The Exponential Generating Functions of Mersenne and Mersenne-Lucas Identities

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Abstract: In this paper, we presented some relations connecting Mersenne and Mersenne-Lucas sequences by applying exponential generating functions.

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1. Introduction

Generating Function is a most powerful application in discrete mathematics and which is used to operate the sequences efficiently. **(Carlitz, L. et al.,1969)** Generating function and its characterization are presented. **(Khoshy.T.2001)** Applications of Fibonacci and Lucas numbers are exhibited. Fibonacci and Lucas identities are established by using exponential generating function **(Church.C.A. & Marjorie Bicknell 1973)**. Generalized Mersenne numbers, properties and its generating functions and so on are investigated **(Ali Boussayoud, Mourad Chelgham & Souhila Boughaba et al.,2018).**

In this communication, we analyze some properties relating Mersenne and Mersenne-Lucas sequences by using exponential generating functions.

Filtering of integers gives some interesting results in Number Theory. In this way, we define Mersenne numbers by the recurrence relation

$$M_n = 3M_{n-1} - 2M_{n-2};\ \ M_0 = 0, M_1 = 1$$

and Mersenne-Lucas numbers by the recurrence relation

$$ML_n = 3ML_{n-1} - 2ML_{n-2};\ \ ML_0 = 2, ML_1 = 3 \text{ for } n \geq 2.$$

The characteristic equation of these recurrence relations are $x^2 - 3x + 2 = 0$ with $\alpha = 2, \beta = 1$.

The Binet formulas for Mersenne and Mersenne-Lucas numbers are

$M_n = \alpha^n - \beta^n$ and $ML_n = \alpha^n + \beta^n$

The ordinary generating functions for these sequences are

$$\sum_{k=0}^{\infty} M_k x^k = \frac{x}{1-3x+2x^2}$$

$$\sum_{k=0}^{\infty} ML_k x^k = \frac{2-3x}{1-3x+2x^2}$$

By using the expansion of Maclaurin series of the exponential function, we have

$$e^{\alpha t} = 1 + \frac{\alpha t}{1!} + \frac{(\alpha t)^2}{2!} + \frac{(\alpha t)^3}{3!} + \cdots + \frac{(\alpha t)^n}{n!} + \cdots$$

$$e^{\alpha t} - e^{\beta t} = (1-1) + \frac{(\alpha-\beta)t}{1!} + \frac{(\alpha^2-\beta^2)t^2}{2!} + \frac{(\alpha^3-\beta^3)t^3}{3!} + \cdots + \frac{(\alpha^n-\beta^n)t^n}{n!} + \cdots$$

$$e^{\alpha t} + e^{\beta t} = (1+1) + \frac{(\alpha+\beta)t}{1!} + \frac{(\alpha^2+\beta^2)t^2}{2!} + \frac{(\alpha^3+\beta^3)t^3}{3!} + \cdots + \frac{(\alpha^n+\beta^n)t^n}{n!} + \cdots$$

We obtain the exponential generating function as

$$e^{\alpha t} - e^{\beta t} = \sum_{n=0}^{\infty} M_n \frac{t^n}{n!}$$

$$e^{\alpha t}+e^{\beta t}=\sum_{n=0}^{\infty} ML_n \frac{t^n}{n!}$$

2. Properties

$$\sum_{k=0}^{n}\binom{n}{k} M_k = 3^n-2^n$$

$$\sum_{k=0}^{n}\binom{n}{k}(-2)^{n-1} M_k = (3^n-2^n)(-2)^{n-1}$$

$$\sum_{k=0}^{n}\binom{n}{k} M_k ML_{n-k} = 2^n M_n$$

Lemma. Let $A(t)=\sum_{n=0}^{\infty} a_n \frac{t^n}{n!}$ and $B(t)=\sum_{n=0}^{\infty} b_n \frac{t^n}{n!}$ then

$A(t)B(t)=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k} a_k b_{n-k}\right]\frac{t^n}{n!}$ and $A(t)B(-t)=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k} a_k b_{n-k}\right]\frac{t^n}{n!}$

3. Exponential Generating Functions for Mersenne Identities

The characteristic equation of Mersenne and Mersenne Lucas numbers are $x^2-3x+2=0$, with roots $\alpha=2,\beta=1$.

$\therefore\ \alpha^2=3\alpha-2$ and $\beta^2=3\beta-2$.

