Reliability Estimation for (2+2) Cascade Model

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Article History: *Do not touch during review process(xxxx)*

Abstract: In this paper, the mathematical formula for the reliability of a special model from cascade models is found. Assuming that the factors of strength and stress follow the binary Weibull distribution, and estimating parameters scale of Weibull distribution by using three different estimation methods (ML, Rg and Pe) to estimate the reliability model and made simulations to compare results by using the mean square error to find out which estimation method is the best for estimating reliability model. As a result, we found that the ML estimator was the best in eight experiments and the Pe estimator was the best in two only experiments and finally the performance of ML and Pe were close.
Keywords: Weibull distribution, Cascade model, Parameter, Percentile estimation, Simulation.

1. Introduction

Obviously in systems, reliability whenever the complexity increases the reliability decreases unless compensative measures are taken. By increasing the reliability of associated units of the system we can increase the reliability system $R = (X \geq Y)$ (Ashok, Devi, & Maheswari, 2019; Chaturvedi & Malhotra, 2020), but sometimes this cannot be accomplished beyond certain limits. An alternative method to increase the system reliability in such case is to have a plus configuration of units in the system. Cascade is one such kind of standby system. Cascade system is a hierarchical standby redundancy (Uma Maheswari, 2013), where an order of units are arranged in the array of activation. The first unit is active withstands stress and remaining units are at standby. If the active unit fails, then the next unit in the order is activating. The stress acting on the next active unit will be "$\mathcal{K}$" times the stress of the prior failed units, where "$\mathcal{K}$" signifies stress attenuation factor (Mutkekar & Munoli, 2016).

Review important studies in this field:(Gogoi & Borah, 2012) considered two states to obtain the reliability cascade model. (Khan & Jan, 2014) assumed the reliabilities of the system have been obtained with the help of the particular forms of density functions of the n-standby system when all stress-strengths are random variables. (Karam & Husieen, 2017) discussed the reliability of n- cascade system when the stress and strength are Frechet distributed random variables. (Rahman, Mohyuddin, Anjum, & Butt, 2016; Vasanti & Venkata Rao, 2016) studied the estimation of reliability for the stress-strength cascade model by comparison between estimators made using data obtained through a simulation experiment. (Siju & Kumar, 2016) studied the strength of the component in the cycle depending on its strength in the (i − 1) the cycle. (Doloi & Gogoi, 2017) discussed two states to get the reliability of the n-cascade system, state one assumed one Lindley stress and one parameter exponential strength, state two assumed one-parameter exponential stress and Lindley strength. (Patowary, Hazarika, & Sriwastav, 2018) attempted to estimate the reliability of a cascade system when strength-stress follow either (gamma, exponential or normal) distributions by using Monte-Carlo Simulation. (Karam & Khaleel, 2019) discussed the reliability of (2+1) cascade model, when the random variables of stress-strength follow distributions are generalized inverse Rayleigh distributions. (Mirajkar, Vadgaon, & Kore, 2015) studied the system of reliability cascade. They assumed that all units are independent and follow exponential stress-strength distributions. (Khaleel & Karam, 2019) studied reliability of (2+1) Cascade model, when the random variables of stress-strength follow distributions are inverse Weibull distribution. (Jebur, Kalaf, & Salman, 2020) estimated the parameter and system reliability in the stress-strength model when the system contains several parallel components. (Hassan, Nagy, Muhammed, & Saad, 2020) studied the estimation problem of a stress-strength model incorporating the multi-component system. (Kanaparthi, Palakurthi, & Narayana, 2020) presented the estimation of stress strength model by considering the cascade stress strength model of New Rayleigh-Pareto.

This paper aims to find out the mathematical formula that expresses the (2 + 2) cascade reliability model when it follows the strength-stress random verbal's of the Weibull distribution and estimates this model by using three different estimation methods (ML, Rg and Pe) as well as comparing the results with the mean square error.

2.Mathematical Model

Assuming the strength random variable of the four units (two units basic $(\mho_1 \; and \; \mho_2)$ and two units $(\mho_3 \; and \; \mho_4)$ redundancy standby) to be $X_r \sim W(\sigma, \eta_r)$; $r = 1,2,3,4$ and to suppose that the strength random variable of the four units $Y_v \sim W(\sigma, \delta_v)$; $v = 1,2,3,4$ respectively, where X_r and Y_v are independently and identically distributedWeibull with common parameter shape σ and scale parameter η_r; $r = 1,2,3,4$ and scale parameter δ_v; $v = 1,2,3,4$.

The CDF and PDF of $W(\sigma, \eta_r)$ are:

$F(x) = 1 - e^{-\eta x^{\sigma}} x > 0; \sigma, \eta_r > 0$; $r = 1,2,3$ …(1)

$f(x) = \sigma \eta x^{\sigma-1} e^{-\eta x^{\sigma}} x > 0; \sigma, \eta_r > 0$; $r = 1,2,3,4$ …(2)

The CDF and PDF of $W(\sigma, \delta_v)$ are :

$G(y) = 1 - e^{-\delta y^{\sigma}} y > 0; \sigma, \; \delta_v > 0$; $v = 1,2,3,4$ …(3)

$\sigma \delta y^{\sigma-1} e^{-\delta y^{\sigma}} y > 0; \sigma, \; \delta_v > 0$; $v = 1,2,3,4$ …(4)

There are six cases for calculating reliability (2+2) cascade model. They can be derived as follows :

$$\mathcal{R} = p[X_1 \geq Y_1, X_2 \geq Y_2] + p[X_1 < Y_1, X_2 \geq Y_2, X_3 \geq Y_3] + p[X_1 < Y_1, X_2 \geq Y_2, X_3 < Y_3, X_4 \geq Y_4]$$
$$+ p[X_1 \geq Y_1, X_2 < Y_2, X_3 \geq Y_3] + p[X_1 \geq Y_1, X_2 < Y_2, X_3 < Y_3, X_4 \geq Y_4]$$
$$+ p[X_1 < Y_1, X_2 < Y_2, X_3 \geq Y_3, X_4 \geq Y_4]$$

$$\mathcal{R} = \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3 + \mathcal{S}_4 + \mathcal{S}_5 + \mathcal{S}_6 \qquad \ldots(5)$$

