Lower Bound for $m_3(2,37)$ and Related Code

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Abstract: In a finite projective plane $PG(2, q)$, an (k, n)-arc is a set of k points of a projective plane such that some n, but no $n + 1$ of them, are collinear. Here, the integer n is the degree of the arc and $k \geq n$. The maximum size of an (k,n)-arc in $PG(2,q)$ is denoted by $m_n(2,q)$. In this paper the classification of the $(k,3)$-arcs in $PG(2,37)$ is presented. It has been obtained using a computer-based exhaustive search that exploits Secant distributions inequivalent $(k,3)$-arcs and produces exactly one representative of each equivalence class. We established that $50 \leq m_3(2,37)$. The constructed $(50,3)$-arcs give the respective lower bounds on $m_3(2,37)$. As a consequence there exist new three-dimensional linear codes over $GF(37)$.

Keywords: Finite projective plane, Arcs, Linear codes, computer search.

1. Introduction

Let $GF(q)$ denote the Galois field of q elements and $V(3,q)$ be the vector space of row vectors of length three with entries in $GF(q)$. Let $PG(2,q)$ be the corresponding projective plane. The points (x_0,x_1,x_2) of $PG(2,q)$ are the 1-dimensional subspaces of $V(3,q)$. Subspaces of dimension two are called lines. The number of points and the number of lines in $PG(2,q)$ is q^2+q+1. There are $q + 1$ points on every line and $q + 1$ lines through every point.

Definition 1.1. In a finite projective plane $PG(2,q)$, a (k,n)-arc $\boldsymbol{A}$ is a set of k points in $PG(2,q)$ where no $n + 1$, but some n of the points in $\boldsymbol{A}$, are collinear.

Definition 1.2. The (k,n)-arc $\boldsymbol{A}$ in $PG(2,q)$ is complete if it admits no extension to a larger arc $\acute{\boldsymbol{A}}$ of the same degree. More precisely, the (k,n)-arc $\boldsymbol{A}$ is complete if it cannot be embedded in any $(\acute{k}$,n)-arc $\acute{\boldsymbol{A}}$ with $\acute{k} > k$.

With respect to a (k,n)-arc $\boldsymbol{A}$, the lines of the plane may be classified according to their incidence with $\boldsymbol{A}$. A line l is an i-secant to $\boldsymbol{A}$ if $|\boldsymbol{A} \cap l| = i$ where $0 \leq i \leq n$, denote by τ_i their total number in $PG(2,q)$. In particular, a line which does not meet $\boldsymbol{A}$ in any point of the plane is an external line, a line meeting $\boldsymbol{A}$ in a singleton is a unisecant or tangent line, while lines meeting $\boldsymbol{A}$ in two and three points are bisecants and trisecants respectively. $\boldsymbol{A}$ point Q $\notin \boldsymbol{A}$ is called of index zero if it does not lie on any n-secant of $\boldsymbol{A}$.

Theorem 1.3. ([10]). For a (k,n)-arc K, the following equations hold:

$$\sum_{i=0}^{n} \tau_i = q^2+q+1$$
$$\sum_{i=1}^{n} i\tau_i = k(q+1),$$
$$\sum_{i=2}^{n} \binom{2}{i}\tau_i = \binom{2}{k}.$$
,

Two (k,n)-arcs K_1 and K_2 are said to be secant distributions inequivalent if they have different i-secant distributions.

Let $m_n(2,q)$ denote the maximal number of points for which an (k,n)arc in $PG(2,q)$ exists. The lower bounds on $m_3(2,q)$ for $13 \leq q \leq 37$, as shown in Table 1, are given in [3], [12], [13],[4],[5], [6],[1], [2], [7].

Table 1. Lower bounds on $m_3(2,q)$ for $7 \leq q \leq 37$

n/q	7	8	9	11	13	16	17	19	23	25	27	29	31	37
3	15	17	17	21	23	28	28	31	37	38	42	44	46	–

2. The main results

2.1 The structure of PG(2,37)

Let $K = GF(q)$. It is well known that $PG(2,q)$ has a cyclic Singer group of order $q^2 + q + 1$, the group generated by the companion matrix C_f, namely

$$\mathcal{C}_f = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_0 & a_1 & a_2 \end{pmatrix},$$

where a_0,a_1,a_2 are the coefficients of primitive polynomial $f(x) = x^3 - a_2x^2 - a_1x - a_0$ over K. The order associated to the Singer group is the following

$$P_1 = (1,0,0), P_{i+1} = C_f(P_i) = P_iC_f, i = 1,2,\ldots,q^2 + q.$$

Let $K = GF(37)$. Consider the polynomial $f(x) = x^3 - 2x^2 - 2x - 2$ in the ring $K[x]$. It is clear that $f(x)$ is primitive polynomial over K. In order to present the results in a more concise form, the points in $PG(2,37)$ are in Singer order and each point is associated with its number. For example some of the points in $PG(2,37)$ and their numbers are given as

$$1 := (1,0,0), 2 := (0,1,0), 3 := (0,0,1), 4 := (1,1,1), \ldots, 1407 := (1,1,18).$$

In Table 2 some of the lines in $PG(2,37)$ and their numbers are presented in Singer order.

Table 2. Some lines in PG(2,37)

Number	a_0,a_1,a_2	List of points of the line $a_0x_0 + a_1x_1 + a_2x_2 = 0$
1	1,0,0	2 3 10 82 110 147 172 195 216 222 242 256 406 460 493 548 557 614 772 790 821 866 895 925 937 1124 1126 1215 1254 1292 1307 1374 1393 1406
2	0,1,0	1 3 92 131 169 184 251 270 283 286 287 294 366 394 431 456 479 500 506 526 540 690 744 777 832 841 898 1056 1074 1105 1150 1179 1209 1221 1231 1272 1277 1350
...	...	...
1406	0,1,12	1 74 132 134 223 262 300 315 382 401 414 417 418 425 497 525 562 587 610 631 637 657 671 821 875 908 963 972 1029 1187 1205 1236 1281 1310 1340 1352 1362 1403
1407	1,1,18	144 162 193 238 267 297 309 319 360 365 438 496 498 587 626 664 679 746 765 778 781 782 789 861 889 926 951 974 995 1001 5 1185 1239 1272 1327 1336 1393

2.2 The construction of $(k,3)$-arcs, $4 \le k \le 10$

In this subsection, the classification of (k,3)-arcs is established by classifying inequivalent $(k,3)$-arcs up to i-secant distributions. Let $\mathcal{A}_4^1 = \{1,2,3,4\}$ be the (4,3)-arc consisting of the points $P_1 = (1,0,0)$, $P_2 = (0,1,0)$, $P_3 = (0,0,1)$ and $P_4 = (1,1,1)$. Let $A^j_k(i) = A^j_k \cup \{i\}$. There is only one projectively inequivalent (3,3)-arc and (4,3)-arc in $PG(2,37)$, While there are two types of (5,3)-arcs during implementation the program (See Table 3), they are corresponding to $\tau_3 = 1$ and $\tau_3 = 2$. According to Theorem 1.3, we can calculated the value of 0-secant, 1-secant, and 2-secant for the (5,3)-arcs:

$$\tau_0 + \tau_1 + \tau_2 + \tau_3 = 1407$$

$$\tau_1 + 2\tau_2 + 3\tau_3 = 190$$

$$\tau_2 + 3\tau_3 = 10$$

By adding one point of index zero to each (5,3)-arc, we get (6,3)-arcs. In fact, there are four types of (6,3)-arcs up to secant distributions, as shown in Table 4:

Table 3. Secant distributions inequivalent (5,3)-arcs

$\mathcal{A}_5^j$	$\mathcal{A}_4^j(i)$	(	$(\tau_3,\tau_2,\tau_1,\tau_0)$
$\mathcal{A}_5^1$	$\mathcal{A}_4^1(287)$	(2	(2,4,176,1225)
$\mathcal{A}_5^2$	$\mathcal{A}_4^1(895)$	(1	(1,7,173,1226)

Table 4. Secant distributions inequivalent (6,3)-arcs

$\mathcal{A}_6^j$	$\mathcal{A}_5^j(i)$	(	$(\tau_3,\tau_2,\tau_1,\tau_0)$
$\mathcal{A}_6^1$	$\mathcal{A}_5^1(1126)$	(4	(4,3,210,1190)
$\mathcal{A}_6^2$	$\mathcal{A}_5^2(1253)$	(3	(3,6,207,1191)
$\mathcal{A}_6^3$	$\mathcal{A}_5^2(7)$	(1	(1,12,201,1193)
$\mathcal{A}_6^4$	$\mathcal{A}_5^2(1407)$	(2	(2,9,204,1192)

Let us define $\mathcal{A}_k^j(i:\tau_3)$ to be the new ($k+1,3$)-arc $\mathcal{A}_k^j(i$) together with the correspond number of 3-secant.

There are 1261 points from *PG*(2,37) which are not on any 3-secant of $\mathcal{A}_6^1$. So, by added each one of them to $\mathcal{A}_6^1$, we get (7,3)-arc. Also, the number of points which are not on any 3-secant of $\mathcal{A}_6^2$ is 1296. So, by added each one of them to $\mathcal{A}_6^2$, we get (7,3)-arc. The number of points of index zero that correspond to $\mathcal{A}_6^3$ is 1366, and adding each one of them to $\mathcal{A}_6^3$ gives (7,3)arc. Finally, there are 1331 points of index zero that correspond to $\mathcal{A}_6^4$, and adding each one of them to $\mathcal{A}_6^4$ gives (7,3)-arc. However, there are 6 secant distributions inequivalent (7,3)-arcs as illustrated in Table 5.

Table 5. Secant distributions inequivalent (7,3)-arcs

$\mathcal{A}_7^j$	$\mathcal{A}_6^j(i:\tau_3)$		$\mathcal{A}_7^j$	$\mathcal{A}_6^j(i:\tau_3)$		$\mathcal{A}_7^j$	$\mathcal{A}_6^j(i:\tau_3)$
$A1_7$	$A^1_6(1125:6)$		$A3_7$	$A^3_6(23:1)$		$A5_7$	$A^4_6(1404:3)$
$A2_7$	$A^2_6(1235:5)$		$A4_7$	$A^4_6(1403:2)$		$A6_7$	$A^4_6(1405:4)$

From Table 5, we have six inequivalent *i*-secant distribution. By adding one point of the points of index zero to each one of them, we get seven inequivalent (8,3)-arc (See Table 6).

Table 6. Secant distributions inequivalent (8,3)-arcs

$\mathcal{A}_8^j$	$\mathcal{A}_7^j(i:\tau_3)$		$\mathcal{A}_8^j$	$\mathcal{A}_7^j(i:\tau_3)$
$A1_8$	$A^2_7(1395:7)$		$A5_8$	$A^6_7(1271:6)$
$A2_8$	$A^3_7(1403:1)$		$A6_8$	$A^6_7(1403:4)$
$A3_8$	$A^3_7(45:2)$		$A7_8$	$A^6_7(1404:5)$
$A4_8$	$A^5_7(1402:3)$			

The data of the secant distributions inequivalent (9,3)-arcs, and (10,3)arcs are given in Table 7, and Table 8 respectively.

Table 7. Secant distributions inequivalent (9,3)-arcs

$\mathcal{A}_9^j$	$\mathcal{A}_8^j(i:\tau_3)$	$\mathcal{A}_9^j$	$\mathcal{A}_8^j(i:\tau_3)$	$\mathcal{A}_9^j$	$\mathcal{A}_8^j(i:\tau_3)$
$A1_9$ $A2_9$	$A^1_8(506:10)$	$A5_9$	$A^4_8(1397:3)$	$A9_9$	$A^7_8(1402:5)$
$A3_9$ $A4_9$	$A^1_8(1169:9)$	$A6_9$	$A^5_8(1391:8)$	$A10_9$	$A^7_8(1403:6)$
	$A^2_8(1401:1)$	$A7_9$	$A^6_8(1401:4)$		
	$A^3_8(67:2)$	$A8_9$	$A^7_8(1383:7)$		

Table 8. Secant distributions inequivalent (10,3)-arcs

$\mathcal{A}_{10}^j$	$\mathcal{A}_9^j(i:\tau_3)$	$\mathcal{A}_{10}^j$	$\mathcal{A}_9^j(i:\tau_3)$	$\mathcal{A}_{10}^j$	$\mathcal{A}_9^j(i:\tau_3)$
$A1_{10}$	$A^2_9(549:12)$	$A5_{10}$	$A^4_9(123:3)$	$A9_{10}$	$A^{10}_9(1187:9)$
$A2_{10}$	$A^2_9(1339:11)$	$A6_{10}$	$A^7_9(1396:4)$	$A10_{10}$	$A^{10}_9(1383:8)$
$A3_{10}$	$A^3_9(1396:1)$	$A7_{10}$	$A^8_9(831:10)$	$A11_{10}$	$A^{10}_9(1398:6)$
$A4_{10}$	$A^4_9(1403:2)$	$A8_{10}$	$A^9_9(1397:5)$	$A12_{10}$	$A^{10}_9(1402:7)$

2.3 Secant distributions inequivalent (k,3)-arcs; $11 \leq k \leq 23$

In this subsection, we classify the inequivalent (k,3)-arcs up to i-secant distributions for all value of k, where $11 \leq k \leq 23$. From Table 8, we have 12 inequivalent (10,3)-arcs. We extend these arcs to (11,3)-arcs by adding one point of the points of the index zero to each (10,3)-arcs. The list of the inequivalent (11,3)-arcs, are shown in Table 9.

Table 9. Secant distributions inequivalent (11,3)-arcs

$\mathcal{A}_{11}^j$	$\mathcal{A}_{10}^j(i:\tau_3)$	$\mathcal{A}_{11}^j$	$\mathcal{A}_{10}^j(i:\tau_3)$	$\mathcal{A}_{11}^j$	$\mathcal{A}_{10}^j(i:\tau_3)$
$A1_{11}$	$A^1_{10}(1277:14)$	$A6_{11}$	$A^7_{10}(811:13)$	$A11_{11}$	$A^{12}_{10}(1271:10)$
$A2_{11}$	$A^3_{10}(1391:1)$	$A7_{11}$	$A^8_{10}(1390:5)$	$A12_{11}$	$A^{12}_{10}(1397:7)$
$A3_{11}$	$A^4_{10}(1396:2)$	$A8_{11}$	$A^9_{10}(1188:12)$	$A13_{11}$	$A^{12}_{10}(1398:8)$
$A4_{11}$	$A^5_{10}(134):3$	$A9_{11}$	$A^{10}_{10}(1271:11)$	$A14_{11}$	$A^{12}_{10}(1401:9)$
$A5_{11}$	$A^6_{10}(1381:4)$	$A10_{11}$	$A^{11}_{10}(1396:6)$		

By the same method using in this subsection and Subsection 2.2, we get Table 10 to 21 that illustrate the inequivalent (12,3)-arcs to (23,3)-arcs respectively.

