EXISTENCE OF SOLUTIONS FOR FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS INVOLVING THE CAPUTO TYPE ATANGANA-BALEANU DERIVATIVE

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Abstract: The existence and uniqueness of solutions for FNIDE in the idea of Atangana-Baleanu derivative in Banach spaces are investigated in this research. In this case, the FD is taken in the Caputo sense. The Banach and Krasnoselskii-Schaefer FPT are used to show the desired results.

Keywords: Triple Neutral Volterra-Fredholm integro-differential equation, Caputo fractional derivative, Atangana-Baleanu derivative, fixed point technique.

Abbreviations: Fractional Integro-Differential Equations (FIDE), Integro-Differential Equations (IDE), Volterra-Fredholm integro-differential equation (VFIDE),Fractional neutral differential equations (FNDE), Caputo fractional integro-differential equations (CFIDE), Fractional neutral integro-differential equations (FNIDE), Fractional Derivative (FD), Fixed Point theorem (FPT)

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1. Introduction

Due to models and intriguing outcomes in actual world occurrences, Atangana-Baleanu derivative has attracted many researchers in the last several years. Until now, however, Atangana et al. has to be conveyed for these operators theory that includes a nonsingular kernel.The application of fractional calculus techniques to IDE broadened the scope of its mathematical modelling and control study. The main distinction between IDE and FIDE is that the first concerns the derivation and integration of integer order, while the second concerns arbitrary order (see Agarwal et al., Dawood et al., Hamoud et al., Ntouyas et al., and Sousa et al.). As Balachanderran et al. and Hamoud et al. have demonstrated, the use of these equations has increased considerably in the modelling of real-life scientific and engineering issues, as integral modelling in terms of efficiency is more precise in translating realistic situations into mathematical formulations.Neutral DE is the DE, which relies on past and current functional values and are found in the mathematical fields.Santos, et al., have done a lot of study on the notion of FNDE and its applications. Baleanu et al. recently explored the existence and uniqueness nature of a solution to the nonlinear problem of fractional limit value by use of FPT,

$$^{c}D^{\nu}\Delta(\omega)=E(\omega,\Delta(\omega)),\omega\in[0,T],0<\nu<1,$$

$$\Delta(0)=\Delta(T),\Delta(0)=\beta_1\Delta(\eta),\Delta(T)=\beta_2\Delta(\eta),0<\eta<T,0<\beta_1<\beta_2<1.$$

Devi and Sreedhar devised the generalised monotone iterative technique for solving CFIDE of type

$$^{c}D^{\nu}\Delta(\omega)=E(\omega,\Delta(\omega),I^{\nu}\Delta(\omega)),\omega\in[0,T],0<\nu<1,$$

$$\Delta(0)=\Delta_0.$$

The results obtained give an explicit mathematical solution of the CFIDE linear IVP which shows that such iterates converge consistently and monotonously to a combined minimal and maximum problem solution.

Ulam stability and data dependency for the Caputo FDE type was studied by Wang and Zhou

$$^{c}D^{\nu}\Delta(\omega)=E(\omega,\Delta(\omega)),\omega\in[0,+\infty),0<\nu<1,$$

$$\Delta(a)=\Delta_0.$$

Dong et al. used Banach and Schauder FPT to obtain the uniqueness and existence of solutions to the problem provided by

$$^{c}D^{\nu}_{0^+}\Delta(\omega)=E(\omega,\Delta(\omega))+\int_0^{\omega}\Theta(\omega,s,\Delta(s))ds,\omega\in[0,T],0<\nu\leq 1,$$

$$\Delta(0) = \Delta_0,$$

Logeswari and Ravichandran investigated the existence of FNIDE in the concept of the Atangana-Baleanu derivative of the form

$$^{ABC}D^{\nu}_{0^+}[\Delta(\omega) - \Lambda(\omega, \Delta(\omega), \Theta\Delta(\omega))] = \Lambda^*(\omega, \Delta(\omega), \Theta^*\Delta(\omega)),\ 0 < \nu < 1,$$

$$\Delta(0) = \Delta_0,$$

We will explore a more general problem of CFIDE termed Caputo fractional neutral VFIDE of the type

$$^{ABC}D^{\nu}_{0^+}[\Delta(\omega) - A(\omega, \Delta(\omega), K\Delta(\omega), H\Delta(\omega))] = B(t, \Delta(\omega), G\Delta(\omega), F\Delta(\omega)) \quad (1)$$

$$\Delta(0) = \Delta_0, \quad (2)$$

where $^{ABC}D^{\nu}_{0^+}$ is the Atangana-Baleanu Caputo FD of order $\nu, 0 < \nu < 1, \omega \in J := [0,1]$,which is motivated by the prior studies.

