Iterative Methods and Non-Linear System Equations

Noorzaman Bawari
Assistant Professor
Science Faculty, Math Department,
Nangarhar University, Nangarhar, Afghanistan
Email: noorzamanstd@uop.edu.pk

Abstract
This paper is based on various techniques that are used to answer a pair of non-linear equations. These techniques assumed that the answer to these equations exists. The solution of non-linear equations includes the Gauss-Seidel Method. This method is using from decades to know how convergence can be obtained. It is a more straightforward method as compared to other methods. The present study compared many non-linear solution techniques.
Keywords: Gauss-Seidel Method, Non-linear Equations

1. Introduction

Let's suppose the non-linear system where,

$$F1\ (x1, x1 \ldots\ldots xn) = 0$$
$$F2\ (x1, x1 \ldots.xn) = 0$$
$$Fn\ (x1, x1 \ldots.xn) = 0$$

A compact shape in this system will be;
F(X) = 0
Here, (f1, f2…..fn=0) are complementing F.
The system of non-linear equations is used in different subjects like Math, CSIT, Robotics, and various engineering schools as most of the physical systems studied in these disciplines are non-linear in their classification. There may be different answers to these polynomial equations. In simple words, we can say that the non-linear equation is not so simple and complicated to solve because only a few of the equations are linear. In general, linear and non-linear equations are scattered which means that the linear coefficient in the proportional matrix is 0 with the number of non-linear coefficient =0 (n) where n represents the number of variables [7].
The iterative technique doesn't use processing setup on the coefficient, which is zero. However, direct techniques or methods are used to fill the process. In such ways, while executing the algorithm, the coefficient value converts from 0 to a non-zero. These kinds of methods usually deal with more than one coefficient; therefore, the processing setup may get slow down. In such a way, indirect techniques/methods are much better than direct techniques/methods.
The iterative process begins with the initial guess, continually iterates between the limitations of equations, and keeps on clarifying the answers until adequate perfection is achieved. Many

methods are designed to solve non-linear equations in recent years. A few of iterative methods are discussed below:

1.1 **Bisection Method**

This method is a primary method that is used to calculate the fundamental roots of the function $f\ (x) = 0$ in which, f shows a continuous function. It is called a Bolzano or Binary search method as well. To commence this method, two initial guesses are required, and its procedure depends upon the Intermediate Value Theorem which says that there will be a minimum one root if $f\ (x) = 0$ is continuous in $f\ (1)\ and\ f\ (u)$ with the different sign and below zero value. For this purpose, we need to bracket out the root and determine the midpoint that is;

$$xm = xl + xu/2$$

The bisection method classifies intervals into 2 parts. After that, we will try to calculate $f(x1)f(xm)$

Figure 1 explains it further.

The Bisection Method has a lot of advantages [14] for those who are using a computer to solve equations. It easily converges. Interval bisects as the iterations start working. Every error is instantly controlled. And because these methods bracket the root, convergence is a hundred per cent assured.

On the contrary, it has some disadvantages as well. First of all, due to converging linearly, the method becomes slow. Whereas when the initial calculations lie near to the root, it will become challenging to find the root because the demand of iterations will be increased.

Choose {x_{l}} and {x_{u}} as the initial guess such that

f(x_{l})f(x_{u})<0

Try the midpoint x_{m}=\frac{x_{l}+x_{u}}{2}

Find

f(x_{l})f(x_{m})

If

f(x_{l})f(x_{m})<0

Then

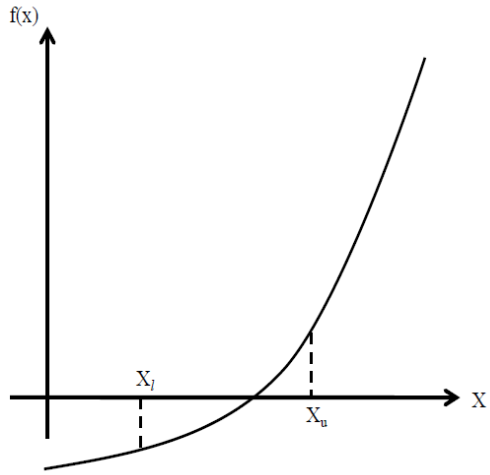

Figure 1: Bi Section Method

$x_{l}=x_{l};x_{u}=x_{m}$

If

$f(x_{l})f(x_{m})>0$

Then

$x_{l}=x_{m};x_{u}=x_{u}$

If

$f(x_{l})f(x_{m})=0$

Then the root is exact root.