2.4 Secant distributions inequivalent (k,3)-arcs; $24 \leq k \leq 50$

In this subsection, we give the main result of our paper. More precisely, we established that $50 \leq m_3(2,37) \leq 75$ by constructed the (50,3)-arcs. Consequently, we show that there exist new three-dimensional linear codes over $GF(37)$.

Let $\delta_k$denotes the number of A^j_karcs obtained from A^j_{k-1} arcs. Because of the big data, we just give the correspond number of arcs. The notation a.p.i.z in Figure 2.4 means adding points of index zero. All the results in thissubsection are illustrated in Figure 2.4.

Table 10. Secant distributions inequivalent (12

$\mathcal{A}_{12}^j$	$\mathcal{A}_{11}^j(i:\tau_3)$	$\mathcal{A}_{12}^j$	$\mathcal{A}_{11}^j(i:\tau_3)$	$\mathcal{A}_{12}^j$	$\mathcal{A}_{11}^j(i:\tau_3)$
$A1_{12}$	$A^1_{11}(833:17)$	$A7_{12}$	$A^7_{11}(1382:5)$	$A13_{12}$	$A^{14}_{11}(1188:13)$
$A2_{12}$	$A^2_{11}(1388:1)$	$A8_{12}$	$A^8_{11}(1189:15)$	$A14_{12}$	$A^{14}_{11}(1283:12)$
$A3_{12}$	$A^3_{11}(1391:2)$	$A9_{12}$	$A^9_{11}(918:14)$	$A15_{12}$	$A^{14}_{11}(1389:11)$
$A4_{12}$	$A^4_{11}(1402:3)$	$A10_{12}$	$A^{10}_{11}(1389:6)$	$A16_{12}$	$A^{14}_{11}(1396:9)$
$A5_{12}$	$A^4_{11}(265:4)$	$A11_{12}$	$A^{12}_{11}(1379:7)$	$A17_{12}$	$A^{14}_{11}(1397:10)$
$A6_{12}$	$A^6_{11}(462:16)$	$A12_{12}$	$A^{13}_{11}(1396:8)$		

Table 11. Secant distributions inequivalent (13,3)-arcs

$\mathcal{A}^j_{13}$	$\mathcal{A}^j_{12}(i:\tau_3)$	$\mathcal{A}^j_{13}$	$\mathcal{A}^j_{12}(i:\tau_3)$	$\mathcal{A}^j_{13}$	$\mathcal{A}^j_{12}(i:\tau_3)$
$A1_{13}$	$A^{1}_{12}(605:20)$	$A8_{13}$	$A^{8}_{12}(1180:18)$	$A15_{13}$	$A^{16}_{12}(1378:9)$
$A2_{13}$	$A^{2}_{12}(1381:1)$	$A9_{13}$	$A^{9}_{12}(1173:17)$	$A16_{13}$	$A^{17}_{12}(1188:14)$
$A3_{13}$	$A^{3}_{12}(1388:2)$	$A10_{13}$	$A^{10}_{12}(1378:6)$	$A17_{13}$	$A^{17}_{12}(1361:13)$
$A4_{13}$	$A^{4}_{12}(1390:3)$	$A11_{13}$	$A^{11}_{12}(1357:7)$	$A18_{13}$	$A^{17}_{12}(1379:10)$
$A5_{13}$	$A^{5}_{12}(1402:4)$	$A12_{13}$	$A^{12}_{12}(1378:8)$	$A19_{13}$	$A^{17}_{12}(1389:12)$
$A6_{13}$	$A^{5}_{12}(307:5)$	$A13_{13}$	$A^{14}_{12}(1188:16)$	$A20_{13}$	$A^{17}_{12}(1396:11)$
$A7_{13}$	$A^{8}_{12}(1121:19)$	$A14_{13}$	$A^{15}_{12}(1204:15)$		

Table 12. Secant distributions inequivalent (14,3)-arcs

$\mathcal{A}^j_{14}$	$\mathcal{A}^j_{13}(i:\tau_3)$	$\mathcal{A}^j_{14}$	$\mathcal{A}^j_{13}(i:\tau_3)$	$\mathcal{A}^j_{14}$	$\mathcal{A}^j_{13}(i:\tau_3)$
$A1_{14}$	$A^{1}_{13}(1167:23)$	$A9_{14}$	$A^{9}_{13}(1364:20)$	$A17_{14}$	$A^{18}_{13}(1348:10)$
$A2_{14}$	$A^{2}_{13}(1357:1)$	$A10_{14}$	$A^{6}_{13}(344:6)$	$A18_{14}$	$A^{20}_{13}(394:16)$
$A3_{14}$	$A^{3}_{13}(1357:2)$	$A11_{14}$	$A^{11}_{13}(1348:7)$	$A19_{14}$	$A^{20}_{13}(1190:15)$
$A4_{14}$	$A^{4}_{13}(1388:3)$	$A12_{14}$	$A^{12}_{13}(1365:8)$	$A20_{14}$	$A^{20}_{13}(1348:11)$
$A5_{14}$	$A^{5}_{13}(1390:4)$	$A13_{14}$	$A^{14}_{13}(1190:19)$	$A21_{14}$	$A^{20}_{13}(1383:14)$
$A6_{14}$	$A^{6}_{13}(1402:5)$	$A14_{14}$	$A^{15}_{13}(1356:9)$	$A22_{14}$	$A^{20}_{13}(1390:12)$
$A7_{14}$	$A^{8}_{13}(1121:22)$	$A15_{14}$	$A^{17}_{13}(394:18)$	$A23_{14}$	$A^{20}_{13}(1395:13)$
$A8_{14}$	$A^{8}_{13}(1318:21)$	$A16_{14}$	$A^{17}_{13}(1203:17)$		

Our work in this paper is to give the correspond values of $m_3(2,37)$ in Table 1. The next theorem gives the results.

Theorem 2.1 (Main Theorem). *There exist a* $(50,3)$*-arc in* $PG(2,37)$. *It follows that* $50 \leq m_3(2,37) \leq 75$.