Consider $K\Delta(\omega) = \int_0^\omega k(\omega, s, \Delta(s))ds, H\Delta(\omega) = \int_0^1 h(\omega, s, \Delta(s))ds, G\Delta(\omega) = \int_0^\omega g(\omega, s, \Delta(s))ds$, and $F\Delta(\omega) = \int_0^1 \chi(\omega, s, \Delta(s))ds$.

The following is how the paper is structured:In Section 2, we review some basic definitions, lemmas, and theorems. In Section 3, we prove the existence and uniqueness results for the problem (1)-(2) using the FPT of Krasnoselskii-Schaefer and Banach. In Section 4, concluding remarks bring the paper to a close.

2. Preliminaries

Here are some definitions, notes and findings utilised throughout this article. (See Kilbas, A., Srivastava, H. and Trujillo, J. (2006), Zhou, Y. (2014))

Definition 2.1 The R-LFD of order $\nu > 0$ of a function $\Delta: (0, \infty) \longrightarrow \mathbb{R}$ is defined by

$$^{RL}D^{\nu}_{\omega}\Delta(\omega) = \frac{1}{\Gamma(m-\nu)}\left(\frac{d}{dt}\right)^m \int_a^\omega (\omega - s)^{m-\nu-1}\Delta(s)ds, m - 1 < \nu \leq m,$$

where $\Gamma(.)$ is the Gamma function.

Definition 2.2 The R-L fractional integral of order $\nu > 0$ of a function $\Delta: (0, \infty) \longrightarrow \mathbb{R}$, according to Riemann-Liouville, the fractional integral that is considered as anti-FD of a function Δ is

$$I^{\nu}_{\omega}\Delta(\omega) = \frac{1}{\Gamma(\nu)}\int_a^\omega (\omega - s)^{\nu-1}\Delta(s)ds, s > a, \quad (3)$$

Definition 2.3 Caputo FD of order $\nu > 0$ of a function $\Delta: (0, \infty) \longrightarrow \mathbb{R}$, according to Caputo, the FD of a continuous and n-time differentiable function Δ is given as

$$^{c}D^{\nu}_{\omega}\Delta(\omega) = \frac{1}{\Gamma(m-\nu)}\int_a^\omega (\omega - s)^{m-\nu-1}\left(\frac{d}{ds}\right)^m\Delta(s)ds, m - 1 < \nu \leq m.$$

Definition 2.4 The R-L AB-derivative of order $0 < \nu \leq 1$ of a function $\Delta \in C([0, T])$ is normally defined as

$$^{AB}D^{\nu}_{0^+}\Delta(\omega) = \frac{\beta(\nu)}{1-\nu}\frac{d}{dt}\left(\int_0^\omega \Delta(s)E_\nu\left[-\nu\frac{(\omega-s)^\nu}{1-\nu}\right]ds\right). \quad (4)$$

Definition 2.5 The Caputo AB-derivative of order $0 < \nu \leq 1$ of a function $\Delta \in C([0, T])$ is normally defined as

$$^{ABC}D^{\nu}_{0^+}\Delta(\omega) = \frac{\beta(\nu)}{1-\nu}\int_0^\omega \Delta'(s)E_\nu\left[-\nu\frac{(\omega-s)^\nu}{1-\nu}\right]ds. \quad (5)$$

Definition 2.6 The associative fractional integral of (5) is

$$^{AB}I^{\nu}_{0^+}\Delta(\omega) = \frac{1-\nu}{\beta(\nu)}\Delta(\omega) + \frac{\nu}{\beta(\nu)}I^{\nu}_{0^+}\Delta(\omega)$$

where $I^{\nu}_{0^+}$ is R-L integral mentioned in (3).