The first case of model work: the two units $(\mho_1 \; and \; \mho_2)$ are activated and the two units $(\mho_3 \; and \; \mho_4)$ act as a standby unit: (Sandhya, 2013)

$$\mathcal{S}_1 = pr[X_1 \geq Y_1, X_2 \geq Y_2]$$

$$\mathcal{S}_1 = \int_0^{\infty} [\overline{F}_{x_1}(y_1)] g(y_1) dy_1 \int_0^{\infty} [\overline{F}_{x_2}(y_2)] g(y_2) dy_2$$

$$\mathcal{S}_1 = \int_0^{\infty} [e^{-\eta_1 y_1^{\sigma}}] \sigma \delta_1 y_1^{\sigma-1} e^{-\delta_1 y_1^{\sigma}} dy_1 \, . \int_0^{\infty} [e^{-\eta_2 y_2^{\sigma}}] \sigma \delta_2 y_2^{\sigma-1} e^{-\delta_2 y_2^{\sigma}} dy_2$$

$$\mathcal{S}_1 = \int_0^{\infty} \sigma \delta_1 y_1^{\sigma-1} e^{-(\eta_1+\delta_1) y_1^{\sigma}} dy_1 \, . \int_0^{\infty} \sigma \delta_2 y_2^{\sigma-1} e^{-(\eta_2+\delta_2) y_2^{\sigma}} dy_2$$

$$\mathcal{S}_1 = \left[\frac{\delta_1}{\eta_1+\delta_1}\right]\left[\frac{\delta_2}{\eta_2+\delta_2}\right] \ldots(6)$$

The second case of the model work: when the unit $\mho_1$ fails and unit $\mho_2$ remains activated, unit standby $\mho_3$ is activated to replace the failed unit $\mho_1$ and unit $\mho_4$ remains in a standby state :

$$\mathcal{S}_2 = pr[X_1 < Y_1, X_2 \geq Y_2, X_3 \geq Y_3] = pr[X_1 < Y_1, \mathcal{M}X_1 \geq \mathcal{K}Y_1] pr[X_2 \geq Y_2]$$

Where "$\mathcal{M}$" and "$\mathcal{K}$" strength – stress attenuation factors: (M. Tirumala Devi, T. Sumathi Uma Maheswari, & N. Swathi, 2016; Sundar, 2012)

$X_3 = \mathcal{M}X_1$ and $Y_3 = \mathcal{K}Y_1$

$$pr[X_1 < Y_1, \mathcal{M}X_1 \geq \mathcal{K}Y_1] = \int_0^{\infty} [F_{x_1}(y_1)] \left[\overline{F}_{x_1}\left(\frac{\mathcal{K}}{\mathcal{M}} y_1\right)\right] g(y_1) dy_1$$

$$pr[X_1 < Y_1, \mathcal{M}X_1 \geq \mathcal{K}Y_1] = \int_0^{\infty} [1 - e^{-\eta_1 y_1^{\sigma}}] \left[e^{-\eta_1 (\frac{\mathcal{K}}{\mathcal{M}})^{\sigma} y_1^{\sigma}}\right] \sigma \delta_1 y_1^{\sigma-1} e^{-\delta_1 y_1^{\sigma}} dy_1$$

$$pr[X_1 < Y_1, \mathcal{M}X_1 \geq \mathcal{K}Y_1] = \int_0^{\infty} \left[e^{-\eta_1 (\frac{\mathcal{K}}{\mathcal{M}})^{\sigma} y_1^{\sigma}} - e^{-\eta_1 \left(1+(\frac{\mathcal{K}}{\mathcal{M}})^{\sigma}\right) y_1^{\sigma}}\right] \sigma \delta_1 y_1^{\sigma-1} e^{-\delta_1 y_1^{\sigma}} dy_1$$

$$pr[X_1 < Y_1, \mathcal{M}X_1 \geq \mathcal{K}Y_1] = \int_0^{\infty} \sigma \delta_1 y_1^{\sigma-1} e^{-\left(\eta_1 (\frac{\mathcal{K}}{\mathcal{M}})^{\sigma} + \delta_1\right) y_1^{\sigma}} dy_1$$
$$- \int_0^{\infty} \sigma \delta_1 y_1^{\sigma-1} e^{-\left(\eta_1 (1+\frac{\mathcal{K}}{\mathcal{M}})^{\sigma} + \delta_1\right) y_1^{\sigma}} dy_1$$

$$pr[X_1 < Y_1, \mathcal{M}X_1 \geq \mathcal{K}Y_1] = \left[\frac{\eta_1 \delta_1}{\left(\eta_1 \left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{\sigma} + \delta_1\right)\left(\eta_1 \left(1 + \left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{\sigma}\right) + \delta_1\right)}\right]$$

As equation (6) can get $\mathrm{pr}[X_2 \geq Y_2]$ as :

$$\mathrm{pr}[X_2 \geq Y_2] = \left[\frac{\delta_2}{\eta_2 + \delta_2}\right]$$

$$\mathcal{S}_2 = \left[\frac{\eta_1\delta_1}{\left(\eta_1\left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{\sigma} + \delta_1\right)\left(\eta_1\left(1+\left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{\sigma}\right)+\delta_1\right)}\right]\left[\frac{\delta_2}{\eta_2+\delta_2}\right] \qquad ...(7)$$

The third case of model work: when the unit $\mho_1$ fails and the replacement unit $\mho_3$ fails and unit $\mho_2$ remains activated, unit standby $\mho_4$ is activated to replace the failed unit $\mho_3$:

$$\mathcal{S}_3 = \mathrm{pr}[X_1 < Y_1, X_2 \geq Y_2, X_3 < Y_3, X_4 \geq Y_4]$$

$$\mathcal{S}_3 = \mathrm{p}r[X_1 < Y_1, \mathcal{M}X_1 < KY_1, \mathcal{M}X_3 \geq \mathcal{K}Y_3]\mathrm{pr}[X_2 \geq Y_2]$$

$$\mathcal{S}_3 = \mathrm{p}r[X_1 < Y_1, \mathcal{M}X_1 < KY_1, \mathcal{M}^2X_1 \geq \mathcal{K}^2Y_1]\mathrm{pr}[X_2 \geq Y_2]$$

Where $X_4 = \mathcal{M}X_3 = \mathcal{M}(\mathcal{M}X_1) = \mathcal{M}^2X_1$

and $Y_4 = \mathcal{K}Y_3 = \mathcal{K}(\mathcal{K}Y_1) = \mathcal{K}^2Y_1$, then :

$$\mathrm{p}r[X_1 < Y_1, \mathcal{M}X_1 < KY_1, \mathcal{M}^2X_1 \geq \mathcal{K}^2Y_1] = \int_0^{\infty}\left[F_{x_1}(y_1)\right]\left[F_{x_1}\left(\frac{\mathcal{K}}{\mathcal{M}}y_1\right)\right]\left[\overline{F}_{x_1}\left(\frac{\mathcal{K}^2}{\mathcal{M}^2}y_1\right)\right].g(y_1)dy_1$$

$$= \int_0^{\infty}\left[1 - e^{-\eta_1 y_1{}^{\sigma}}\right]\left[1 - e^{-\eta_1(\frac{\mathcal{K}}{\mathcal{M}})^{\sigma}y_1{}^{\sigma}}\right].\left[e^{-\eta_1(\frac{\mathcal{K}}{\mathcal{M}})^{2\sigma}y_1{}^{\sigma}}\right]\sigma\delta_1 y_1{}^{\sigma-1}e^{-\delta_1 y_1{}^{\sigma}}dy_1$$

