Solution of Linear Fractional Programming Problem by Fourier-Motzkin Elimination Technique

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Abstract: In this research paper, Fourier-Motzkin elimination technique is suggested to solve LFPP, which is based upon the concept of bounds. The proposed approach is computationally more efficient and easy to understand as compared to the traditional simplex method. An illustration has been given at the end.

Keywords: Linear fractional programming problem, optimal solution, inequalities, Fourier-Motzkin elimination technique.

1. Introduction

Linear fractional programming is an important subclass of Mathematical Optimization. A linear fractional programming problem is basically the ratio of two linear functions subject to linear constraints along with non-negative restrictions. Earlier Charnes et al (1962), Swarup (1963), Gass (1985), Hirche (1996) and Chadha (1999) solved fractional programming problems and gave iterative algorithms. Various elimination techniques to solve linear programming problems are given earlier by Williams (1986), Karmarker (1984), Kohler (1973), Sharma et al (2003) , Kanniappan et al (1998) and Jain et al (2004, 2008a, 2008b, 2008c, 2012a, 2012b , 2014, 2018, 2009, 2013). Puri et al (1974) proposed an enumerative procedure to find an optimal solution of LFPP.

The paper unfolds as follows:- Section 2 describes the application of Fourier-Motzkin elimination technique on equations as well as inequalities. Problem formulation is given in the section 3 which is to be solved by proposed technique. To demonstrate the whole procedure of the proposed technique, an illustration is given in the section 4. This concludes the paper.

2. Fourier-Motzkin Elimination Technique (F-ME Technique)

F-ME technique is an important process to solve small LPPs. In this process, the number of variables reduces one by one in each iteration. Initially, F-ME technique was applied on equations.

There is exactly one solution when we solve an equation provided that the equation is solvable; but in the case of an inequality, there may be many solutions possible in bounded form. Among these solutions, we have to choose an optimize solution according to the requirement of the problem under consideration. Here, we have applied F-ME technique to solve the system of inequalities of the same nature either ($\leq$) or ($\geq$). This process consists three different classes of inequalities w.r.t. each variable x_i e.g. first class, second class and third class.

First class: If the coefficient of the variable y_i is $+1$, then the inequalities having this type of coefficients may be classified in this class.

Second class: If the coefficient of the variable y_i is -1, then the inequalities having this type of coefficients may be classified in this class.

Third class : If the coefficient of the variable y_i is 0, then the inequalities having this type of coefficients may be classified in this class.

Now, we have to eliminate the variables to achieve the optimality of the problem. We can eliminate the variables by combining the inequalities in such a way that the variables reduced one by one in each iteration. Proceeding in the similar fashion, at the end of the process there remains a single variable with bounded values. From all the bounded values of the last variable, we can find the permissible value of that particular variable.

Now, from the process of back-substitution we can obtain the values of other variables and thus the optimal solution is reached.

3. The Problem

Consider the linear fractional programming problem:

Max. $Z = \frac{(ay + \propto)}{(c\,y + \beta)}$

Such that $A\,y \leq b$

and $y \geq 0$

It is assumed that the feasible solution set constitutes a convex polyhedral with finite extreme points along with $(\,c\,y + \beta) \neq 0$.

First of all, we have to reframe the above problem in standard form by converting objective function to a constraint inequality. Then, we have

Max. Z

$$(\,cy + \beta)\,Z - (\,a\,y + \alpha) \geq 0$$
$$-(\,cy + \beta)\,Z + (\,a\,y + \alpha) \geq 0$$

Further, we have to make all the inequalities (involved in the problem along with above inequality) of the same nature. Therefore, the above problem becomes:

Max. Z

$$(\,cy + \beta)\,Z - (\,a\,y + \alpha) \geq 0$$
$$-(\,cy + \beta)\,Z + (\,a\,y + \alpha) \geq 0$$
$$-A\,y \geq -b$$
$$y \geq 0$$

Now, we have to form three classes of the variable y_i. After that, we have to combine the inequalities in such a way that we could be able to reduce the variables one by one in each iteration. If at any stage of this process we found $0 \leq d$ where d is not a positive number, then it can be concluded that an infeasible solution exists of the problem under consideration, otherwise the solution achieved is feasible.

4. Example

Let us take a LFPP mentioned below which have been solved earlier by different elimination techniques by Jain et al [6 & 9].

