part of the main circle. On a generalRiemannian manifold, this is the property of geodesic curves.However, in a more general context, the norm does not haveto come from the Euclidean scalar product. In the more general setting, a Minkowski norm on a vector space is not symmetric,hence the length of a vector may be different than the length ofthe opposite vector. Consequently, in a Finsler space, the length of a curve travelled in one direction may be different than thelength of the same trajectory travelled in the opposite direction.However, geodesics are still well defined as shortest curves connecting their points that are close enough.Imagine a ship sailing in a sea without any current andwithout any wind. The situation can be well described by thetools of Riemannian geometry, where shortest curves (fastesttrajectories) are geodesics. However, if a wind or a current is present, then obviously the time required to travel some distancein one direction is different than the time necessary to travelthe same distance in the opposite direction. There are various geometrical techniques how to describe this situation. One ofthem is the mentioned Finsler geometry, where the wind or thecurrent modify the Riemannian metric to a general Finsler metric.

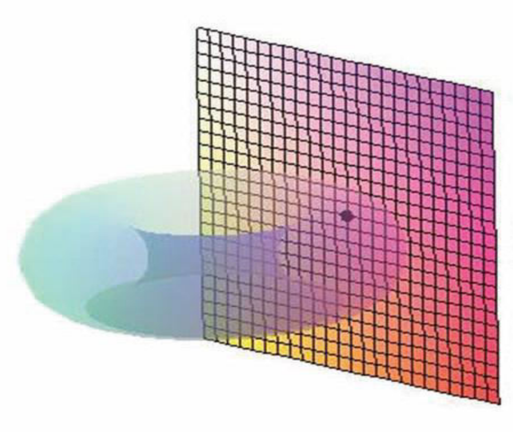
We shall introduce this description and some interesting related phenomenon.

ELEMENTARY RIEMANNIAN AND FINSLERGEOMETRY

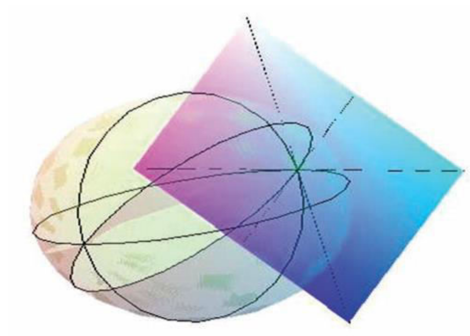
Let us recall that the Euclidean vector space is a vector space witha symmetric and positively definite bilinear form . This bilinearform induces the norm (length) of vectors by the formula

$$|v| = \sqrt{g(v,v)}$$





length of the curve is the time necessary for traveling along thecurve with the unit speed. Using the length of curves, we candefine the distance of two points x and y of M as the infimum of lengths of all curves connecting and y. For any two points x and$y$ in a Riemannian manifold, the distance from x to $y$is the same asthe distance from y to x.Geodesics in the Riemannian manifold (M, g) are curveswhich are extremals of the length functional and they are the best possible analogues of lines in the Euclidean space. Theimportant property of a geodesic γ is that for any two pointson γ which are close enough, γ is the shortest curve connecting these two points. Hence, the distance of these two points isrealized by this geodesic γ. Geodesics are obtained as solutionsof a system of differential equations. Another important property is that, given a point x in M and a tangent vector v inthe tangent space of M at$x$, there exist a unique local geodesic$\gamma(t)$ such that $\gamma(0) = x$ and $\gamma'(0) = v$ in other words, the position at the beginning is x and the velocity at the beginning is v. Ifthe Riemannian manifold (M, g) is complete, then for any x of Mthere is a neighbourhood$U$ of x such that any point y of U can be joined with x by the unique minimizing geodesic lying in U. Inthe Euclidean space with the standard norm, geodesics are thestraight lines. In the sphere with the natural Riemannian metric, geodesics are the main circles. The above mentioned propertiesof geodesics allow us to consider geodesics as the most naturaltrajectories in general Riemannian manifolds.