$$= \int_0^{\infty}\left[e^{-\eta_1(\frac{\mathcal{K}}{\mathcal{M}})^{2\sigma}y_1{}^{\sigma}} - e^{-\eta_1\left(1+(\frac{\mathcal{K}}{\mathcal{M}})^{2\sigma}\right)y_1{}^{\sigma}}\right]\sigma\delta_1 y_1{}^{\sigma-1}e^{-\delta_1 y_1{}^{\sigma}}dy_1$$

$$.\int_0^{\infty}\left[e^{-\eta_1(\frac{\mathcal{K}}{\mathcal{M}})^{2\sigma}y_1{}^{\sigma}} - e^{-\eta_1(\frac{\mathcal{K}}{\mathcal{M}})^{\sigma}y_1{}^{\sigma}}\right]\sigma\delta_1 y_1{}^{\sigma-1}e^{-\delta_1 y_1{}^{\sigma}}dy_1$$

$$= \left[\int_0^{\infty}\sigma\delta_1 y_1{}^{\sigma-1}e^{-\left(\eta_1(\frac{\mathcal{K}}{\mathcal{M}})^{2\sigma}+\delta_1\right)y_1{}^{\sigma}}dy_1 - \sigma\delta_1 y_1{}^{\sigma-1}e^{-\left(\eta_1\left(1+(\frac{\mathcal{K}}{\mathcal{M}})^{2\sigma}\right)+\delta_1\right)y_1{}^{\sigma}}dy_1\right]$$

$$.\left[\int_0^{\infty}\sigma\delta_1 y_1{}^{\sigma-1}e^{-\left(\eta_1(\frac{\mathcal{K}}{\mathcal{M}})^{2\sigma}+\delta_1\right)y_1{}^{\sigma}}dy_1 - \sigma\delta_1 y_1{}^{\sigma-1}e^{-\left(\eta_1\left((\frac{\mathcal{K}}{\mathcal{M}})^{\sigma}+(\frac{\mathcal{K}}{\mathcal{M}})^{2\sigma}\right)+\delta_1\right)y_1{}^{\sigma}}dy_1\right]$$

$$\mathrm{p}r[X_1 < Y_1, \mathcal{M}X_1 < KY_1, \mathcal{M}^2X_1 \geq \mathcal{K}^2Y_1] = \left[\left(\frac{\eta_1^2\left(\frac{k}{m}\right)^{\sigma}\delta_1^2}{\left(\eta_1\left(\frac{k}{m}\right)^{2\sigma}+\delta_1\right)^2\left(\eta_1\left(1+\left(\frac{k}{m}\right)^{2\sigma}\right)+\delta_1\right)\left(\eta_1\left(\left(\frac{k}{m}\right)^{\sigma}+\left(\frac{k}{m}\right)^{2\sigma}\right)+\delta_1\right)}\right)\right]$$

and$\mathrm{pr}[X_2 \geq Y_2] = \left[\frac{\delta_2}{\eta_2+\delta_2}\right]$

$$\mathcal{S}_3 = \left[\left(\frac{\eta_1^2\left(\frac{k}{m}\right)^{\sigma}\delta_1^2}{\left(\eta_1\left(\frac{k}{m}\right)^{2\sigma}+\delta_1\right)^2\left(\eta_1\left(1+\left(\frac{k}{m}\right)^{2\sigma}\right)+\delta_1\right)\left(\eta_1\left(\left(\frac{k}{m}\right)^{\sigma}+\left(\frac{k}{m}\right)^{2\sigma}\right)+\delta_1\right)}\right)\right]\left[\frac{\delta_2}{\eta_2+\delta_2}\right] \qquad ...(8)$$

The fourth case of the model work: when the unit $\mho_1$ remains activated but the unit $\mho_2$fails, unit standby $\mho_3$ is activated to replace the failed unit $\mho_2$ and unit $\mho_4$ remains in a standby state :

$$\mathcal{S}_4 = \mathrm{pr}[X_1 \geq Y_1, X_2 < Y_2, X_3 \geq Y_3] = \mathrm{pr}[X_1 \geq Y_1]\mathrm{p}r[X_2 < Y_2, \mathcal{M}X_2 \geq \mathcal{K}Y_2]$$

Where $X_3 = \mathcal{M}X_2$ and $Y_3 = \mathcal{K}Y_2$, then :

As equation (6) can get $\mathrm{pr}[X_1 \geq Y_1]$ as :

$$\mathrm{pr}[X_1 \geq Y_1] = \left[\frac{\delta_1}{\eta_1 + \delta_1}\right]$$

$$pr[X_2 < Y_2, \mathcal{M}X_2 \geq \mathcal{K}Y_2] = \int_0^\infty [F_{x_2}(y_2)] \left[\bar{F}_{x_2}\left(\frac{\mathcal{K}}{\mathcal{M}} y_2\right)\right] g(y_2) dy_2$$

$$pr[X_2 < Y_2, \mathcal{M}X_2 \geq \mathcal{K}Y_2] = \int_0^\infty \left[1 - e^{-\eta_2 y_2^{\sigma}}\right] \left[e^{-\eta_2 \left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{\sigma} y_2^{\sigma}}\right] \sigma\delta_2 y_2^{\sigma-1} e^{-\delta_2 y_2^{\sigma}} dy_2$$

$$pr[X_2 < Y_2, \mathcal{M}X_2 \geq \mathcal{K}Y_2] = \int_0^\infty \left[e^{-\eta_2 \left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{\sigma} y_2^{\sigma}} - e^{-\eta_2 \left(1+\left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{\sigma}\right) y_2^{\sigma}}\right] \sigma\delta_2 y_2^{\sigma-1} e^{-\delta_2 y_2^{\sigma}} dy_2$$

$$pr[X_2 < Y_2, \mathcal{M}X_2 \geq \mathcal{K}Y_2] = \int_0^\infty \sigma\delta_2 y_2^{\sigma-1} e^{-\left(\eta_2 \left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{\sigma} + \delta_2\right) y_2^{\sigma}} dy_2$$

$$- \int_0^\infty \sigma\delta_2 y_2^{\sigma-1} e^{-\left(\eta_2 \left(1+\left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{\sigma}\right) + \delta_2\right) y_2^{\sigma}} dy_2$$

$$pr[X_2 < Y_2, \mathcal{M}X_2 \geq \mathcal{K}Y_2] = \left[\frac{\eta_2\delta_2}{\left(\eta_2 \left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{\sigma} + \delta_2\right)\left(\eta_2 \left(1 + \left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{\sigma}\right) + \delta_2\right)}\right]$$

$$\mathcal{S}_4 = \left[\frac{\delta_1}{\eta_1+\delta_1}\right]\left[\frac{\eta_2\delta_2}{\left(\eta_2 \left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{\sigma} + \delta_2\right)\left(\eta_2 \left(1 + \left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{\sigma}\right) + \delta_2\right)}\right] \qquad ...(9)$$