Proof. First, we know from [10] that $m_3(2,q) \leq 2q+1$. So, $m_3(2,37) \leq 75$. Secondly, by our method of classification, the set of points (1,0,0), (0,1,0),

(0,0,1), (1,1,1), (0,1,32), (1,26,22), (1,10,8), (1,6,7), (1,15,25), (1,22,4), (1,12,36),(1,33,15), (1,17,6), (1,14,29), (1,5,14), (1,16,17), (1,20,16), (1,29,34), (1,31,32),(1,35,4), (1,34,3), (1,30,27), (1,25,19), (1,2,26), (1,8,10), (1,19,13), (1,18,35), (1,13,23), (1,11,8), (1,9,24), (1,27,31), (1,7,14), (1,26,17), (1,23,5), (1,9,32),(1,1,6), (1,4,10), (1,28,30), (1,10,22), (1,7,11), (1,21,12), (1,18,25), (1,3,20),(1,4,9), (1,22,18), (1,24,33), (1,16,28), (1,30,5), (1,32,21), (1,21,16) forms a(50,3)-arc in $PG(2,37)$ with secant distribution$\tau_3 = 228, \tau_2 = 541, \tau_1 = 134$, and $\tau_0 = 504$.

Table 13. Secant distributions inequivalent (15

$\mathcal{A}^j_{15}$	$\mathcal{A}^j_{14}(i:\tau_3)$	$\mathcal{A}^j_{15}$	$\mathcal{A}^j_{14}(i:\tau_3)$	$\mathcal{A}^j_{15}$	$\mathcal{A}^j_{14}(i:\tau_3)$
$A1_{15}$	$A^{1}_{14}(211:27)$	$A10_{15}$	$A^{10}_{14}(1402:6)$	$A19_{15}$	$A^{20}_{14}(1329:11)$
$A2_{15}$	$A^{1}_{14}(1401:26)$	$A11_{15}$	$A^{10}_{14}(346:7)$	$A20_{15}$	$A^{21}_{14}(394:19)$
$A3_{15}$	$A^{2}_{14}(1342:1)$	$A12_{15}$	$A^{12}_{14}(1332:8)$	$A21_{15}$	$A^{22}_{14}(1348:12)$
$A4_{15}$	$A^{3}_{14}(1331:2)$	$A13_{15}$	$A^{13}_{14}(1188:23)$	$A22_{15}$	$A^{23}_{14}(394:18)$
$A5_{15}$	$A^{4}_{14}(1380:3)$	$A14_{15}$	$A^{14}_{14}(1348:9)$	$A23_{15}$	$A^{23}_{14}(1329:13)$
$A6_{15}$	$A^{5}_{14}(1388:4)$	$A15_{15}$	$A^{15}_{14}(1187:22)$	$A24_{15}$	$A^{23}_{14}(1353:16)$
$A7_{15}$	$A^{6}_{14}(1356:5)$	$A16_{15}$	$A^{16}_{14}(1189:21)$	$A25_{15}$	$A^{23}_{14}(1382:14)$
$A8_{15}$	$A^{8}_{14}(251:25)$	$A17_{15}$	$A^{17}_{14}(1337:10)$	$A26_{15}$	$A^{23}_{14}(1383:17)$
$A9_{15}$	$A^{9}_{14}(1069:24)$	$A18_{15}$	$A^{18}_{14}(1376:20)$	$A27_{15}$	$A^{23}_{14}(1390:15)$

Table 14. Secant distributions inequivalent (16,3)-arcs

$\mathcal{A}^j_{16}$	$\mathcal{A}^j_{15}(i:\tau_3)$	$\mathcal{A}^j_{16}$	$\mathcal{A}^j_{15}(i:\tau_3)$	$\mathcal{A}^j_{16}$	$\mathcal{A}^j_{15}(i:\tau_3)$
$A1_{16}$	$A^{2}_{15}(211:30)$	$A11_{16}$	$A^{11}_{15}(363:8)$	$A21_{16}$	$A^{23}_{15}(1323:13)$
$A2_{16}$	$A^{3}_{15}(1323:1)$	$A12_{16}$	$A^{14}_{15}(1338:9)$	$A22_{16}$	$A^{25}_{15}(1329:14)$
$A3_{16}$	$A^{4}_{15}(1237:2)$	$A13_{16}$	$A^{15}_{15}(185:27)$	$A23_{16}$	$A^{26}_{15}(394:22)$
$A4_{16}$	$A^{5}_{15}(1370:3)$	$A14_{16}$	$A^{15}_{15}(825:26)$	$A24_{16}$	$A^{26}_{15}(1359:21)$
$A5_{16}$	$A^{6}_{15}(1357:4)$	$A15_{16}$	$A^{16}_{15}(1340:25)$	$A25_{16}$	$A^{27}_{15}(1230:20)$
$A6_{16}$	$A^{7}_{15}(1321:5)$	$A16_{16}$	$A^{17}_{15}(1319:10)$	$A26_{16}$	$A^{27}_{15}(1329:15)$
$A7_{16}$	$A^{8}_{15}(930:29)$	$A17_{16}$	$A^{18}_{15}(1281:24)$	$A27_{16}$	$A^{27}_{15}(1381:17)$
$A8_{16}$	$A^{9}_{15}(1273:28)$	$A18_{16}$	$A^{19}_{15}(1223:11)$	$A28_{16}$	$A^{27}_{15}(1382:16)$
$A9_{16}$	$A^{10}_{15}(1303:6)$	$A19_{16}$	$A^{21}_{15}(1329:12)$	$A29_{16}$	$A^{27}_{15}(1383:19)$
$A10_{16}$	$A^{11}_{15}(1402:7)$	$A20_{16}$	$A^{22}_{15}(1069:23)$	$A30_{16}$	$A^{27}_{15}(1389:18)$

Table 15. Secant distributions inequivalent (17,3)-arcs

$\mathcal{A}^j_{17}$	$\mathcal{A}^j_{16}(i:\tau_3)$	$\mathcal{A}^j_{17}$	$\mathcal{A}^j_{16}(i:\tau_3)$	$\mathcal{A}^j_{17}$	$\mathcal{A}^j_{16}(i:\tau_3)$
$A1_{17}$	$A^{1}_{16}(1363:34)$	$A13_{17}$	$A^{14}_{16}(495:31)$	$A25_{17}$	$A^{26}_{16}(1193:15)$
$A2_{17}$	$A^{2}_{16}(1280:1)$	$A14_{17}$	$A^{14}_{16}(1238:30)$	$A26_{17}$	$A^{28}_{16}(1329:16)$
$A3_{17}$	$A^{3}_{16}(1166:2)$	$A15_{17}$	$A^{15}_{16}(1338:29)$	$A27_{17}$	$A^{28}_{16}(1348:17)$
$A4_{17}$	$A^{4}_{16}(1357:3)$	$A16_{17}$	$A^{16}_{16}(1316:10)$	$A28_{17}$	$A^{30}_{16}(1230:24)$
$A5_{17}$	$A^{5}_{16}(1356:4)$	$A17_{17}$	$A^{18}_{16}(1193:11)$	$A29_{17}$	$A^{30}_{16}(1329:18)$
$A6_{17}$	$A^{6}_{16}(1262:5)$	$A18_{17}$	$A^{19}_{16}(1193:12)$	$A30_{17}$	$A^{30}_{16}(1353:22)$
$A7_{17}$	$A^{7}_{16}(1091:33)$	$A19_{17}$	$A^{20}_{16}(457:28)$	$A31_{17}$	$A^{30}_{16}(1377:21)$
$A8_{17}$	$A^{8}_{16}(1158:32)$	$A20_{17}$	$A^{21}_{16}(1193:13)$	$A32_{17}$	$A^{30}_{16}(1381:20)$
$A9_{17}$	$A^{9}_{16}(1265:6)$	$A21_{17}$	$A^{22}_{16}(1193:14)$	$A33_{17}$	$A^{30}_{16}(1382:19)$
$A10_{17}$	$A^{10}_{16}(1303:7)$	$A22_{17}$	$A^{23}_{16}(1069:27)$	$A34_{17}$	$A^{30}_{16}(1383:23)$
$A11_{17}$	$A^{11}_{16}(1402:8)$	$A23_{17}$	$A^{24}_{16}(554:26)$		
$A12_{17}$	$A^{11}_{16}(368:9)$	$A24_{17}$	$A^{25}_{16}(963:25)$		