Theorem 1. For n positive integers

$$\sum_{k=0}^{n}\binom{n}{k} 3^k(-2)^{n-k} M_k = M_{2n}$$

$$\sum_{k=0}^{n}\binom{n}{k} 3^k(-2)^{n-k} ML_k = ML_{2n}$$

Proof. Let $A(t)=e^{3\alpha t}-e^{3\beta t}=\sum_{n=0}^{\infty} 3^n M_n \frac{t^n}{n!}$ and $B(t)=e^{-2t}=\sum_{n=0}^{\infty}(-2)^n \frac{t^n}{n!}$

$A(t)B(t)=(e^{3\alpha t}-e^{3\beta t})\,(e^{-2t})$

$=\left[e^{(3\alpha-2)t}-e^{(3\beta-2)t}\right]=e^{\alpha^2 t}-e^{\beta^2 t}$

$=\sum_{n=0}^{\infty} M_{2n}\frac{t^n}{n!}$

By using Lemma, we have

$$A(t)B(t)=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k} 3^k(\alpha^k-\beta^k)(-2)^{n-k}\right]\frac{t^n}{n!}$$

$=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k} 3^k M_k(-2)^{n-k}\right]\frac{t^n}{n!}$

$$\therefore \sum_{k=0}^{n}\binom{n}{k} 3^k M_k(-2)^{n-k} = M_{2n}$$

Similarly, let $A(t)=e^{3\alpha t}+e^{3\beta t}$ and $B(t)=e^{-2t}$

$$A(t)B(t) = \left[e^{(3\alpha-2)t} + e^{(3\beta-2)t}\right] = e^{\alpha^2 t} + e^{\beta^2 t}$$

$$= \sum_{n=0}^{\infty} ML_{2n} \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}3^k ML_k (-2)^{n-k}\right]\frac{t^n}{n!}$$

$$\therefore \sum_{k=0}^{n}\binom{n}{k}3^k ML_k(-2)^{n-k} = ML_{2n}$$

Theorem 2. For n positive integers

$$\sum_{k=0}^{n}\binom{n}{k}(-3)^{n-k}M_k = (-1)^{n+1}M_n$$

$$\sum_{k=0}^{n}\binom{n}{k}(-3)^{n-k}ML_k = (-1)^{n}ML_n$$

Proof. Assume that $A(t) = e^{\alpha t} - e^{\beta t}$ and $B(t) = e^{-3t}$

$$A(t)B(t) = \left[e^{(\alpha-3)t} - e^{(\beta-3)t}\right] = e^{-\beta t} - e^{-\alpha t}$$

$$= \sum_{n=0}^{\infty}\frac{(-\beta t)^n - (-\alpha t)^n}{n!}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^{n+1}(\alpha^n - \beta^n)t^n}{n!}$$

$$= \sum_{n=0}^{\infty}(-1)^{n+1} M_n \frac{t^n}{n!}$$

From the multiplication of series,

$$A(t)B(t) = \left[\sum_{n=0}^{\infty}\frac{M_n t^n}{n!}\right]\left[\sum_{n=0}^{\infty}\frac{(-3t)^n}{n!}\right]$$

$$= \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}M_k(-3)^{n-k}\right]\frac{t^n}{n!}$$

$$\therefore \sum_{k=0}^{n}\binom{n}{k}M_k(-3)^{n-k} = (-1)^{n+1}M_n$$

Similarly, choose $A(t) = e^{\alpha t} + e^{\beta t}$ and $B(t) = e^{-3t}$

$$A(t)B(t) = \left[e^{(\alpha-3)t} + e^{(\beta-3)t}\right] = e^{-\beta t} + e^{-\alpha t}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n(\alpha^n + \beta^n)t^n}{n!}$$

$$= \sum_{n=0}^{\infty}(-1)^n ML_n \frac{t^n}{n!}$$

By using the Lemma, we have

$$A(t)B(t) = \left[\sum_{n=0}^{\infty}\frac{ML_n t^n}{n!}\right]\left[\sum_{n=0}^{\infty}\frac{(-3t)^n}{n!}\right]$$

$$= \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \binom{n}{k} ML_k (-3)^{n-k} \right] \frac{t^n}{n!}$$

$$\therefore \sum_{k=0}^{n} \binom{n}{k} ML_k (-3)^{n-k} = (-1)^n ML_n$$

Theorem 3. For n positive integers,

$$\sum_{k=0}^{n} \binom{n}{k} (2)^{n-k} M_{2k} = (3)^n M_n$$

$$\sum_{k=0}^{n} \binom{n}{k} (2)^{n-k} ML_{2k} = (3)^n ML_n$$

Proof. Let $A(t) = e^{\alpha^2 t} - e^{\beta^2 t}$ and $B(t) = e^{2t}$

$$A(t)B(t) = \left[e^{(\alpha^2+2)t} - e^{(\beta^2+2)t}\right] = e^{3\alpha t} - e^{3\beta t}$$