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NAVIGATION DATA AND RELATED RANDERSMETRIC

Let us discussed the real problem of the ship on the sea.Let us consider the surface of the sea to bethe two-dimensional space E^2 equipped with the standard norm. In the more general case, the sea canbe modelled by the two-dimensional sphere with the natural Riemannian metric. However, in full mathematical generalitythe sea can be any Riemannian manifold of arbitrary dimension.The possible tracks of the ship are curves on the manifold andit is natural to aim at the destination in the shortest possibletime. To measure the length of particular trajectories, or thetime necessary for travelling these particular trajectories,respectively, we use the Riemannian metric h on M and theintegral above. Naturally, the curves of interest are geodesicswith respect to this metric

In 1931, E. Zermelo studied the following problem ([4][5]):

Suppose that a ship sails on the open sea in calm waters and a mild breeze comes up. How must the ship be steered in order to reach a given destination in the shortest time?

The problem was solved by Zermelo himself for the Euclidean flat plane and by Z. Shen([9][10]) in the case when the sea is a Riemannian manifold (M, h)under the assumption that the wind Wis a time-independent mild breeze, i.e. $h(x, W) < 1$. Recently, Zermelo navigation problems on Finsler manifolds have been discussed and applied widely by many scholars (e.g. [4][8][6]).

Essentially, Zermelo navigation problem is tightly related to the geometry of indicatrix on a Finsler manifold. Let us consider a Finsler manifold (M, Φ). For each x ∈M, the indicatrix$S_\Phi$of at xis a closed hypersurface of $T_xM$around the origin xdefined by

$$S_\Phi = \{y \in T_xM | \emptyset(x, y) = 1\}$$

Let W=W(x)be a vector field on M. Consider the parallel shift $S_\emptyset$+{W}of $S_\emptyset$along W. It is easy to see that y∈$S_\emptyset$+{W}if and only if Φ(x, y−W_x) =1. Further, when Φ(x, $-W_x$) <1, $S_\emptyset$+{W}contains the origin xofT_xM. In this case, it is easy to see that, for any y∈T_xM\{0}, there is a unique positive number $F = F(x, y) > 0$such that

$$\frac{y}{F(x,y)} \in S_\emptyset+\{W\}$$

that is, F=F(x, y)satisfies the following

$$\emptyset\left(x, \frac{y}{F(x,y)} - W_x\right) = 1$$

It is not difficult to show that F=F(x, y)is a regular Finsler metric. On the other hand, when Φ(x, $-W_x$) =1, $S_\emptyset$+{W} must pass thorough the origin xof T_xM. In this case, for any y∈$A_{x:}$={y∈T_xM |gw(y,W)>0}, there is still a unique positive number $F = F(x, y)$such that $F = F(x, y)$satisfies (1.1), equivalently, $F = F(x, y)$satisfies (1.2). Obviously, such $F = F(x, y)$is a Finsler metric with singularity and not regular on whole TM. We call $F = F(x, y)$a conic Finsler metric (for the definition and fundamental properties of conic Finsler metrics, see [7]. Actually, the conic Finsler metric $F = F(x, y)$is regular on conic domain $A := \cup_{x \in M} A_x \subset TM$. Inevitably, it is natural to meet conic Finsler metrics in studying natural sciences.

In general, Finsler metric $F = F(x, y)$obtained from (1.2)is called a solution of the Zermelo navigation problem with navigation data (Φ, W)([8]).

Modeling of the aircraft motion

In order to define the place of an aircraft in space, the coordinates of the center of mass of the aircraft and the position of the connected coordinate system with respect to a stationary coordinate system (for example that one can be the coordinate system connected to the ground) have to be given. To describe aircraft dynamics can be defined six ordinary differential equations (ODEs). The first three of themare the so called Euler ODEs

$$\frac{d\gamma}{dt} = \omega_x - (\omega_y cos\gamma - \omega_z sin\gamma)tanv$$

$$\frac{dv}{dt} = \omega_y sin\gamma + \omega_z cos\gamma$$

$$\frac{d\psi}{dt} = (\omega_y cos\gamma - \omega_z sin\gamma)secv$$

whereγ denotes roll angle, v the pitch angle, ψ the yaw angle and ω_x,ω_y, $\omega_z$are the angular velocities.