The fifth case of model work: when the unit $\mho_1$ remains activated and unit $\mho_2$ fails, the replacement unit $\mho_3$ fails and unit standby $\mho_4$ is activated to replace the failed unit $\mho_3$:

$$\mathcal{S}_5 = pr[X_1 \geq Y_1, X_2 < Y_2, X_3 < Y_3, X_4 \geq Y_4]$$

$$= pr[X_1 \geq Y_1] pr[X_2 < Y_2, \mathcal{M}X_2 < KY_2, \mathcal{M}X_3 \geq \mathcal{K}Y_3]$$

$$= pr[X_1 \geq Y_1] pr[X_2 < Y_2, \mathcal{M}X_2 < KY_2, \mathcal{M}^2X_2 \geq \mathcal{K}^2Y_2]$$

Where $X_4 = \mathcal{M}X_3 = \mathcal{M}(\mathcal{M}X_2) = \mathcal{M}^2X_2$

and $Y_4 = \mathcal{K}Y_3 = \mathcal{K}(\mathcal{K}Y_2) = \mathcal{K}^2Y_2$, then :

$$pr[X_1 \geq Y_1] = \left[\frac{\delta_1}{\eta_1 + \delta_1}\right]$$

and

$$pr[X_2 < Y_2, \mathcal{M}X_2 < KY_2, \mathcal{M}^2X_2 \geq \mathcal{K}^2Y_2] = \int_0^\infty [F_{x_2}(y_2)] \left[F_{x_2}\left(\frac{\mathcal{K}}{\mathcal{M}} y_2\right)\right] \left[\bar{F}_{x_2}\left(\frac{\mathcal{K}^2}{\mathcal{M}^2} y_2\right)\right] . g(y_1) dy_1$$

$$= \int_0^\infty \left[1 - e^{-\eta_2 y_2^{\sigma}}\right] \left[1 - e^{-\eta_2 \left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{\sigma} y_2^{\sigma}}\right] . \left[e^{-\eta_2 \left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{2\sigma} y_2^{\sigma}}\right] \sigma\delta_2 y_2^{\sigma-1} e^{-\delta_2 y_2^{\sigma}} dy_2$$

$$= \int_0^\infty \left[e^{-\eta_2 \left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{2\sigma} y_2^{\sigma}} - e^{-\eta_2 \left(1+\left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{2\sigma}\right) y_2^{\sigma}}\right] \sigma\delta_2 y_2^{\sigma-1} e^{-\delta_2 y_2^{\sigma}} dy_2$$

$$. \int_0^\infty \left[e^{-\eta_2 \left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{2\sigma} y_2^{\sigma}} - e^{-\eta_2 \left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{\sigma} y_2^{\sigma}}\right] \sigma\delta_2 y_2^{\sigma-1} e^{-\delta_2 y_2^{\sigma}} dy_2$$

$$= \left[\int_0^\infty \sigma\delta_2 y_2^{\sigma-1} e^{-\left(\eta_2 \left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{2\sigma} + \delta_2\right) y_2^{\sigma}} dy_2 - \sigma\delta_2 y_2^{\sigma-1} e^{-\left(\eta_2 \left(1+\left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{2\sigma}\right) + \delta_2\right) y_2^{\sigma}} dy_2\right]$$

$$. \left[\int_0^\infty \sigma\delta_2 y_2^{\sigma-1} e^{-\left(\eta_2 \left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{2\sigma} + \delta_2\right) y_2^{\sigma}} dy_2 - \sigma\delta_2 y_2^{\sigma-1} e^{-\left(\eta_2 \left(\left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{\sigma} + \left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{2\sigma}\right) + \delta_2\right) y_2^{\sigma}} dy_2\right]$$

$$\mathrm{pr}[X_2 < Y_2, \mathcal{M}X_2 < KY_2, \mathcal{M}^2X_2 \geq \mathcal{K}^2Y_2] = \left[\left(\frac{\eta_2^2\left(\frac{k}{m}\right)^{\sigma}\delta_2^2}{\left(\eta_2\left(\frac{k}{m}\right)^{2\sigma}+\delta_2\right)^2\left(\eta_2\left(1+\left(\frac{k}{m}\right)^{2\sigma}\right)+\delta_2\right)\left(\eta_2\left(\left(\frac{k}{m}\right)^{\sigma}+\left(\frac{k}{m}\right)^{2\sigma}\right)+\delta_2\right)}\right)\right]$$

$$\mathcal{S}_5 = \left[\frac{\delta_1}{\eta_1+\delta_1}\right]\left[\left(\frac{\eta_2^2\left(\frac{k}{m}\right)^{\sigma}\delta_2^2}{\left(\eta_2\left(\frac{k}{m}\right)^{2\sigma}+\delta_2\right)^2\left(\eta_2\left(1+\left(\frac{k}{m}\right)^{2\sigma}\right)+\delta_2\right)\left(\eta_2\left(\left(\frac{k}{m}\right)^{\sigma}+\left(\frac{k}{m}\right)^{2\sigma}\right)+\delta_2\right)}\right)\right] \quad ...(10)$$

The sixth case of the model work: when the units $\mho_1$ and $\mho_2$ are fails, the standby units $\mho_3$ and $\mho_4$ are activated to replace the failed units $\mho_1$ and $\mho_4$:

$$\mathcal{S}_6 = \mathrm{pr}[X_1 < Y_1, X_2 < Y_2, X_3 \geq Y_3, X_4 \geq Y_4]$$

$$= \mathrm{pr}[X_1 < Y_1, \mathcal{M}X_1 \geq \mathcal{K}Y_1]\mathrm{pr}[X_2 < Y_2, \mathrm{m}X_2 \geq \mathrm{k}Y_2]$$

$$\mathrm{pr}[X_1 < Y_1, \mathcal{M}X_1 \geq \mathcal{K}Y_1] = \int_0^{\infty}[F_{x_1}(y_1)]\left[\overline{F}_{x_1}\left(\frac{\mathcal{K}}{\mathcal{M}}y_1\right)\right]g(y_1)dy_1$$

$$\mathrm{pr}[X_1 < Y_1, \mathcal{M}X_1 \geq \mathcal{K}Y_1] = \int_0^{\infty}\left[1-e^{-\eta_1 y_1{}^{\sigma}}\right]\left[e^{-\eta_1(\frac{\mathcal{K}}{\mathcal{M}}y_1)^{\sigma}}\right]\sigma\delta_1 y_1{}^{\sigma-1}e^{-\delta_1 y_1{}^{\sigma}}dy_1$$

$$\mathrm{pr}[X_1 < Y_1, \mathcal{M}X_1 \geq \mathcal{K}Y_1] = \int_0^{\infty}\left[e^{-\eta_1\left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{\sigma}y_1{}^{\sigma}} - e^{-\eta_1\left(1+\left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{\sigma}\right)y_1{}^{\sigma}}\right]\sigma\delta_1 y_1{}^{\sigma-1}e^{-\delta_1 y_1{}^{\sigma}}dy_1$$