Table 16. Secant distributions inequivalent (18

$\mathcal{A}^j_{18}$	$\mathcal{A}^j_{17}(i:\tau_3)$	$\mathcal{A}^j_{18}$	$\mathcal{A}^j_{17}(i:\tau_3)$	$\mathcal{A}^j_{18}$	$\mathcal{A}^j_{17}(i:\tau_3)$
$A1_{18}$	$A^{1}_{17}(50:38)$	$A14_{18}$	$A^{14}_{17}(495:35)$	$A27_{18}$	$A^{28}_{17}(963:30)$
$A2_{18}$	$A^{1}_{17}(439:39)$	$A15_{18}$	$A^{15}_{17}(1158:34)$	$A28_{18}$	$A^{29}_{17}(1107:18)$
$A3_{18}$	$A^{2}_{17}(1148:1)$	$A16_{18}$	$A^{12}_{17}(470:10)$	$A29_{18}$	$A^{33}_{17}(1329:19)$
$A4_{18}$	$A^{3}_{17}(829:2)$	$A17_{18}$	$A^{17}_{17}(1014:11)$	$A30_{18}$	$A^{33}_{17}(1348:20)$
$A5_{18}$	$A^{4}_{17}(1052:3)$	$A18_{18}$	$A^{18}_{17}(1014:12)$	$A31_{18}$	$A^{33}_{17}(1350:21)$
$A6_{18}$	$A^{5}_{17}(1186:4)$	$A19_{18}$	$A^{19}_{17}(519:33)$	$A32_{18}$	$A^{33}_{17}(1381:22)$
$A7_{18}$	$A^{6}_{17}(1140:5)$	$A20_{18}$	$A^{20}_{17}(1138:13)$	$A33_{18}$	$A^{34}_{17}(963:29)$
$A8_{18}$	$A^{7}_{17}(1142:37)$	$A21_{18}$	$A^{21}_{17}(1138:14)$	$A34_{18}$	$A^{34}_{17}(1271:28)$
$A9_{18}$	$A^{9}_{17}(1245:6)$	$A22_{18}$	$A^{22}_{17}(1271:32)$	$A35_{18}$	$A^{34}_{17}(1329:23)$
$A10_{18}$	$A^{10}_{17}(1245:7)$	$A23_{18}$	$A^{24}_{17}(394:31)$	$A36_{18}$	$A^{34}_{17}(1348:24)$
$A11_{18}$	$A^{11}_{17}(1303:8)$	$A24_{18}$	$A^{25}_{17}(1138:15)$	$A37_{18}$	$A^{34}_{17}(1358:26)$
$A12_{18}$	$A^{12}_{17}(1303:9)$	$A25_{18}$	$A^{26}_{17}(1193:16)$	$A38_{18}$	$A^{34}_{17}(1377:27)$
$A13_{18}$	$A^{13}_{17}(185:36)$	$A26_{18}$	$A^{27}_{17}(1329:17)$	$A39_{18}$	$A^{34}_{17}(1382:25)$

Table 17. Secant distributions inequivalent (19,3)-arcs

$\mathcal{A}_{19}^{j}$	$\mathcal{A}_{18}^{j}(i:\tau_3)$	$\mathcal{A}_{19}^{j}$	$\mathcal{A}_{18}^{j}(i:\tau_3)$	$\mathcal{A}_{19}^{j}$	$\mathcal{A}_{18}^{j}(i:\tau_3)$
$A1_{19}$	$A^{2}_{18}(50:44)$	$A16_{19}$	$A^{16}_{18}(1142:10)$	$A31_{19}$	$A^{32}_{18}(1329:22)$
$A2_{19}$	$A^{2}_{18}(1198:43)$	$A17_{19}$	$A^{17}_{18}(828:11)$	$A32_{19}$	$A^{33}_{18}(898:35)$
$A3_{19}$	$A^{2}_{18}(1381:42)$	$A18_{19}$	$A^{18}_{18}(800:12)$	$A33_{19}$	$A^{34}_{18}(963:34)$
$A4_{19}$	$A^{3}_{18}(656:1)$	$A19_{19}$	$A^{19}_{18}(1244:38)$	$A34_{19}$	$A^{35}_{18}(1107:23)$
$A5_{19}$	$A^{4}_{18}(689:2)$	$A20_{19}$	$A^{16}_{18}(563:13)$	$A35_{19}$	$A^{36}_{18}(1329:24)$
$A6_{19}$	$A^{5}_{18}(428:3)$	$A21_{19}$	$A^{21}_{18}(967:14)$	$A36_{19}$	$A^{38}_{18}(564:33)$
$A7_{19}$	$A^{6}_{18}(392:4)$	$A22_{19}$	$A^{23}_{18}(179:37)$	$A37_{19}$	$A^{38}_{18}(1229:32)$
$A8_{19}$	$A^{7}_{18}(605:5)$	$A23_{19}$	$A^{21}_{18}(1014:15)$	$A38_{19}$	$A^{39}_{18}(963:31)$
$A9_{19}$	$A^{9}_{18}(1011:6)$	$A24_{19}$	$A^{25}_{18}(1138:16)$	$A39_{19}$	$A^{39}_{18}(1329:25)$
$A10_{19}$	$A^{10}_{18}(688:7)$	$A25_{19}$	$A^{26}_{18}(1193:17)$	$A40_{19}$	$A^{39}_{18}(1348:26)$
$A11_{19}$	$A^{11}_{18}(1005:8)$	$A26_{19}$	$A^{27}_{18}(394:36)$	$A41_{19}$	$A^{39}_{18}(1350:27)$
$A12_{19}$	$A^{12}_{18}(554:9)$	$A27_{19}$	$A^{28}_{18}(960:18)$	$A42_{19}$	$A^{39}_{18}(1363:30)$
$A13_{19}$	$A^{13}_{18}(604:41)$	$A28_{19}$	$A^{29}_{18}(797:19)$	$A43_{19}$	$A^{39}_{18}(1377:29)$
$A14_{19}$	$A^{13}_{18}(1237:40)$	$A29_{19}$	$A^{30}_{18}(1329:20)$	$A44_{19}$	$A^{39}_{18}(1381:28)$
$A15_{19}$	$A^{15}_{18}(744:39)$	$A30_{19}$	$A^{31}_{18}(1011:21)$		

2.5 The related linear codes

A linear $[n,k,d]$-code C over $GF(q)$ is a k-dimensional subspace of the $n$dimensional vector space $GF(q)^n$ with minimum distance d. The Hamming distance between to codewords $\mathbf{x},\mathbf{y} \in GF(q)^n$, denoted $d(\mathbf{x},\mathbf{y})$ is the number