$$= \sum_{n=0}^{\infty} (3)^n M_n \frac{t^n}{n!}$$

From the multiplication of series,

$$A(t)B(t) = \left[\sum_{n=0}^{\infty} \frac{M_{2n} t^n}{n!}\right] \left[\sum_{n=0}^{\infty} \frac{(2t)^n}{n!}\right]$$

$$= \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \binom{n}{k} M_{2k} (2)^{n-k} \right] \frac{t^n}{n!}$$

$$\therefore \sum_{k=0}^{n} \binom{n}{k} M_{2k} (2)^{n-k} = (3)^n M_n$$

Similarly, let $A(t) = e^{\alpha^2 t} + e^{\beta^2 t}$ and $B(t) = e^{2t}$

$$A(t)B(t) = \left[e^{(\alpha^2+2)t} + e^{(\beta^2+2)t}\right] = e^{3\alpha t} + e^{3\beta t}$$

$$= \sum_{n=0}^{\infty} (3)^n ML_n \frac{t^n}{n!}$$

By using the Lemma, we have

$$A(t)B(t) = \left[\sum_{n=0}^{\infty} \frac{ML_{2n} t^n}{n!}\right] \left[\sum_{n=0}^{\infty} \frac{(2t)^n}{n!}\right]$$

$$= \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \binom{n}{k} ML_{2k} (2)^{n-k} \right] \frac{t^n}{n!}$$

$$\therefore \sum_{k=0}^{n} \binom{n}{k} ML_{2k} (2)^{n-k} = (3)^n ML_n$$

Theorem 4. For n positive integers,

$$\sum_{k=0}^{n} \binom{n}{k} M_k ML_{n-k} = (2)^n M_n$$

Proof. Assume that $A(t) = e^{\alpha t} - e^{\beta t}$ and $B(t) = e^{\alpha t} + e^{\beta t}$

$A(t)B(t) = \left[e^{2\alpha t} - e^{2\beta t}\right] = \sum_{n=0}^{\infty} (2)^n M_n \frac{t^n}{n!}$

From the multiplication of series,

$$A(t)B(t) = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \binom{n}{k} (\alpha^k - \beta^k)(\alpha^{n-k} + \beta^{n-k})\right] \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \binom{n}{k} M_k ML_{n-k}\right] \frac{t^n}{n!}$$

$$\therefore \sum_{k=0}^{n} \binom{n}{k} M_k ML_{n-k} = (2)^n M_n$$

Theorem 5. For n positive integers,

$$\sum_{k=0}^{n} \binom{n}{k} M_k M_{n-k} = [2^n ML_n - 2(3)^n]$$

Proof. Assume that $A(t) = B(t) = e^{\alpha t} - e^{\beta t}$

$A(t)B(t) = \left[e^{\alpha t} - e^{\beta t}\right]^2 = e^{2\alpha t} + e^{2\beta t} - 2e^{(\alpha+\beta)t}$

$= \sum_{n=0}^{\infty} [(2)^n ML_n - 2(3^n)] \frac{t^n}{n!}$

From the multiplication of series,

$$A(t)B(t) = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \binom{n}{k} (\alpha^k - \beta^k)(\alpha^{n-k} - \beta^{n-k})\right] \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \binom{n}{k} M_k M_{n-k}\right] \frac{t^n}{n!}$$

$$\therefore \sum_{k=0}^{n} \binom{n}{k} M_k M_{n-k} = (2)^n ML_n - 2(3)^n$$

Theorem 6. For n positive integers,

$$\sum_{k=0}^{n} \binom{n}{k} ML_k ML_{n-k} = [2^n ML_n + 2(3)^n]$$

Proof. Assume that $A(t) = B(t) = e^{\alpha t} + e^{\beta t}$

$A(t)B(t) = \left[e^{\alpha t} + e^{\beta t}\right]^2 = e^{2\alpha t} + e^{2\beta t} + 2e^{(\alpha+\beta)t}$

$$= \sum_{n=0}^{\infty}[(2)^n ML_n + 2(3^n)]\frac{t^n}{n!}$$

By using Lemma, we have

$$A(t)B(t) = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k} ML_k ML_{n-k}\right]\frac{t^n}{n!}$$