$$\mathrm{pr}[X_1 < Y_1, \mathcal{M}X_1 \geq \mathcal{K}Y_1] = \int_0^{\infty}\sigma\delta_1 y_1{}^{\sigma-1}e^{-\left(\eta_1\left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{\sigma}+\delta_1\right)y_1{}^{\sigma}}dy_1$$

$$.\int_0^{\infty}\sigma\delta_1 y_1{}^{\sigma-1}e^{-\left(\eta_1\left(1+\left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{\sigma}\right)+\delta_1\right)y_1{}^{\sigma}}dy_1$$

$$\mathrm{pr}[X_1 < Y_1, \mathcal{M}X_1 \geq \mathcal{K}Y_1] = \left[\left(\frac{\delta_1}{\eta_1\left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{\sigma}+\delta_1}\right)-\left(\frac{\delta_1}{\eta_1\left(1+\left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{\sigma}\right)+\delta_1}\right)\right]$$

$$\mathrm{pr}[X_1 < Y_1, \mathcal{M}X_1 \geq \mathcal{K}Y_1] = \left[\frac{\eta_1\delta_1}{\left(\eta_1\left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{\sigma}+\delta_1\right)\left(\eta_1\left(1+\left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{\sigma}\right)+\delta_1\right)}\right]$$

and

$$\mathrm{pr}[X_2 < Y_2, \mathrm{m}X_2 \geq \mathrm{k}Y_2] = \int_0^{\infty}[F_{x_2}(y_2)]\left[\overline{F}_{x_2}\left(\frac{\mathcal{K}}{\mathcal{M}}y_2\right)\right]g(y_2)dy_2$$

$$\mathrm{pr}[X_2 < Y_2, \mathrm{m}X_2 \geq \mathrm{k}Y_2] = \int_0^{\infty}\left[1-e^{-\eta_2 y_2{}^{\sigma}}\right]\left[e^{-\eta_2(\frac{\mathcal{K}}{\mathcal{M}}y_2)^{\sigma}}\right]\sigma\delta_2 y_2{}^{\sigma-1}e^{-\delta_2 y_2{}^{\sigma}}dy_2$$

$$\mathrm{pr}[X_2 < Y_2, \mathcal{M}X_2 \geq \mathcal{K}Y_2] = \int_0^{\infty}\left[e^{-\eta_2\left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{\sigma}y_2{}^{\sigma}} - e^{-\eta_2\left(1+\left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{\sigma}\right)y_2{}^{\sigma}}\right]\sigma\delta_2 y_2{}^{\sigma-1}e^{-\delta_2 y_2{}^{\sigma}}dy_2$$

$$\mathrm{pr}[X_2 < Y_2, \mathcal{M}X_2 \geq \mathcal{K}Y_2] = \int_0^{\infty}\sigma\delta_2 y_2{}^{\sigma-1}e^{-\left(\eta_2\left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{\sigma}+\delta_2\right)y_2{}^{\sigma}}dy_2$$

$$.\int_0^{\infty}\sigma\delta_2 y_2{}^{\sigma-1}e^{-\left(\eta_2\left(1+\left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{\sigma}\right)+\delta_2\right)y_2{}^{\sigma}}dy_2$$

$$\mathrm{pr}[X_2 < Y_2, mX_2 \geq kY_2] = \left[\left(\frac{\delta_2}{\eta_2\left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{\sigma} + \delta_2}\right) - \left(\frac{\delta_2}{\eta_2\left(1+\left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{\sigma}\right) + \delta_2}\right)\right]$$

$$\mathrm{pr}[X_2 < Y_2, mX_2 \geq kY_2] = \left[\frac{\eta_2\delta_2}{\left(\eta_2\left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{\sigma} + \delta_2\right)\left(\eta_2\left(1+\left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{\sigma}\right) + \delta_2\right)}\right]$$

$$\mathcal{S}_6 = \left[\frac{\eta_1\delta_1}{\left(\eta_1\left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{\sigma} + \delta_1\right)\left(\eta_1\left(1+\left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{\sigma}\right) + \delta_1\right)}\right]\left[\frac{\eta_2\delta_2}{\left(\eta_2\left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{\sigma} + \delta_2\right)\left(\eta_2\left(1+\left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{\sigma}\right) + \delta_2\right)}\right] \quad ...(11)$$

Now , substituting (6),(7),(8),(9),(10) and (11) in (5) ;will get reliability function for (2+2) cascade model of Weibull distribution :

$$\mathcal{R} = \left[\frac{\delta_1}{\eta_1+\delta_1}\right]\left[\frac{\delta_2}{\eta_2+\delta_2}\right] + \left[\frac{\eta_1\delta_1}{\left(\eta_1\left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{\sigma} + \delta_1\right)\left(\eta_1\left(1+\left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{\sigma}\right) + \delta_1\right)}\right]\left[\frac{\delta_2}{\eta_2+\delta_2}\right]$$

$$+\left[\left(\frac{\eta_1^2\left(\frac{k}{m}\right)^{\sigma}\delta_1^2}{\left(\eta_1\left(\frac{k}{m}\right)^{2\sigma} + \delta_1\right)^2\left(\eta_1\left(1+\left(\frac{k}{m}\right)^{2\sigma}\right) + \delta_1\right)\left(\eta_1\left(\left(\frac{k}{m}\right)^{\sigma} + \left(\frac{k}{m}\right)^{2\sigma}\right) + \delta_1\right)}\right)\right]\left[\frac{\delta_2}{\eta_2+\delta_2}\right]$$

$$+\left[\frac{\delta_1}{\eta_1+\delta_1}\right]\left[\frac{\eta_2\delta_2}{\left(\eta_2\left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{\sigma} + \delta_2\right)\left(\eta_2\left(1+\left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{\sigma}\right) + \delta_2\right)}\right]$$

$$+\left[\frac{\delta_1}{\eta_1+\delta_1}\right]\left[\left(\frac{\eta_2^2\left(\frac{k}{m}\right)^{\sigma}\delta_2^2}{\left(\eta_2\left(\frac{k}{m}\right)^{2\sigma} + \delta_2\right)^2\left(\eta_2\left(1+\left(\frac{k}{m}\right)^{2\sigma}\right) + \delta_2\right)\left(\eta_2\left(\left(\frac{k}{m}\right)^{\sigma} + \left(\frac{k}{m}\right)^{2\sigma}\right) + \delta_2\right)}\right)\right]$$

$$+\left[\frac{\eta_1\delta_1}{\left(\eta_1\left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{\sigma} + \delta_1\right)\left(\eta_1\left(1+\left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{\sigma}\right) + \delta_1\right)}\right]\left[\frac{\eta_2\delta_2}{\left(\eta_2\left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{\sigma} + \delta_2\right)\left(\eta_2\left(1+\left(\frac{\mathcal{K}}{\mathcal{M}}\right)^{\sigma}\right) + \delta_2\right)}\right] \quad \ldots(14)$$

3.Parameter Estimation:

3-1Maximum likelihood Estimation Method (ML):-

If the random sample $x_1, x_2, x_3, \ldots, x_n$ from $W(\sigma, \eta)$, the likelihood function "L" , is:(Bhattacharya & Bhattacharjee, 2009; Ismail, 2012)

$$L(x_1, x_2, x_3 \ldots, x_n, \sigma, \eta) = f(x_1; \sigma, \eta) f(x_2; \sigma, \eta) f(x_3; \sigma, \eta) \ldots f(x_n; \sigma, \eta)$$

$$= \prod_{r=1}^{n} f(x_r; \sigma, \eta)$$

likelihood function will be:

$L(x_1, x_2, x_3 \ldots, x_n; \sigma, \eta) = \sigma^n \eta^n \prod_{r=1}^{n} x_i^{\sigma-1} e^{-\sum_{r=1}^{n} \eta x_r^{\sigma}}$ …(15)

By taking natural logarithm for equation (15) it can be written as:

$$\ln L = n\mathrm{Ln}\sigma + n\mathrm{Ln}\eta + (\sigma - 1)\sum_{r=1}^{n} \ln x_r - \eta \sum_{r=1}^{n} x_r^{\sigma}$$

$$\frac{\partial \ln L}{\partial \eta} = \frac{n}{\eta} - \sum_{r=1}^{n} x_r^{\sigma}$$

$\frac{n}{\hat{\eta}} - \sum_{r=1}^{n} x_r^{\sigma} = 0$ ...(16)

From (16) we obtain the maximum likelihood estimator of η:

$\hat{\eta}_{(ML)} = \frac{n}{\sum_{r=1}^{n} x_r^{\sigma}}$ ...(17)

Assume that strength random samples $X_{1_{r_1}}; r_1 = 1, 2, \ldots, n_1, X_{2_{r_2}}; r_2 = 1, 2, \ldots, n_2, X_{3_{r_3}}; r_3 = 1, 2, \ldots, n_3$and $X_{4_{r_4}}; r_4 = 1, 2, \ldots, n_3$from $W(\sigma, \eta_1)$, $W(\sigma, \eta_2)$, $W(\sigma, \eta_3)$and $W(\sigma, \eta_4)$, with samples size(n_1 , n_2, n_3 and n_4) respectively where $\eta_1, \eta_2, \eta_3$and η_4 are unknown parameters:

$\hat{\eta}_{\zeta(ML)} = \frac{n_\zeta}{\sum_{r_\zeta=1}^{n_\zeta} x_{\zeta_{r_\zeta}}^{\sigma}}, \zeta = 1,2,3,4$ ...(18)

Like the above steps, for the random stress variables with the samples size (m_1, m_2, m_3 and m_4) the maximum likelihood estimators for unknown parameters $\delta_1, \delta_2, \delta_3$ and δ_4 will be as :

$\hat{\delta}_{\zeta(ML)} = \frac{m_\zeta}{\sum_{v_\zeta=1}^{m_\zeta} y_{\zeta_{v_\zeta}}^{\sigma}}, \zeta = 1,2,3,4$ …(19)

3-2 Regression Estimation Method (Rg) :-

Assume the random sample$x_1, x_2, x_3, \dots, x_n$ from $W(\sigma, \eta)$: (Al-nasser & Radaideh, 2008; Lewis & Linzer, 2005)

$$F(x_{(r)}) = 1 - e^{-\eta x_{(r)}^{\sigma}}$$
$$\left(1 - F(x_{(r)})\right)^{-1} = e^{\eta x_{(r)}^{\sigma}}$$

$\mathrm{Ln}\left[\left(1 - F(x_{(r)})\right)^{-1}\right] = \eta x_{(r)}^{\sigma}$

Changing $F(x_{(r)})$ by the plotting position P_r, where $P_r = \frac{r}{n+1}$

$\mathrm{Ln}[(1 - P_r)^{-1}] = \eta x_{(r)}^{\sigma}$...(20)

by using equation standard regression :

$z_r = a + bu_r + e_r$…(21)

Compare equation (21) with equation (20) :

$z_r = \mathrm{Ln}[(1 - P_r)^{-1}], a = 0, b = \eta, u_r = x_{(r)}^{\sigma}$ where ; $r = 1,2, \dots, n$ ...(22)

Where b can be estimated by minimizing summation of the squared error for b :

$\hat{b} = \frac{n\sum_{r=1}^{n} z_r u_r - \sum_{r=1}^{n} z_r \sum_{r=1}^{n} u_r}{n\sum_{r=1}^{n}(u_r)^2 - \left(\sum_{r=1}^{n} u_r\right)^2}$ ...(23)

Substation equation (22) in equation (23), the estimator for β :

$\hat{\eta}_{(Rg)} = \frac{n\sum_{r=1}^{n} x_{(r)}^{\sigma}\,\mathrm{Ln}[(1-P_r)^{-1}] - \sum_{r=1}^{n} x_{(r)}^{\sigma} \sum_{r=1}^{n} \mathrm{Ln}[(1-P_r)^{-1}]}{n\sum_{r=1}^{n}\left[x_{(r)}^{\sigma}\right]^2 - \left[\sum_{r=1}^{n} x_{(r)}^{\sigma}\right]^2}$ ...(24)

Now, the Regression estimators of the unknown scale parameters ($\eta_1, \eta_2, \eta_3$and η_4) and ($\delta_1, \delta_2, \delta_3$ and δ_4)are :

$$\hat{\eta}_{\zeta(Rg)} = \frac{n_\zeta \sum_{r_\zeta=1}^{n_\zeta} x_{\zeta_{(r_\zeta)}}^{\sigma} \ln\left[\left(1-P_{r_\zeta}\right)^{-1}\right] - \sum_{r_\zeta=1}^{n_\zeta} x_{\zeta_{(r_\zeta)}}^{\sigma} \sum_{r_\zeta=1}^{n_\zeta} \ln\left[\left(1-P_{r_\zeta}\right)^{-1}\right]}{n_\zeta \sum_{r_\zeta=1}^{n_\zeta}\left[x_{\zeta_{(r_\zeta)}}^{\sigma}\right]^2 - \left[\sum_{r_\zeta=1}^{n_\zeta} x_{\zeta_{(r_\zeta)}}^{\sigma}\right]^2} ; \zeta = 1,2,3,4 \quad \text{…(25)}$$

and

$$\hat{\delta}_{\zeta(Rg)} = \frac{m_\zeta \sum_{v_\zeta=1}^{m_\zeta} y_{\zeta_{(v_\zeta)}}^{\sigma} \ln\left[\left(1-P_{v_\zeta}\right)^{-1}\right] - \sum_{v_\zeta=1}^{m_\zeta} y_{\zeta_{(v_\zeta)}}^{\sigma} \sum_{v_\zeta=1}^{m_\zeta} \ln\left[\left(1-P_{v_\zeta}\right)^{-1}\right]}{m_\zeta \sum_{v_\zeta=1}^{m_\zeta}\left[y_{\zeta_{(v_\zeta)}}^{\sigma}\right]^2 - \left[\sum_{v_\zeta=1}^{m_\zeta} y_{\zeta_{(v_\zeta)}}^{\sigma}\right]^2} ; \zeta = 1,2,3,4 \quad \text{...(26)}$$