Lemma 2.1(Ascoli-Arzela theorem). Let $S = \{s(\omega)\}$ is a function family of continuous mappings $s: J \longrightarrow X$. If S is uniformly bounded and equicontinuous, and for any $\omega^* \in J$, the set $\{s(\omega^*)\}$ is relatively compact, then there exists a uniformly convergent function sequence $\{s_n(\omega)\}(n = 1,2,\ldots, \omega \in J)$ in S.

Theorem 2.1 (Banach FPT). Let $(S, \|.\|)$ be a complete normed space, and let the mapping $F: S \longrightarrow S$ be a contraction mapping. Then F has exactly one fixed point.

Theorem 2.2 (Krasnoselskii-Schaefer FPT). Let S be nonempty, closed, bounded and convex subset of a real Banach space X and let T_1 and T_2 be operators on S satisfying the following conditions

1. T_1 is contraction on S,
2. T_2 is completely continuous on S.

Then either

(I) There exists a $x \in S$ s.t. $T_1 x + T_2 x = x$, or

(II) The set $\epsilon = \{\Delta \in X : \lambda T_1(\frac{\Delta}{\lambda}) + \lambda T_2(\Delta)\}$ is unbounded for $\lambda \in (0,1)$.

Lemma 2.2 Let $\Delta(\omega), \chi(\omega), q(\omega) \in C(J, \mathbb{R}_+)$ and let $n(\omega) \in C(J, \mathbb{R}_+)$ be nondecreasing for which the inequality

$$\Delta(\omega) \leq n(\omega) + \int_0^\omega \chi(s)\Delta(s)ds + \int_0^\omega \chi(s) \int_0^s q(r)\Delta(r)drds,$$

holds for any $t \in J$. Then

$$\Delta(\omega) \leq n(\omega)[1 + \int_0^\omega \chi(s)(\int_0^s (\chi(r) + q(r))dr)ds].$$

3. Existence and uniqueness results

Now, we provide the following hypotheses before starting and establishing the major results:

(A1) $B: J \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous function, and there exist a positive constant M_1 such that

$$\| B(\omega, \Delta_1, w_1, \Phi_1) - B(\omega, \Delta_2, w_2, \Phi_2) \|_2 \leq M_1(\| \Delta_1 - \Delta_2 \| + \| w_1 - w_2 \| + \| \Phi_1 - \Phi_2 \|),$$

for all $\Delta_1, \Delta_2, w_1, w_2, \Phi_1$ and $\Phi_2 \in \mathbb{R}$ are continuous functions on J in the Banach spaces. Let $M_2 = \max_{\omega \in J} \| B(\omega, 0,0,0) \|$ and $M = \max\{M_1, M_2\}$.

(A2) $A: J \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous function, and there exist a positive constant L_1 such that

$$\| A(\omega, \Delta_1, w_1, \Phi_1) - A(\omega, \Delta_2, w_2, \Phi_2) \|_2 \leq L_1(\| \Delta_1 - \Delta_2 \| + \| w_1 - w_2 \| + \| \Phi_1 - \Phi_2 \|),$$

for all $\Delta_1, \Delta_2, w_1, w_2, \Phi_1$ and $\Phi_2 \in \mathbb{R}$ are continuous functions on J in the Banach spaces. Let $L_2 = \max_{\omega \in J} \| A(\omega, 0,0,0) \|$ and $L = \max\{L_1, L_2\}$.

(A3) There exist $N_1^k > 0$ and $N_1^h > 0$ such that

$$\| k(\omega, s, \Delta) - k(\omega, s, \Phi) \|_2 \leq N_1^k \| \Delta - \Phi \|$$

$$\| h(\omega, s, \Delta) - h(\omega, s, \Phi) \|_2 \leq N_1^h \| \Delta - \Phi \|$$

for all Δ and $\Phi \in \mathbb{R}$ are continuous function on J in the Banach spaces. Let $N_2^k = \max_{\omega \in J} \| k(\omega, s, 0) \|$, $N^k = \max\{N_1^k, N_2^k\}$, and $N_2^h = \max_{\omega \in J} \| h(\omega, s, 0) \|$, $N^h = \max\{N_1^h, N_2^h\}$.