$\epsilon_{x}^{i}|=|x_{i}-x_{i+1}|$

Check if

$|\epsilon_{x}^{i}|\leq\epsilon$

1.2 Newton Raphson Method

This method [15] is a very famous method to determine the roots of every non-linear equation. Its function is estimated with the tangent line shown in fig. 2. Newton's Method always begins with the initial calculations which are adjacent to the root. In contrast to other methods, the Newton method required only one basic calculation, and if that calculation is near to root, this method will work very effectively. Its convergence is always fast and a very good point of Newton is that it takes very little time and fewer iterations to determine the root. Newton and bisection both calculate consistently so that the accurate number of iterations used to find the root will be determined.

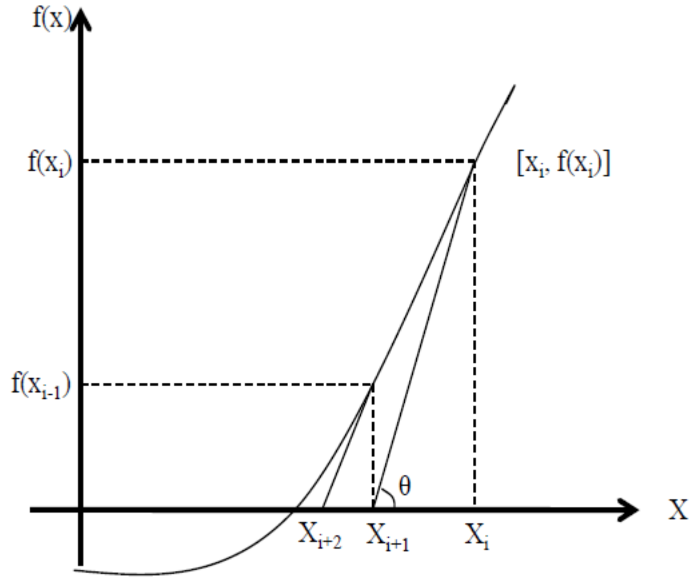

Figure 2: Newton Raphson Method

Newton needs to apply the function and derivative of this function to approach the root. The procedure occurred by the basic calculation that is more near to the root. Then, limit the function by tangent line to determine the exact place of the root. The x-intercept of the tangent line is estimated too. The ideal estimation is x-intercept than the basic calculation. This exercise is going on until or unless the appropriate root is obtained. Error calculations can also occur during this process. It is defined as the variation between the estimated and the actual values. Newton Raphson Method is very speedy as compared to the Bisection Method. But on the other hand, it needs a derivative of the function that is the most complicated task. At some points, when these functions couldn't resolved by using Newton's method than the False Position Method will be preferable. To fulfill the convergence criteria the following state must be execute i.e.*"*$f\ (x) \times f\ 00\ (x) < [f\ (x)]\ 2$*"*. yet its convergence is still not guaranteed.

1.2.1 Algorithm

Write out {f(x)}

Calculate {f'(x)}

Choose an initial guess

Find

x_{i+1}=x_{i}-\frac{f(x_{i})}{f'(x_{i})}

Continue iterating till

|\epsilon_{x}^{i}|=|x_{i}-x_{i+1}|

Check if

|\epsilon_{x}^{i}|\leq\epsilon

1.2 Secant Method

This technique [15] is known as the root-finding algorithm. Unlike the Newton Method, it doesn't require a derivative of the functions because the derivative is replaced with the finite difference formula in it. To begin the process of the secant method, we must have two values. The value of the function at the beginning is determined, which provides the two locations on the curve. There is no need for opposite signs for the fundamental importance in the secant method. A new point is achieved when a straight line intersects the x-axis as shown in fig.3. The unique point is then replaced with the old one. This operation continues until the desired root is obtained. In the secant method, the process starts with two initial guesses. Therefore, there is no need to bracket the root. The convergence of the secant method is fastest than the rest of the methods. To calculate zeroes of continuous function for basic computer programs [1] secant methods joins with such a method whose convergence is guaranteed (False Position Method). The disadvantage of secant technique is that the inclined of the secant line gets compact that forces secant line to go away from the points we have. To get the desired convergence, the initial guess must be near to the roots which can cause it to move far from the points you have. For the convergence of the Secant Method, the initial guess must be near to the root. The format of the convergence is "$\alpha, where\ \alpha = 1 + \sqrt{5}\ 2 = 1.618$"