$$\therefore \sum_{k=0}^{n}\binom{n}{k} ML_k ML_{n-k} = (2)^n ML_n + 2(3)^n$$

Theorem 7. Let m, n be any positive integers

$$\sum_{k=0}^{n}\binom{n}{k} M_{mk} ML_{m(n-k)} = [2^n M_{mn}]$$

Proof. If we choose $A(t) = e^{\alpha^m t} - e^{\beta^m t}$ and $B(t) = e^{\alpha^m t} + e^{\beta^m t}$

$$A(t)B(t) = e^{2\alpha^m t} - e^{2\beta^m t}$$

$$= \sum_{n=0}^{\infty}(2)^n M_{mn}\frac{t^n}{n!}$$

From the multiplication of series

$$A(t)B(t) = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k} M_{mk} ML_{m(n-k)}\right]\frac{t^n}{n!}$$

$$\therefore \sum_{k=0}^{n}\binom{n}{k} M_{mk} ML_{m(n-k)} = (2)^n M_{mn}$$

Theorem 8. Let m, n be any positive integers

$$\sum_{k=0}^{n}\binom{n}{k} M_{mk} M_{m(n-k)} = [2^n ML_{mn} - 2ML_m^n]$$

Proof. If we choose $A(t) = B(t) = e^{\alpha^m t} - e^{\beta^m t}$

$$A(t)B(t) = e^{2\alpha^m t} + e^{2\beta^m t} - 2e^{(\alpha^m + \beta^m)t}$$

$$= \sum_{n=0}^{\infty}(2)^n ML_{mn}\frac{t^n}{n!} - 2\sum_{n=0}^{\infty} ML_m^n \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty}[2^n ML_{mn} - 2ML_m^n]\frac{t^n}{n!}$$

$$A(t)B(t) = \left[\sum_{n=0}^{\infty}\frac{M_{mn}t^n}{n!}\right]\left[\sum_{n=0}^{\infty}\frac{M_{mn}t^n}{n!}\right]$$

$$= \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k} M_{mk} M_{m(n-k)}\right]\frac{t^n}{n!}$$

$$\therefore \sum_{k=0}^{n}\binom{n}{k} M_{mk} M_{m(n-k)} = (2)^n ML_{mn} - 2ML_m^n$$

Theorem 9. Let m, n be any positive integers

$$\sum_{k=0}^{n}\binom{n}{k} ML_{mk}M_{m(n-k)} = [2^n M_{mn}]$$

Proof. If we choose $A(t) = e^{\alpha^m t} + e^{\beta^m t}$, $B(t) = e^{\alpha^m t} - e^{\beta^m t}$

$$A(t)B(t) = e^{2\alpha^m t} - e^{2\beta^m t}$$

$$= \sum_{n=0}^{\infty}(2)^n M_{mn}\frac{t^n}{n!}$$

$$A(t)B(t) = \left[\sum_{n=0}^{\infty}\frac{ML_{mn}t^n}{n!}\right]\left[\sum_{n=0}^{\infty}\frac{M_{mn}t^n}{n!}\right]$$

$$= \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k} ML_{mk}M_{m(n-k)}\right]\frac{t^n}{n!}$$

$$\therefore \sum_{k=0}^{n}\binom{n}{k} ML_{mk}M_{m(n-k)} = (2)^n M_{mn}$$

Theorem 10. Let m, n be any positive integers

$$\sum_{k=0}^{n}\binom{n}{k} ML_{mk}ML_{m(n-k)} = [2^n ML_{mn} - 2ML_m^n]$$

Proof. If we choose $A(t) = B(t) = e^{\alpha^m t} + e^{\beta^m t}$

$$A(t)B(t) = e^{2\alpha^m t} + e^{2\beta^m t} + 2e^{(\alpha^m+\beta^m)t}$$

$$= \sum_{n=0}^{\infty}(2)^n ML_{mn}\frac{t^n}{n!} + 2\sum_{n=0}^{\infty} ML_m^n\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty}[2^n ML_{mn} + 2ML_m^n]\frac{t^n}{n!}$$

$$A(t)B(t) = \left[\sum_{n=0}^{\infty}\frac{M_{mn}t^n}{n!}\right]\left[\sum_{n=0}^{\infty}\frac{ML_{mn}t^n}{n!}\right]$$

$$= \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k} ML_{mk}ML_{m(n-k)}\right]\frac{t^n}{n!}$$

$$\therefore \sum_{k=0}^{n}\binom{n}{k} ML_{mk}ML_{m(n-k)} = (2)^n ML_{mn} - 2ML_m^n$$