(A4) There exist $C_1^g > 0$ and $C_1^f > 0$ such that

$$\| g(\omega, s, \Delta) - g(\omega, s, \Phi) \|_2 \leq C_1^g \| \Delta - \Phi \|$$

$$\| \chi(\omega, s, \Delta) - \chi(\omega, s, \Phi) \|_2 \leq C_1^f \| \Delta - \Phi \|$$

for all Δ and $\Phi \in \mathbb{R}$ are continuous function on J in the Banach spaces X. Let $C_2^g = \max_{\omega \in J} \| g(\omega, s, 0 \|$, $C^g = \max\{C_1^g, C_2^g\}$, and $C_2^f = \max_{\omega \in J} \| \chi(\omega, s, 0 \|$, $C^f = \max\{C_1^f, C_2^f\}$.

(A5) For each $r, B_r = \{\Delta \in C[J, X] : \| \Delta \| \leq r\} \subseteq C[J, X]$, then B_r is clearly a bounded closed and convex subset in $C([0,1], X)$ where $r \geq (1 - 2U)^{-1}(\| \Delta_0 \| + U)$ and consider $U = \max\{L, M\}$ and $U < \frac{1}{2}$.

(A6) There exist two functions $q, p \in L^1(J, R_+)$ such that

(i) $|B(\omega, \Delta, w, \Phi)| \leq q(\psi(\| \Delta \|)) + |w| + |\Phi|$, foreach $(\omega, \Delta, w, \Phi) \in J \times D \times E \times E$.

(ii) $|A(\omega, \Delta, w, \Phi)| \leq p(\psi(\| \Delta \|)) + |w| + |\Phi|$, foreach $(\omega, \Delta, w, \Phi) \in J \times D \times E \times E$,

where E is measurable function and $\psi: [0, \infty) \longrightarrow (0, \infty)$ will be continuous non-decreasing function.

(A7) There exist constants $M_*, \vartheta^* > 0$ such that

$$\frac{(1-L-\vartheta^*)M_*}{\|\phi+\vartheta^*\|+(L(N^k+N^h)\omega+\vartheta^*(C^g+C^f)\omega)(\psi M_*+1)} > 1.$$

Lemma 3.1 If (A3) and (A4) are satisfied, then the estimate

$$\| K\Delta(\omega) \| \leq \omega(N_1^k \| \Delta \| + N_2^k), \| K\Delta(\omega) - K\Phi(\omega) \| \leq N^k\omega \| \Delta - \Phi \|,$$

$$\| H\Delta(\omega) \| \leq \omega(N_1^h \| \Delta \| + N_2^h), \| H\Delta(\omega) - H\Phi(\omega) \| \leq N^h\omega \| \Delta - \Phi \|,$$

$$\| G\Delta(\omega) \| \leq \omega(C_1^g \| \Delta \| + C_2^g), \| G\Delta(\omega) - G\Phi(\omega) \| \leq C^g\omega \| \Delta - \Phi \|,$$

and

$$\| F\Delta(\omega) \| \leq \omega(C_1^f \| \Delta \| + C_2^f), \| F\Delta(\omega) - F\Phi(\omega) \| \leq C^f\omega \| \Delta - \Phi \|, \omega \in J.$$

Proposition 3.1 For $0 < \nu < 1$, $\omega \in J$, we conclude that

$$({}^{AB}I_{0+}^{\nu}({}^{AB}D_{0+}^{\nu}u))(\omega) = \Delta(\omega) - \Delta(0)E_\nu(\lambda\omega^\nu) - \frac{\nu}{1-\nu}\Delta(0)E_{\nu,\nu+1}(\lambda\omega^\nu)$$
$$= \Delta(\omega) - \Delta(0).$$

Lemma 3.2 Let $0 < \nu < 1, \omega \in J$ and $\Delta \in C[0,1]$ is called a mild solution of the problem (1)-(2) if and only if Δ satisfies the following equation:

$$\Delta(\omega) = \Delta_0 - A(0,\Delta(0),0,0) + A(\omega,\Delta(\omega),K\Delta(\omega),H\Delta(\omega)) \tag{6}$$
$$+ {}^{AB}I_{0+}^{\nu}B(\omega,\Delta(\omega),G\Delta(\omega),F\Delta(\omega)).$$