Max. $Z = \frac{5\,x_1 + 3\,x_2}{5\,x_1 + 2\,x_2 + 1}$

s.t. $3\,x_1 + 5\,x_2 \leq 15$

$5\,x_1 + 2\,x_2 \leq 10$

and $x_1\,, x_2 \geq 0.$

First of all, we have to rewrite the above problem in standard form by converting objective function to a constraint inequality. Then, we have

Max. Z

$$5(Z-1)x_1 + (2Z-3)\,x_2 + Z \geq 0$$
$$-5(Z-1)x_1 - (2Z-3)\,x_2 - Z \geq 0$$

Further, we have to make all the inequalities (involved in the problem along with above inequality) of the same nature.

Therefore, the above problem becomes:

Max. Z

$$5(Z-1)x_1 + (2Z-3)\,x_2 + Z \geq 0$$
$$-5(Z-1)x_1 - (2Z-3)\,x_2 - Z \geq 0$$

$$-3\,x_1 - 5\,x_2 \geq -15$$
$$-5\,x_1 - 2\,x_2 \geq -10$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$

Now our object is to find the optimal value of Z for non-negative values of decision variables x_1 and x_2. Therefore for maximum value of Z, we assume that $1 < Z < \frac{3}{2}$.

Looking into the above bounds of Z, above equations reduces to:

$$-x_1 - \frac{(2Z-3)}{5(Z-1)} x_2 - \frac{Z}{5(Z-1)} \geq 0 \quad (1)$$

$$x_1 + \frac{(2Z-3)}{5(Z-1)} x_2 + \frac{Z}{5(Z-1)} \geq 0 \quad (2)$$

$$-x_1 - \frac{5}{3} x_2 \geq -5 \quad (3)$$

$$-x_1 - \frac{2}{5} x_2 \geq -2 \quad (4)$$

$$x_1 \geq 0 \quad (5)$$

$$x_2 \geq 0 \quad (6)$$

After eliminating x_1 from the above inequalities, we have

$$-x_2 + \frac{3Z}{(19Z-16)} \geq -\frac{75(Z-1)}{(19Z-16)} \quad (7)$$

$$-x_2 + Z \geq -10(Z-1) \quad (8)$$

$$x_2 - \frac{Z}{(3-2Z)} \geq 0 \quad (9)$$

$$-x_2 \geq -3 \quad (10)$$

$$-x_2 \geq -5 \quad (11)$$

$$x_2 \geq 0 \quad (12)$$

After eliminating x_2 from the above inequalities, we have

$$Z \geq \frac{75}{78} \quad (13)$$

$$Z \geq \frac{10}{11} \quad (14)$$

$$-7Z^2 + 16Z - 9 \geq 0 \quad \text{or} \quad 1 \leq Z \leq \frac{9}{7} \quad (15)$$

$$-11Z^2 + 24Z - 15 \geq 0 \quad (16)$$

$$Z \leq \frac{9}{7} \quad (17)$$

$$Z \leq \frac{15}{11} \quad (18)$$

Out of these values of Z, $Z = \frac{15}{11}$ is the maximum amongst all the values but it doesn't satisfy the inequality $-7Z^2 + 16Z - 9 \geq 0$. Therefore, $Z = \frac{9}{7}$ is an optimal solution which satisfies all the inequalities.

By putting $Z = \frac{9}{7}$ into the inequalities (7) - (12), we have

$$x_2 \leq 3$$
$$x_2 \leq \frac{29}{7}$$
$$x_2 \geq 3$$
$$x_2 \leq 3$$
$$x_2 \leq 5$$
$$x_2 \geq 0$$

Out of these values, $x_2 = 3$ satisfies all the inequalities altogether.

By putting $x_2 = 3$ and $Z = \frac{9}{7}$ into the inequalities (1)-(6), we get

$$x_1 \geq 0$$
$$x_1 \leq 0$$
$$x_1 \leq \frac{4}{5}$$

The solution of above three inequalities comes out to be $x_1 = 0$.

Hence, $x_1 = 0$ and $x_2 = 3$ is an optimal solution of LFPP under consideration with Max. Z = $\frac{9}{7}$.

5. Conclusion

The proposed Fourier-Motzkin elimination technique is quite easy to understand and apply. It takes least computation time as compared to traditional simplex method to solve linear programming problem. An illustration is given at the end of the paper to simplify the whole procedure and the optimal solution thus obtained can be verified by the methods available in the literature so far e.g. graphical method, simplex method and other existing methods.

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