Table 18. Secant distributions inequivalent (20

$\mathcal{A}_{20}^{j}$	$\mathcal{A}_{19}^{j}(i:\tau_3)$	$\mathcal{A}_{20}^{j}$	$\mathcal{A}_{19}^{j}(i:\tau_3)$	$\mathcal{A}_{20}^{j}$	$\mathcal{A}_{19}^{j}(i:\tau_3)$
$A1_{20}$	$A^{2}_{19}(50:48)$	$A17_{20}$	$A^{19}_{19}(1187:43)$	$A33_{20}$	$A^{36}_{19}(899:39)$
$A2_{20}$	$A^{3}_{19}(1198:47)$	$A18_{20}$	$A^{20}_{19}(1140:13)$	$A34_{20}$	$A^{38}_{19}(360:38)$
$A3_{20}$	$A^{4}_{19}(1094:2)$	$A19_{20}$	$A^{21}_{19}(350:14)$	$A35_{20}$	$A^{39}_{19}(797:25)$
$A4_{20}$	$A^{6}_{19}(94:3)$	$A20_{20}$	$A^{23}_{19}(830:15)$	$A36_{20}$	$A^{40}_{19}(1329:26)$
$A5_{20}$	$A^{6}_{19}(1237:4)$	$A21_{20}$	$A^{20}_{19}(575:16)$	$A37_{20}$	$A^{41}_{19}(1011:27)$
$A6_{20}$	$A^{7}_{19}(1404:5)$	$A22_{20}$	$A^{25}_{19}(800:17)$	$A38_{20}$	$A^{42}_{19}(874:36)$
$A7_{20}$	$A^{8}_{19}(1170:6)$	$A23_{20}$	$A^{26}_{19}(1079:42)$	$A39_{20}$	$A^{42}_{19}(963:37)$
$A8_{20}$	$A^{9}_{19}(1390:7)$	$A24_{20}$	$A^{27}_{19}(623:18)$	$A40_{20}$	$A^{43}_{19}(1229:35)$
$A9_{20}$	$A^{10}_{19}(1404:8)$	$A25_{20}$	$A^{28}_{19}(352:19)$	$A41_{20}$	$A^{44}_{19}(1204:34)$
$A10_{20}$	$A^{11}_{19}(1321:9)$	$A26_{20}$	$A^{29}_{19}(797:20)$	$A42_{20}$	$A^{44}_{19}(1329:28)$
$A11_{20}$	$A^{14}_{19}(604:46)$	$A27_{20}$	$A^{30}_{19}(1010:21)$	$A43_{20}$	$A^{44}_{19}(1343:31)$
$A12_{20}$	$A^{14}_{19}(954:45)$	$A28_{20}$	$A^{31}_{19}(797:22)$	$A44_{20}$	$A^{44}_{19}(1348:29)$
$A13_{20}$	$A^{16}_{19}(1042:10)$	$A29_{20}$	$A^{32}_{19}(563:41)$	$A45_{20}$	$A^{44}_{19}(1350:30)$
$A14_{20}$	$A^{17}_{19}(732:11)$	$A30_{20}$	$A^{33}_{19}(898:40)$	$A46_{20}$	$A^{44}_{19}(1363:33)$
$A15_{20}$	$A^{18}_{19}(529:12)$	$A31_{20}$	$A^{34}_{19}(663:23)$	$A47_{20}$	$A^{44}_{19}(1377:32)$
$A16_{20}$	$A^{18}_{19}(360:44)$	$A32_{20}$	$A^{35}_{19}(1107:24)$		

Table 19. Secant distributions inequivalent (21,3)-arcs

$\mathcal{A}^j_{21}$	$\mathcal{A}^j_{20}(i:\tau_3)$	$\mathcal{A}^j_{21}$	$\mathcal{A}^j_{20}(i:\tau_3)$	$\mathcal{A}^j_{21}$	$\mathcal{A}^j_{20}(i:\tau_3)$
$A1_{21}$	$A^{1}_{20}(973:53)$	$A18_{21}$	$A^{20}_{20}(800:15)$	$A35_{21}$	$A^{39}_{20}(360:44)$
$A2_{21}$	$A^{3}_{20}(390:3)$	$A19_{21}$	$A^{21}_{20}(1140:16)$	$A36_{21}$	$A^{39}_{20}(850:43)$
$A3_{21}$	$A^{4}_{20}(418:4)$	$A20_{21}$	$A^{22}_{20}(529:17)$	$A37_{21}$	$A^{39}_{20}(1326:42)$
$A4_{21}$	$A^{5}_{20}(751:5)$	$A21_{21}$	$A^{21}_{20}(604:18)$	$A38_{21}$	$A^{40}_{20}(1326:41)$
$A5_{21}$	$A^{6}_{20}(1385:6)$	$A22_{21}$	$A^{23}_{20}(527:48)$	$A39_{21}$	$A^{42}_{20}(797:28)$
$A6_{21}$	$A^{7}_{20}(709:7)$	$A23_{21}$	$A^{23}_{20}(1191:47)$	$A40_{21}$	$A^{44}_{20}(1329:29)$
$A7_{21}$	$A^{8}_{20}(1028:8)$	$A24_{21}$	$A^{24}_{20}(1132:19)$	$A41_{21}$	$A^{45}_{20}(1010:30)$
$A8_{21}$	$A^{9}_{20}(1186:9)$	$A25_{21}$	$A^{26}_{20}(352:20)$	$A42_{21}$	$A^{45}_{20}(1348:31)$
$A9_{21}$	$A^{10}_{20}(1156:10)$	$A26_{21}$	$A^{27}_{20}(722:21)$	$A43_{21}$	$A^{46}_{20}(963:40)$
$A10_{21}$	$A^{11}_{20}(489:51)$	$A27_{21}$	$A^{27}_{20}(1266:22)$	$A44_{21}$	$A^{46}_{20}(1326:39)$
$A11_{21}$	$A^{12}_{20}(604:52)$	$A28_{21}$	$A^{30}_{20}(834:46)$	$A45_{21}$	$A^{47}_{20}(1101:32)$
$A12_{21}$	$A^{12}_{20}(805:50)$	$A29_{21}$	$A^{31}_{20}(33:23)$	$A46_{21}$	$A^{47}_{20}(1329:33)$
$A13_{21}$	$A^{14}_{20}(517:11)$	$A30_{21}$	$A^{32}_{20}(634:24)$	$A47_{21}$	$A^{47}_{20}(1336:38)$
$A14_{21}$	$A^{14}_{20}(1135:12)$	$A31_{21}$	$A^{33}_{20}(281:45)$	$A48_{21}$	$A^{47}_{20}(1343:35)$
$A15_{21}$	$A^{17}_{20}(416:49)$	$A32_{21}$	$A^{35}_{20}(33:25)$	$A49_{21}$	$A^{47}_{20}(1348:34)$
$A16_{21}$	$A^{18}_{20}(733:13)$	$A33_{21}$	$A^{36}_{20}(797:26)$	$A50_{21}$	$A^{47}_{20}(1357:36)$
$A17_{21}$	$A^{18}_{20}(1252:14)$	$A34_{21}$	$A^{37}_{20}(1010:27)$	$A51_{21}$	$A^{47}_{20}(1363:37)$

ofpostions in which x_i 6= y_i, for $\mathbf{x} = (x_1,...,x_n)$ and $\mathbf{y} = (y_1,...,y_n)$.