Theorem 3.1 If the assumptions (A1)-(A5) are satisfied and if $A(0,\Delta(0),0,0) = B(0,\Delta(0),0,0) = 0$ and

$$\left((N^k + N^h)\omega + \frac{1-\nu}{\beta(\nu)}(1 + (C^g + C^f)\omega) + \frac{1^\nu}{\beta(\nu)\Gamma(\nu)}(1 + (C^g + C^f)\omega)\right) < 1, \omega \in J,$$

then the problem (1)-(2) has a unique solution on J.

Proof. First, we will show that $\Delta(\omega)$ satisfies (1)-(2) iff $\Delta(\omega)$ satisfies (6).

Consider $\Delta(\omega)$ satisfy (1), then by using the AB-integral of (1), we get

$$({}^{AB}I_{0+}^{\nu} \; {}^{AB}D_{0+}^{\nu}(\Delta(\omega) - A(\omega,\Delta(\omega),K\Delta(\omega),H\Delta(\omega)))) = {}^{AB}I_{0+}^{\nu}B(\omega,\Delta(\omega),G\Delta(\omega),F\Delta(\omega)). \tag{7}$$

Now, by using Proposition 3.1, we obtain

$$\Delta(\omega) - A(\omega,\Delta(\omega),K\Delta(\omega),H\Delta(\omega)) - (\Delta_0 - A(0,\Delta(0),0,0)) =$$
$${}^{AB}I_{0+}^{\nu}B(\omega,\Delta(\omega),G\Delta(\omega),F\Delta(\omega)).$$

Since $\Delta(0) = \Delta_0$ from (2) and $B(0,x(0),0,0) = 0$, then (6) satisfied. Now, if $\Delta(\omega)$ satisfy (6), then taking $B(0,x(0),0,0) = 0$, it is visibly that $\Delta(0) = \Delta_0$. In R-L sense using the AB-derivative of (6) and substitute $({}^{AB}D_{0+}^{\nu}({}^{AB}I_{0+}^{\nu}u))(\omega) = \Delta(\omega)$, we obtain

$$({}^{ABR}D_{0+}^{\nu}u)(\omega) = \Delta_0({}^{ABR}D_{0+}^{\nu}1)(\omega) + ({}^{ABR}D_{0+}^{\nu})(A(\omega,\Delta(\omega),K\Delta(\omega),H\Delta(\omega)))$$
$$-A(0,\Delta(0),0,0)({}^{ABR}D_{0+}^{\nu}1)(\omega) + ({}^{ABR}D_{0+}^{\nu}({}^{AB}I_{0+}^{\nu}))B(\omega,\Delta(\omega),G\Delta(\omega),F\Delta(\omega)).$$

Thus, we have

$${}^{ABR}D_{0+}^{\nu}(u(\omega) - A(\omega,\Delta(\omega),K\Delta(\omega),H\Delta(\omega)))) = (\Delta_0 - A(0,\Delta(0),0,0)))E_\nu(\frac{-\nu}{1-\nu}\omega^\nu)$$
$$+B(\omega,\Delta(\omega),G\Delta(\omega),F\Delta(\omega)).$$

Now, define the operator

$$T\Delta(\omega) =$$
$$\Delta_0 - A(0,\Delta(0),0,0) + A(\omega,\Delta(\omega),K\Delta(\omega),H\Delta(\omega)) + {}^{AB}I_{0+}^{\nu}B(\omega,\Delta(\omega),G\Delta(\omega),F\Delta(\omega)).$$