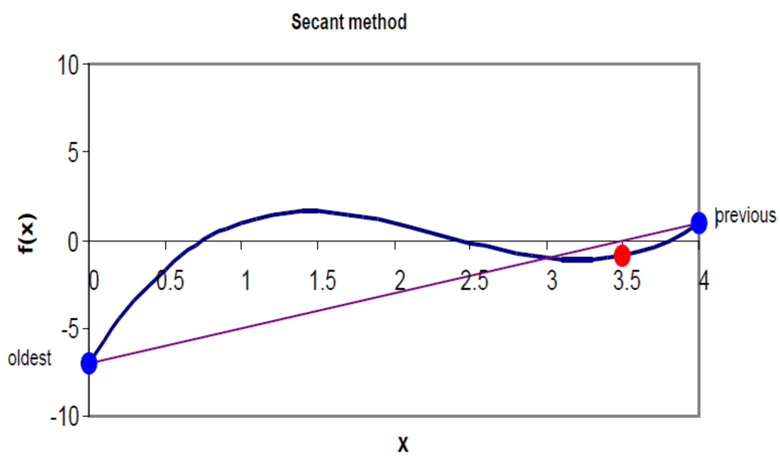

Figure 3: Secant Method

1.3.1 Algorithm

From Newton-Raphson Method,we have

$$x_{n+1}=x_{n}-\frac{f(x_{n})}{f'(x_{n})}$$

We replace the derivative with this formula

$$f'(x_{n})=x_{1}-\frac{f(x_{n}-f(x_{n-1}))}{x_{n}-x_{n-1}}$$

After substituting the value of x_{n} in Newton's Method formula, we obtain

$$x_{n+1}=x_{n}-f(x_{n})\times\frac{x_{n}-x_{n-1}}{f(x_{n}-f(x_{n-1}))}$$

$$x_{n+1}=\frac{x_{n-1}f(x_{n})-x_{n}f(x_{n-1})}{f(x_{n})-f(x_{n-1})}$$

1.2 False Position Method

False Position Method is the technique that integrates the elements of Secant and Bisection Methods [15]. The other name of the False Position Method is Falsi or Interpolation Method. It requires two initial guesses to start the process. The function of the initial guess should not below the zero which shows that the function should have opposite signs. A new value can be achieved by intersecting the chord joining of initial guess and the x-axis as shown in fig.4.

The code of False Position Method is:

$$“xn + 1 = xn - 1f\ (xn) - xnf\ (xn - 1)\ /\ f\ (x)\ n - f\ (xn - 1)$$

The reason behind the discovery of this method [5] was that Bisection Method Converges slowly. So FalsePosition Method is more efficient than the Bisection Method.

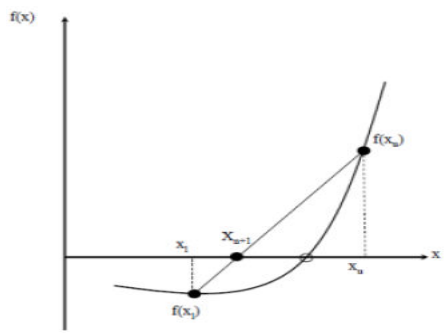

Figure 4: False Position Method [15]