3-2 Percentile Estimation Method (PE) :

Suppose that random sample $x_r; r = 1,2, \dots, n$ with size n from $W(\sigma, \eta)$, Since the CDF defined in equation: (Gupta & Kundu, 2001)

$$F(x_{(r)}) = 1 - e^{-\eta x_{(r)}^{\sigma}}$$
$$\ln\left(1 - F(x_{(r)})\right) = -\eta x_{(r)}^{\sigma}$$

$x_{(r)} = \left(\frac{-\ln(1-F(x_{(r)}))}{\eta}\right)^{\frac{1}{\sigma}}$ ...(27)

Since P_r;$r = 1,2, \dots, n$ denotes some estimate of $F(x_{(r)}; \sigma, \eta)$, then :

$x_{(r)} = \left(\frac{-\ln(1-P_r)}{\eta}\right)^{\frac{1}{\sigma}}$...(28)

Minimizing equation defined as :

$\sum_{r=1}^{n}\left[x_{(r)} - F(x_{(r)})\right]^2$ ...(29)

Substitution (28) in (29), will get as :

$\sum_{r=1}^{n}\left[x_{(r)} - \left(\frac{-\ln(1-P_r)}{\eta}\right)^{\frac{1}{\sigma}}\right]^2$ ...(30)

The partial derivative of equation (30):

$$\sum_{r=1}^{n} 2\left[\left(x_{(r)}\right) - \eta^{-\frac{1}{\sigma}}(-\ln(1-P_r))^{\frac{1}{\sigma}}\right]\left(\frac{1}{\sigma}\eta^{-\left(\frac{1}{\sigma}+1\right)}\right)(-\ln(1-P_r))^{\frac{1}{\sigma}} = 0$$

The Percentile estimator of η :

$$\hat{\eta}_{(PE)} = \left[\frac{\sum_{r=1}^{n}(-\ln(1-P_r))^{\frac{2}{\sigma}}}{\sum_{r=1}^{n}(x_{(r)})(-\ln(1-P_r))^{\frac{1}{\sigma}}}\right]^{\sigma} \quad ...(31)$$

Now the Percentile estimators parameters $(\eta_1, \eta_2, \eta_3 \text{and } \eta_4)$ and $(\delta_1, \delta_2, \delta_3 \text{ and } \delta_4)$are :

$$\hat{\eta}_{\zeta(PE)} = \left[\frac{\sum_{r_\zeta=1}^{n_\zeta}\left(-\ln\left(1-P_{r_\zeta}\right)\right)^{\frac{2}{\sigma}}}{\sum_{r_\zeta=1}^{n_\zeta}\left(x_{\zeta_{(r_\zeta)}}\right)\left(-\ln\left(1-P_{r_\zeta}\right)\right)^{\frac{1}{\sigma}}}\right]^{\sigma} \ ; \ \zeta = 1,2,3,4 \quad ...(32)$$

and

$$\hat{\delta}_{\zeta(PE)} = \left[\frac{\sum_{\upsilon_\zeta=1}^{m_\zeta}\left(-\ln\left(1-P_{\upsilon_\zeta}\right)\right)^{\frac{2}{\sigma}}}{\sum_{\upsilon_\zeta=1}^{m_\zeta}\left(y_{\zeta_{(\upsilon_\zeta)}}\right)\left(-\ln\left(1-P_{\upsilon_\zeta}\right)\right)^{\frac{1}{\sigma}}}\right]^{\sigma} \ ; \ \zeta = 1,2,3,4 \quad ...(33)$$

4.The simulation study and discussions

We conduct extensive simulations to compare the performances of the different methods, mainly for their mean square errors, for different sample sizes and different parameters values. The parameters of the Weibull distribution can be estimated with ten experiments, while the best estimation method to estimate $\mathcal{R}$ described in equation (14) is being explored. Experiments performed were based on run size K=10000. Results have been recorded for $(n_1, n_2, m_1, m_2) = (10,10,10,10)$ (small samples) , $(25,25,25,25)$ (moderate samples) and $(75,75,75,75)$(large samples).

The following different values of parameters $(\sigma, \eta_1, \eta_2, \eta_3, \delta_1, \delta_2, \delta_3)$ and attenuation factors $(\mathcal{K} \text{ and } \mathcal{M})$ in the table (1):

Experiment	σ	η_1	η_2	δ_1	δ_2	$\mathcal{K}$	$\mathcal{M}$	$\mathcal{R}$
1	0.7	0.7	0.7	0.7	0.7	1.8	0.2	0.2773
2	1.8	0.7	0.7	0.7	0.7	1.8	0.2	0.2503
3	0.7	1.9	1.9	0.7	0.7	1.8	0.2	0.0792
4	0.7	0.7	0.7	1.5	1.5	1.8	0.2	0.5220
5	0.7	0.7	0.7	0.7	0.7	1.2	0.9	0.4599
6	1.8	0.7	0.7	0.7	0.7	1.2	0.9	0.3765
7	0.7	1.9	1.9	0.7	0.7	1.2	0.9	0.1464
8	0.7	0.7	0.7	1.5	1.5	1.2	0.9	0.8528
9	1.6	1.2	1.6	0.6	0.9	1.1	0.95	0.2311
10	0.7	0.9	0.7	1.5	1.7	1.1	0.95	0.9002

For the ten experiments, $(n_1 = n_2 = m_1 = m_2)$should be noted, where $(n_1, n_2, m_1 \, and \, m_2)$ are the sample sizes drawn from stress and strength variables :

Table (2): Values (Mean and MSE) for an experiment (1)

Simple size	Criterion	ML	Rg	Pr
$(10,10,10,10)$	Mean	0.2791	0.2789	0.2790
	MSE	0.0078	0.0121	0.0092
$(25,25,25,25)$	Mean	0.2774	0.2772	0.2772
	MSE	0.0032	0.0056	0.0042
$(75,75,75,75)$	Mean	0.2769	0.2777	0.2776
	MSE	0.0010	0.0021	0.0016

Table (3): Values (Mean and MSE) for an experiment (2)

Simple size	Criterion	ML	Rg	Pr
$(10,10,10,10)$	Mean	0.2509	0.2512	0.2508
	MSE	0.0062	0.0098	0.0063
$(25,25,25,25)$	Mean	0.2496	0.2502	0.2497
	MSE	0.0025	0.0045	0.0027
$(75,75,75,75)$	Mean	0.2504	0.2508	0.2506
	MSE	0.0008	0.0017	0.0009

Table (4): Values (Mean and MSE) for an experiment (3)