The other three kinematic equations give a relation between the derivatives of the coordinates of the center of mass in normal ground coordinate system $OX_gY_gZ_g$with projection of the linear and angular velocities.

$$\frac{dx_g}{dt} = V_x cos\psi \cos v + V_z(sin\psi \sin\gamma - \sin v \cos\psi \cos\gamma) - V_y(sinvcos\psi \sin\gamma + \sin\psi \cos\gamma)$$

$$\frac{dy_g}{dt} = V_x cosvsin\psi - V_z(cos\psi \sin\gamma + \sin v \sin\psi \cos\gamma) - V_y(cos\psi \cos\gamma - \sin v \sin\psi \sin\gamma)$$

$$\frac{dz_g}{dt} = V_x sinv + V_z cosvcos\gamma + V_y cosv\, sin\gamma$$

Many of the components of the above stated equations are not a matter of interest of the present problem, since it concerns the airplane dynamics as a moving mass point and there is no need to be considered the characteristics of the body shape (γ = 0, the yaw angle- ψ=θ -heading angle, the pitch angle- ϑ=α -angle of attack). Then the system of ODEs transforms into:

$$\frac{dx_g}{dt} = Vcos\theta cos\alpha$$

$$\frac{dy_g}{dt} = V\ cos\alpha\ sin\theta$$

$$\frac{dz_g}{dt} = V\, sin\alpha$$

Passing through a plane inclined at $\alpha = \pi$:

$$\frac{dx_g}{dt} = -Vcos\theta$$

$$\frac{dy_g}{dt} = -V\, sin\theta$$

During the flight of an aircraft as a mass point, except the coordinates, a change of mass is observed, caused by fuel consumption. For a short range flight this change of mass is negligibly small and the assumption as w = const, $\frac{dw}{dt}$= 0 is a very suitable one. But for long range flights the change of mass is very significant and must be taken into consideration.

$\frac{dm}{dt} = \frac{-C_s}{g}$, where C_s-fuel flow rate and g- gravitational acceleration.

Therefore, assuming g = const, and putting f = $\frac{-C_s}{g}$, the expression can be simplified.

$$\frac{dw}{dt} = -f$$

In the process of model preparation one more assumption have to be analyzed, namely that the main part of the passenger aircraft flight is the so called cruise flight. Cruise flight is an airplane flight in a horizontal plane with cruising velocity. The characteristics of this velocity are:

• balance of the acting forces, which in horizontal plane is given by T h = D, where T h - thrust and D-drag

, • very slight dependence of fuel flow rate on the velocity of the airplane, and

• Minimum fuel burn per kilometer (maximum range).

As a result of that overview, a system of three ODEs is achieved, such that the optimal trajectory, should solve them under some additional conditions.

$$\dot{x} = -Vcos\theta$$

$$\dot{y} = -V\, sin\theta$$

$$\dot{w} = -f$$

Statement of the Optimal Control Problem

The purpose of the current work is to determine an aircraft trajectory, from an initial point to a final destination point and more precisely to determine a trajectory of the center of mass of an aircraft in a 2-D horizontal coordinate system. In here the approach of [2,3](Sridhar, Chen, Ng and Linke) is followed. It is required by the authors that to establish such kind of trajectory travelling time between the two points, taking into consideration the presence of wind, while avoiding regions of airspace, which are forbidden to be used must be minimized. These regions can be no-go-areas or parts of the airspace, which represent any kind of danger for the aircraft or facilitate persistant contrail formation. First the wind is introduced in the ODEs, characterizing the motion of a mass point in horizontal plane.

$$\dot{x} = -V cos\theta + u(x, y)$$

$$\dot{y} = -V\ sin\theta + v(x, y)$$

$$\dot{w} = -f$$

where$u(x, y)$ and $v(x, y)$ are the wind velocity components along the x- and y - axis, which can be functions of both x and y coordinates and also sometimes of the time t. The horizontal plane trajectory is optimized as finding the control variable - heading angle θ, for which cost functional attains minimum. The undesirable regions are formulated as soft state constraints. As a result a constrained optimization problem needs to be solved. To do that there are numerous approaches as: the Lagrange multiplier method, the augmented Lagrange multiplier method, penalty function method and so on. Considering the nature of the particular problem a penalty function method was chosen.

Conclusion

The Mathematical formulation of the Zermelo Navigation problem using Riemannian and Finsler geometry was introduced.Some special situations with the complete solutionof the problem were mentioned. The research in other special situations and other possible ways of the description of the problem were indicated. The potential for the further study is huge because the problem is too complex to be solved explicitly in full generality.

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