Table 20. Secant distributions inequivalent (22

$\mathcal{A}^j_{22}$	$\mathcal{A}^j_{21}(i:\tau_3)$	$\mathcal{A}^j_{22}$	$\mathcal{A}^j_{21}(i:\tau_3)$	$\mathcal{A}^j_{22}$	$\mathcal{A}^j_{21}(i:\tau_3)$
$A1_{22}$	$A^{3}_{21}(247:5)$	$A19_{22}$	$A^{21}_{21}(1367:19)$	$A37_{22}$	$A^{41}_{21}(730:30)$
$A2_{22}$	$A^{4}_{21}(555:6)$	$A20_{22}$	$A^{22}_{21}(713:54)$	$A38_{22}$	$A^{42}_{21}(804:31)$
$A3_{22}$	$A^{5}_{21}(598:7)$	$A21_{22}$	$A^{23}_{21}(656:53)$	$A39_{22}$	$A^{43}_{21}(360:47)$
$A4_{22}$	$A^{5}_{21}(1347:8)$	$A22_{22}$	$A^{21}_{21}(688:20)$	$A40_{22}$	$A^{44}_{21}(963:46)$
$A5_{22}$	$A^{7}_{21}(354:9)$	$A23_{22}$	$A^{26}_{21}(42:21)$	$A41_{22}$	$A^{45}_{21}(516:32)$
$A6_{22}$	$A^{8}_{21}(1042:10)$	$A24_{22}$	$A^{26}_{21}(1131:22)$	$A42_{22}$	$A^{46}_{21}(797:33)$
$A7_{22}$	$A^{9}_{21}(301:11)$	$A25_{22}$	$A^{27}_{21}(731:23)$	$A43_{22}$	$A^{47}_{21}(1070:45)$
$A8_{22}$	$A^{9}_{21}(1407:12)$	$A26_{22}$	$A^{28}_{21}(965:52)$	$A44_{22}$	$A^{49}_{21}(804:34)$
$A9_{22}$	$A^{11}_{21}(564:58)$	$A27_{22}$	$A^{30}_{21}(88:24)$	$A45_{22}$	$A^{49}_{21}(1329:35)$
$A10_{22}$	$A^{11}_{21}(1390:56)$	$A28_{22}$	$A^{30}_{21}(1332:25)$	$A46_{22}$	$A^{50}_{21}(1101:36)$
$A11_{22}$	$A^{12}_{21}(604:57)$	$A29_{22}$	$A^{33}_{21}(33:26)$	$A47_{22}$	$A^{51}_{21}(1010:37)$
$A12_{22}$	$A^{14}_{21}(517:13)$	$A30_{22}$	$A^{34}_{21}(42:27)$	$A48_{22}$	$A^{51}_{21}(1159:43)$
$A13_{22}$	$A^{15}_{21}(997:55)$	$A31_{22}$	$A^{34}_{21}(1266:28)$	$A49_{22}$	$A^{51}_{21}(1326:44)$
$A14_{22}$	$A^{17}_{21}(733:14)$	$A32_{22}$	$A^{35}_{21}(721:51)$	$A50_{22}$	$A^{51}_{21}(1329:38)$
$A15_{22}$	$A^{17}_{21}(1069:15)$	$A33_{22}$	$A^{36}_{21}(281:49)$	$A51_{22}$	$A^{51}_{21}(1344:40)$
$A16_{22}$	$A^{19}_{21}(733:16)$	$A34_{22}$	$A^{36}_{21}(360:50)$	$A52_{22}$	$A^{51}_{21}(1348:39)$
$A17_{22}$	$A^{19}_{21}(1252:17)$	$A35_{22}$	$A^{37}_{21}(850:48)$	$A53_{22}$	$A^{51}_{21}(1352:42)$
$A18_{22}$	$A^{21}_{21}(1140:18)$	$A36_{22}$	$A^{40}_{21}(797:29)$	$A54_{22}$	$A^{51}_{21}(1355:41)$

A central problem in coding theory is that of optimizing one of the parameters $n,k$and d for given values of the other two and q-fixed. There are two versions introduced in [9], namely

1. Find $d_q(n,k)$, the largest value of d for which there exists an $[n,k,d]_q$code.
2. Find $n_q(k,d)$, the smallest value of n for which there exists an $[n,k,d]_q$code.

A code which achieves one of these two values is called d-optimal or $n$optimal respectively. The well-known lower bound for $n_q(k,d)$ is the Griesmer bound [8], [16]

$$n_q(k,d) \geq g_q(k,d) = \sum_{j=0}^{k-1} \lceil \frac{d}{q^j} \rceil$$

($\lceil x \rceil$ denotes the smallest integer $\geq x$). Codes with parameters $[g_q(k,d),k,d]_q$, are called Griesmer codes.

Theorem 2.2 (Griesmer Bound [9]). ***Let C be a linear*** $[n,k,d]$***-code over*** *$GF(q)$. Then we must have that* $n_q(k,d) \geq \sum_{j=0}^{k-1} \lceil \frac{d}{q^j} \rceil$.

In [9], we see that $n_q(k,d) = g_q(k,d)$ for all d when k = 1 or 2. The problem of finding $n_q(k,d)$ for all d has been solved only in the next cases

(See [14], [15]):