Simple size	Criterion	ML	Rg	Pr
$(10, 10, 10, 10)$	Mean	0.0847	0.0879	0.0857
	MSE	0.0016	0.0028	0.0020
$(25, 25, 25, 25)$	Mean	0.0814	0.0835	0.0824
	MSE	0.0006	0.0011	0.0008
$(75, 75, 75, 75)$	Mean	0.0800	0.0809	0.0805
	MSE	0.0002	0.0008	0.0004

Table (5): Values (Mean and MSE) for an experiment (4)

Simple size	Criterion	ML	Rg	Pr
$(10, 10, 10, 10)$	Mean	0.5093	0.5029	0.5073
	MSE	0.0111	0.0175	0.0132
$(25, 25, 25, 25)$	Mean	0.5164	0.5118	0.5144
	MSE	0.0045	0.0082	0.0060
$(75, 75, 75, 75)$	Mean	0.5205	0.5190	0.5198
	MSE	0.0016	0.0030	0.0023

Table (6): Values (Mean and MSE) for an experiment (5)

Simple size	Criterion	ML	Rg	Pr
$(10, 10, 10, 10)$	Mean	0.4569	0.4368	0.4445
	MSE	0.0174	0.0251	0.0196
$(25, 25, 25, 25)$	Mean	0.4581	0.4465	0.4510
	MSE	0.0072	0.0121	0.0092
$(75, 75, 75, 75)$	Mean	0.4599	0.4545	0.4564
	MSE	0.0024	0.0046	0.0035

Table (7): Values (Mean and MSE) for an experiment (6)

Simple size	Criterion	ML	Rg	Pr
$(10, 10, 10, 10)$	Mean	0.3766	0.3707	0.3748
	MSE	0.0128	0.0195	0.0127
$(25, 25, 25, 25)$	Mean	0.3767	0.3721	0.3757
	MSE	0.0053	0.0093	0.0050
$(75, 75, 75, 75)$	Mean	0.3766	0.3744	0.3762
	MSE	0.0018	0.0034	0.0016

Table (8): Values (Mean and MSE) for an experiment (7)

Simple size	Criterion	ML	Rg	Pr
$(10, 10, 10, 10)$	Mean	0.1561	0.1565	0.1545
	MSE	0.0055	0.0086	0.0064
$(25, 25, 25, 25)$	Mean	0.1511	0.1520	0.1509
	MSE	0.0020	0.0036	0.0026
$(75, 75, 75, 75)$	Mean	0.1480	0.1474	0.1473
	MSE	0.0006	0.0012	0.0009

Table (9): Values (Mean and MSE) for an experiment (8)

Simple size	Criterion	ML	Rg	Pr
$(10, 10, 10, 10)$	Mean	0.8376	0.7772	0.8007
	MSE	0.0227	0.0349	0.0261
$(25, 25, 25, 25)$	Mean	0.8479	0.8080	0.8231
	MSE	0.0085	0.0155	0.0113
$(75, 75, 75, 75)$	Mean	0.8507	0.8285	0.8362
	MSE	0.0028	0.0058	0.0043

Table (10): Values (Mean and MSE) for an experiment(9)

Simple size	Criterion	ML	Rg	Pr
$(10, 10, 10, 10)$	Mean	0.2402	0.2367	0.2367
	MSE	0.0097	0.0144	0.0093
$(25, 25, 25, 25)$	Mean	0.2356	0.2341	0.2339
	MSE	0.0037	0.0064	0.0036
$(75, 75, 75, 75)$	Mean	0.2323	0.2308	0.2315
	MSE	0.0012	0.0023	0.0010

Table (11):Values (Mean and MSE) for an experiment(10)

Simple size	Criterion	ML	Rg	Pr
$(10, 10, 10, 10)$	Mean	0.8875	0.8097	0.8385
	MSE	0.0279	0.0429	0.0319
$(25, 25, 25, 25)$	Mean	0.8948	0.8423	0.8617
	MSE	0.0105	0.0194	0.0140
$(75, 75, 75, 75)$	Mean	0.8973	0.8695	0.8791
	MSE	0.0034	0.0072	0.0053

Tables (1,2,3,4,5,6,7,8,9,10 and 11) show the results of the ten experiments. Tables showing both of the ten experiments reveal some very clear common points :

1. It was noticed that when the amount of the shape parameter σ was increased, the model reliability decreased, this can be noticed clear when comparing experiment (1) with experiment (2) and also when comparing experiment (5) with experiment (6) in a table (1).

2. It was noticed that when increasing the amount of the two-scale parameters $\eta_1$and η_2 , the model reliability decreased. This is clear when comparing experiment (1) with experiment (3) and also when comparing experiment (5) with experiment (7) in a table (1).
3. The model's reliability value increased with the increase in the value of the scale parameters δ_1 and δ_2, and this can be seen when comparing experiment (1) with experiment (4) and also when comparing experiment (5) with experiment (8) in a table (1)
4. As for the attenuation ($\mathcal{K}$ and $\mathcal{M}$) factors, it should be noted that the inverse relationship between the value of the reliability model with the value of $(\mathcal{K}/\mathcal{M})$so that if the value of $(\mathcal{K}/\mathcal{M})$ increases, the value of the reliability model decreases, and if the value of $(\mathcal{K}/\mathcal{M})$ decreases the reliability of the model increases. This can be seen when comparing experiments (1, 2,3,4) with experiments (5,6,7,8) respectively in a table (1).

The following can be listed to compare the performances of all three methods to estimate the reliability for (2+2) cascade model of Weibull distribution :

1. The decrease in the mean square errors with increases in sample size in all estimation methods. It confirms asymptotic impartiality and consistency of all estimators.
2. The performances of Rg's and Pe's are according to their order.
3. The performances of ML's and Pe's are close to each other.
4. The ML estimator is the best of the three deferent estimation methods and this is shown in the simulation of the results of the tables (2,3,4,5,7,8,10).
5. The Pe estimator is the best of the three deferent estimation methods and appears in the simulation results of tables (6 and 9).

5. Conclusions

A- In this study, we conclude from ten experiments for different values of parameters that are presented in Table 1 include:

1. The numerical value of reliability increases with decreasing value of the saucepan of the shape parameter (σ) , and the relationship between them is inverse.
2. The numerical value of reliability decreases with increasing the value of the amount of the two-scale parameters$(\eta_1\ and\ \eta_2)$, and the relationship between them is inverse.
3. The numerical value of reliability increases with the increase in the value of the two-scale parameters $(\delta_1\ and\ \delta_2)$, and the relationship between them is positive.
4. The numerical value of reliability increases when the ratio between the two attenuation factors $(\mathcal{K}/\mathcal{M})$ decreases, meaning that the relationship between them is inverse.

B-We concludes from simulation study results of the in tables (2,3,4,5,6,7,8,9,10 and 11) the following:

1. The performance of the ML estimator and the Pe estimator are approximate in estimating model reliability in most experimental results and for different sizes sample.
2. The ML estimator is best for estimating the reliability of model in 80% of simulation results.
3. Pe estimator is the best at estimating the reliability of model in 20% of simulation results.
4. This model can be applied to real data and different estimation methods can be used to estimate model reliability.

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