Theorem 11. Let n, r be any positive integers then

$$\sum_{k=0}^{n}\binom{n}{k} 3^{k+r}(-2)^{n-k}M_{k+r} = 3^r M_{2n+r}$$

Proof. If we choose $A(t) = e^{3\alpha t} - e^{3\beta t}$ and $\mathrm{B}(t) = e^{-2t}$

$$D_t^r A(t) = D_t^r\left[e^{3\alpha t} - e^{3\beta t}\right]$$

$$= (3\alpha)^r e^{3\alpha t} - (3\beta)^r e^{3\beta t}$$

$$D_t^r A(t)B(t) = \left[(3\alpha)^r e^{3\alpha t} - (3\beta)^r e^{3\beta t}\right](e^{-2t})$$

$$= (3\alpha)^r\left[e^{(3\alpha-2)t}\right] - (3\beta)^r\left[e^{(3\beta-2)t}\right]$$

$$= (3)^r\left[\alpha^r e^{\alpha^2 t} - \beta^r e^{\beta^2 t}\right]$$

$$= \sum_{n=0}^{\infty} 3^r\, M_{2n+r}\frac{t^n}{n!}$$

$$D_t^r A(t)B(t) = \left[\sum_{n=0}^{\infty} 3^{n+r}\, M_{n+r}\frac{t^n}{n!}\right]\left[\sum_{n=0}^{\infty}(-2)^n\frac{t^n}{n!}\right]$$

$$= \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}3^{k+r}M_{k+r}(-2)^{n-k}\right]\frac{t^n}{n!}$$

$$\therefore \sum_{k=0}^{n}\binom{n}{k}3^{k+r}(-2)^{n-k}M_{k+r} = 3^r M_{2n+r}$$

Theorem 12. Let m, n, r be any positive integers then

$$\sum_{k=0}^{n}\binom{n}{k}2^{2m(n-k)}M_{4m(k+r)} = ML_{2m}^{n}M_{4mr+2mn}$$

Proof. Let $A(t) = e^{\alpha^{4m}t} - e^{\beta^{4m}t}$ and $\mathrm{B}(t) = e^{2^{2m}t}$

$$D_t^r A(t) = D_t^r\left[e^{\alpha^{4m}t} - e^{\beta^{4m}t}\right]$$

$$= (\alpha^{4m})^r e^{\alpha^{4m}t} - (\beta^{4m})^r e^{\beta^{4m}t}$$

$$D_t^r A(t)B(t) = \left[\alpha^{4mr}e^{(\alpha^{4m}+2^{2m})t} - \beta^{4mr}e^{(\beta^{4m}+2^{2m})t}\right]$$

$$= \alpha^{4mr}e^{(\alpha^{4m}+(\alpha\beta)^{2m})t} - \beta^{4mr}e^{(\beta^{4m}+(\alpha\beta)^{2m})t}$$

$$= \alpha^{4mr}e^{\alpha^{2m}(\alpha^{2m}+\beta^{2m})t} - \beta^{4mr}e^{\beta^{2m}(\alpha^{2m}+\beta^{2m})t}$$

$$= \alpha^{4mr}e^{(\alpha^{2m}ML_{2m})t} - \beta^{4mr}e^{(\beta^{2m}ML_{2m})t}$$

$$= \left[\sum_{n=0}^{\infty}\alpha^{2mn+4mr}\, ML_{2m}^{n}\frac{t^n}{n!}\right] - \left[\sum_{n=0}^{\infty}\beta^{2mn+4mr}\, ML_{2m}^{n}\frac{t^n}{n!}\right]$$

$$= \sum_{n=0}^{\infty} M_{4mr+2mn}\, ML_{2m}^{n}\frac{t^n}{n!}$$

$$D_t^r A(t)B(t) = \left[\sum_{n=0}^{\infty} M_{4m(n+r)}\frac{t^n}{n!}\right]\left[\sum_{n=0}^{\infty}(2)^{2mn}\frac{t^n}{n!}\right]$$

$$= \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}M_{4m(k+r)}(2)^{2m(n-k)}\right]\frac{t^n}{n!}$$

$$\therefore \sum_{k=0}^{n}\binom{n}{k}(2)^{2m(n-k)}M_{4m(k+r)} = ML_{2m}^{n}M_{4mr+2mn}$$

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