- $k \leq 8$ for codes over $GF(2)$,

Table 21. Secant distributions inequivalent (23

$\mathcal{A}^j_{23}$	$\mathcal{A}^j_{22}(i:\tau_3)$	$\mathcal{A}^j_{23}$	$\mathcal{A}^j_{22}(i:\tau_3)$	$\mathcal{A}^j_{23}$	$\mathcal{A}^j_{22}(i:\tau_3)$
$A1_{23}$	$A^{1}_{22}(752:7)$	$A21_{23}$	$A^{22}_{22}(729:22)$	$A41_{23}$	$A^{43}_{22}(1345:51)$
$A2_{23}$	$A^{2}_{22}(1172:8)$	$A22_{23}$	$A^{24}_{22}(835:23)$	$A42_{23}$	$A^{44}_{22}(730:34)$
$A3_{23}$	$A^{3}_{22}(1003:9)$	$A23_{23}$	$A^{25}_{22}(730:24)$	$A43_{23}$	$A^{45}_{22}(797:35)$
$A4_{23}$	$A^{4}_{22}(1321:10)$	$A24_{23}$	$A^{26}_{22}(314:59)$	$A44_{23}$	$A^{46}_{22}(516:36)$
$A5_{23}$	$A^{5}_{22}(1182:11)$	$A25_{23}$	$A^{26}_{22}(1096:58)$	$A45_{23}$	$A^{47}_{22}(730:37)$
$A6_{23}$	$A^{6}_{22}(1166:12)$	$A26_{23}$	$A^{28}_{22}(88:25)$	$A46_{23}$	$A^{47}_{22}(1137:38)$
$A7_{23}$	$A^{8}_{22}(301:13)$	$A27_{23}$	$A^{28}_{22}(1094:26)$	$A47_{23}$	$A^{49}_{22}(1337:50)$
$A8_{23}$	$A^{8}_{22}(1265:14)$	$A28_{23}$	$A^{29}_{22}(1004:27)$	$A48_{23}$	$A^{52}_{22}(804:39)$
$A9_{23}$	$A^{9}_{22}(732:64)$	$A29_{23}$	$A^{30}_{22}(1047:28)$	$A49_{23}$	$A^{52}_{22}(1329:40)$
$A10_{23}$	$A^{11}_{22}(564:63)$	$A30_{23}$	$A^{31}_{22}(731:29)$	$A50_{23}$	$A^{53}_{22}(1326:49)$
$A11_{23}$	$A^{11}_{22}(1390:62)$	$A31_{23}$	$A^{32}_{22}(1098:57)$	$A51_{23}$	$A^{54}_{22}(1010:41)$
$A12_{23}$	$A^{13}_{22}(1027:61)$	$A32_{23}$	$A^{34}_{22}(1013:56)$	$A52_{23}$	$A^{54}_{22}(1070:48)$
$A13_{23}$	$A^{15}_{22}(733:5)$	$A33_{23}$	$A^{35}_{22}(281:55)$	$A53_{23}$	$A^{54}_{22}(1329:42)$
$A14_{23}$	$A^{15}_{22}(1042:16)$	$A34_{23}$	$A^{36}_{22}(848:30)$	$A54_{23}$	$A^{54}_{22}(1338:45)$
$A15_{23}$	$A^{17}_{22}(733:17)$	$A35_{23}$	$A^{38}_{22}(730:31)$	$A55_{23}$	$A^{54}_{22}(1344:44)$
$A16_{23}$	$A^{18}_{22}(733:18)$	$A36_{23}$	$A^{38}_{22}(828:32)$	$A56_{23}$	$A^{54}_{22}(1348:43)$
$A17_{23}$	$A^{18}_{22}(1069:19)$	$A37_{23}$	$A^{39}_{22}(1043:54)$	$A57_{23}$	$A^{54}_{22}(1351:46)$
$A18_{23}$	$A^{20}_{22}(704:60)$	$A38_{23}$	$A^{39}_{22}(1070:53)$	$A58_{23}$	$A^{54}_{22}(1352:47)$
$A19_{23}$	$A^{22}_{22}(1042:20)$	$A39_{23}$	$A^{41}_{22}(1093:33)$		
$A20_{23}$	$A^{22}_{22}(1140:21)$	$A40_{23}$	$A^{43}_{22}(1229:52)$		

- $k \leq 5$ for codes over $GF(3)$,
- $k \leq 4$ for codes over $GF(4)$,
- $k = 3$ for codes over $GF(q), 5 \leq q \leq 9$.

Thus, in the case of three-dimensional codes the problem remains open when $q \geq 11$. It is well known that there exists a projective $[n,3,d]_q$ code if and only if there exists an $(n,n-d)$-arc in $PG(2,q)$ (See [9]).

Theorem 2.3 ([9]). *There exist a projective* $[n,3,d]_q$*-code if and only if there exist an* $(n,n-d)$*-arc in* $PG(2,q)$.

Consequently, we get our next corollary.

Corollary 2.4. *There exist Griesmer codes with parameters* $[50,3,47]_{37}$.

Proof. From Theorem 2.1 and Theorem 2.3 we get the results.

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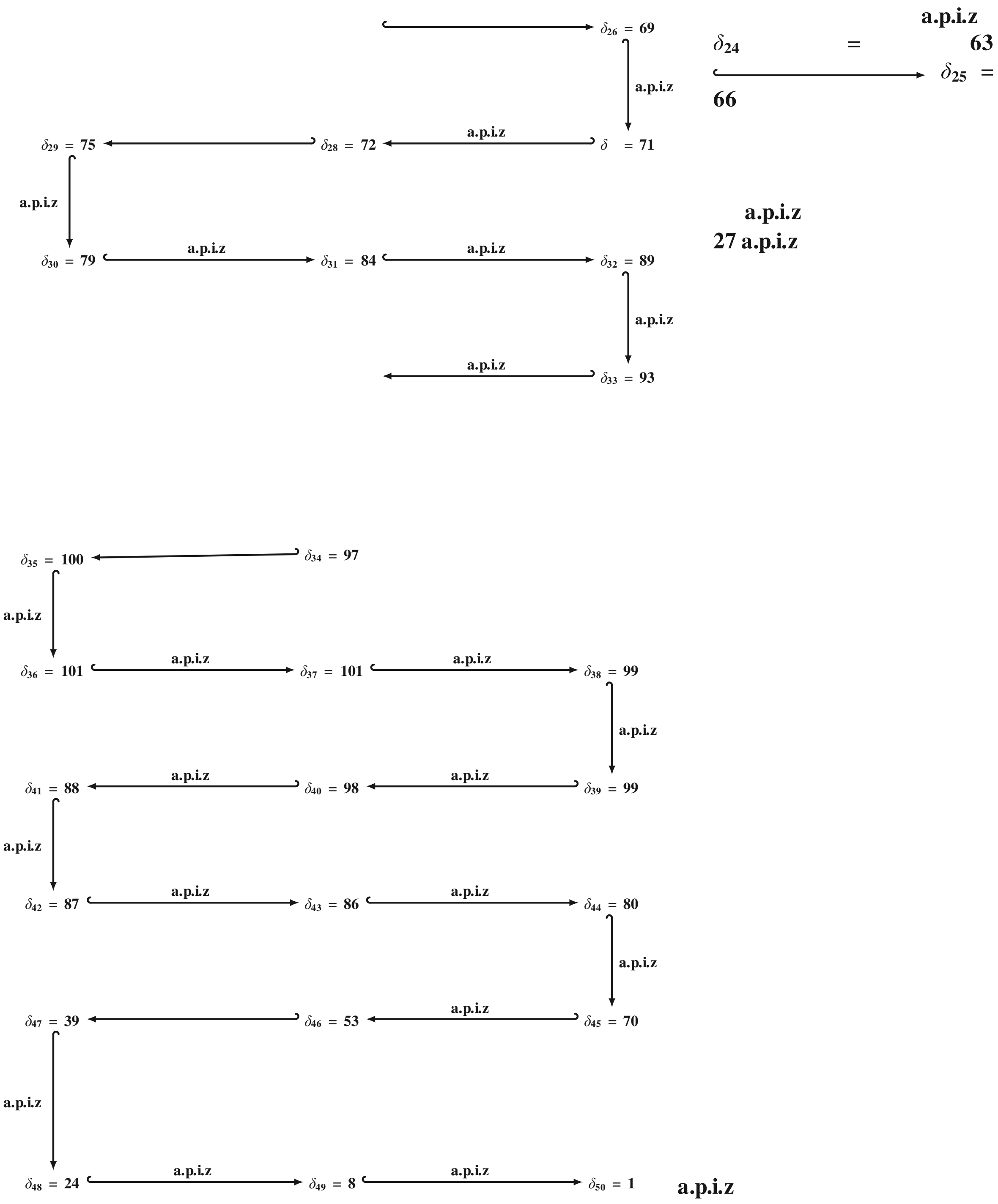

Figure 1. The number of Secant distributions inequivalent $\mathcal{A}_k^j$ arcs, $24 \leq k \leq 50$

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