1.4.1 Algorithm

For interval[l,u] first value is calculated

By using this formula

x_{n+1}=\frac{x_{n-1}f(x_{n})-x_{n}f(x_{n-1})}{f(x_{n})-f(x_{n-1})}

Find

f(x_{l})f(x_{u}) and multiply

If

f(x_{l})f(x_{u})<0

then take first half of the new interval

If

f(x_{l})f(x_{u})>0

then take second half as new interval

|\epsilon_{x}^{i}|=|x_{i}-x_{i+1}|

Check if

|\epsilon_{x}^{i}|\leq\epsilon

1.3 Gauss-Seidel Method

This is such an iterative method that is used to solve a non-linear equation system. Figure 5 represents the graphical explanation of this method. The primary step of this method is the initial guess. Then replace the answer within the equation and choose the updated value of the already computed $(x\ k+1\ 1\ , xk+1\ 2\ , \ldots, xk+1\ i\ )\ in\ the\ update\ of\ xi+1\ such\ that\ x\ k+1\ i+1\ =\ gi+1(x\ k+1\ 1\ , xk+1\ 2\ , \ldots, xk+1\ i\ , xk\ i+1, \ldots, xk\ n\ )$

Gauss-Seidel Method always has linear convergence. Programming becomes very easy with this method and it needs a little time to perform. Its hardcoded example is shown in the following program:

1.5.1 PROGRAM

```
% This function applies N iterations of Gauss Seidel method
to the system of non-linear equations Ax = b
% x0 is intial value
tol =0.01
x0=[0 0 0];
N=100; n = length(x0);
x=x0; X=[x0];
for k=1:N %for N iterations
%Consider non-linear systems of equations as follows.
x(1)=1/4*(11-x0(2)^2-x0(3));
x(2)=1/4*(18-x(1)-x0(3)^2);
x(3)=1/4*(15-x(2)-x(1)^2);
% breaking loop if tollerance condition meets
X=[X;x];
if norm(x-x0,inf) < tol
disp('TOLRENCE met in k interations, where')
k=k
break;
end
x0 = x
end if (k==N)
disp('TOLRENCE not met in maximum number of interations')
end
```

The above system has solution:(0.999, 1.999, 2.999).

1.4 Related Work

To figure out the solution of nonlinear equations is indeed a challenging and tough task in the numerical analysis [5]. The nature of the methods used in these solutions is iterative [14]. The

iterative nature assured a solution with already determined accuracy [1]. It has discussed and compared many methods with each other [1] [8] [3]. A remarkable victory has been accomplished in the form of solving a few nonlinear equations with Newton Methods [4].
In this paper, two latest iterative techniques are being introduced, and their analysis with the old techniques was done. As a result, new techniques were found to be more structured and systematic as compared to the Newton-Raphson Method, the Method of Cordero and Torregrosa [9], and the Method of Darvishi and Barati [11]. In high dimensions, the bisection method was formulated. In some conditions, it has expanded to a multi-dimensional Bisection Method towards the solution of abandoned issues [10]. Among many proposed techniques for the solution of the nonlinear equation system, the Gauss-Seidel method was considered as a very efficient model. It has been introduced by [15]. It has been used in multi-dimensions. And mainly worked for network-based problems [12]. They carried out a comparison between Jacobi and Gauss's techniques and showed that the Gauss method was highly good choice than the Jacobi method. By inspiring with this research, we would suggest the use of the Gauss-Seidel Method for the solution of the nonlinear equation system.

2. Comparison between Nonlinear Solvers

Although the Newton method is very famous for solving the nonlinear equations; however, it always needs an initial guess which is near to the predicted solution and its every iteration required specific derivation. It might be possible that a derivative of a few equation's functions is not found. In this situation, the Newton Method is more than worst, to begin with. Researchers found Newton Methods as the most unstable method. Similarly, when we talk about the Secant Method, we found that it is somehow a modified version of a Newton Method. Unlike the Newton Method, it doesn't need derivative at all. However, the convergence in the Secant Method is doubtful. It can fail in the case of flat function. On the other hand, the Bisection Method and False Position Methods are considered as the easiest method to determine the roots of one-dimensional nonlinear equations only. Thus it is not recommended to use these methods in finding the solution of multi-dimensional nonlinear equations. In the end, the best method to solve the nonlinear equation system is Gauss-Seidel Method. It's an easy, convergent, and derivative-free method which does not require any bracket of root to get the solution.

3. Conclusion

This paper presented different methods for the solution of a nonlinear equation system. These methods can also be used for multi-dimensional nonlinear equations as well depending upon the nature of the problem. However, the Gauss-Seidel method is found to be the best method because of its guaranteed convergence and easiness. Therefore, we concluded that the Gauss-Seidel Method is an appropriate method to solve the nonlinear equation system.

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