Then by Lemma 3.1, we have for $\Delta \in B_r$, where $r > 0$,

$$\| T\Delta(\omega) \| \leq \| \Delta_0 \| + L(\| \Delta \| + t(N_1^k + N_1^h) \| \Delta \| + N_2^k + N_2^h)) + \frac{1-\nu}{\beta(\nu)}(M_1(\| \Delta \| + (C^g + C^f)\omega \| \Delta \|)) + \frac{1-\nu}{\beta(\nu)} M_2 + \frac{\nu}{\beta(\nu)}(M_1(\| \Delta \| + (C^g + C^f)\omega \| \Delta \|)({}^{AB}I_{0+}^{\nu})(\omega)) + \frac{\nu}{\beta(\nu)} M_2({}^{AB}I_{0+}^{\nu})(\omega)$$

$$\leq \| \Delta_0 \| + U \| \Delta \| + U((N_1^k + N_1^h)\omega + \frac{1-\nu}{\beta(\nu)}(1 + (C^g + C^f)\omega) + \frac{1^\nu}{\beta(\nu)\Gamma(\nu)}(1 + (C^g + C^f)\omega)) \| \Delta \|$$

$$+U((N_1^k + N_1^h)\omega + \frac{1-\nu}{\beta(\nu)}(1 + (C^g + C^f)\omega) + \frac{1^\nu}{\beta(\nu)\Gamma(\nu)}(1 + (C^g + C^f)\omega))$$

$$\leq r(1 - 2U) + 2Ur$$

$$\leq r.$$

Now, for any Δ_1 and $\Delta_2 \in C[J, X]$

$$\| T\Delta_1(\omega) - T\Delta_2(\omega) \| \leq \| \Delta_0 - A(0, \Delta_1(0), 0, 0) + A(\omega, \Delta_1(\omega), K\Delta_1(\omega), H\Delta_1(\omega))$$

$$+ \ {}^{AB}I_{0+}^{\nu} B(\omega, \Delta_1(\omega), G\Delta_1(\omega), F\Delta_1(\omega)) \|$$

$$+\| \Delta_0 - A(0, \Delta_2(0), 0, 0) + A(\omega, \Delta_2(\omega), K\Delta_2(\omega), H\Delta_2(\omega))$$

$$+ \ {}^{AB}I_{0+}^{\nu} B(\omega, \Delta_2(\omega), G\Delta_2(\omega), F\Delta_2(\omega)) \|$$

$$\leq L(\| \Delta_1 - \Delta_2 \| + (N_1^k + N_1^h)\omega \| \Delta_1 - \Delta_2 \|) + \frac{1-\nu}{\beta(\nu)}[M(\| \Delta_1 - \Delta_2 \|$$

$$+(C^g + C^f)\omega \| \Delta_1 - \Delta_2 \|)]$$

$$+\frac{\nu}{\beta(\nu)}[M(\| \Delta_1 - \Delta_2 \| + (C^g + C^f)\omega \| \Delta_1 - \Delta_2 \|)]({}^{AB}I_{0+}^{\nu} 1)(\omega)$$

$$\leq U \| \Delta_1 - \Delta_2 \| + U((N_1^k + N_1^h)\omega + \frac{1-\nu}{\beta(\nu)}(1 + (C^g + C^f)\omega)$$

$$+\frac{1^\nu}{\beta(\nu)\Gamma(\nu)}(1 + (C^g + C^f)\omega)) \| \Delta_1 - \Delta_2 \|$$

$$\leq 2U \| \Delta_1 - \Delta_2 \|.$$

Since $U < \frac{1}{2}$, it follows that the operator T is contraction on J. The application of Theorem 2.1gives the existence of a uniqueness of solution of the problem (1)-(2). This completes the proof.

Theorem 3.2 Assume that the assumptions (A1)-(A7) are satisfied and

$$q(\omega_2 - \omega_1) = [M(\| \Delta(\omega_2) - \Delta(\omega_1) \| + (C^g + C^f)\omega \| \Delta(\omega_2) - \Delta(\omega_1) \|)].$$

Then the problem (1)-(2) has at least one solution $\Delta(\omega)$ on J.

Proof. Define two operators T_1 and T_2 on B_{r_0}, where $r_0$is an positive constant, as follows

$$(T_1\Delta)(\omega) = \Delta_0 - A(0, \Delta(0), 0, 0) + A(\omega, \Delta(\omega), K\Delta(\omega), H\Delta(\omega)), \tag{8}$$

$$(T_2\Delta)(\omega) = \ {}^{AB}I_{0+}^{\nu} B(\omega, \Delta(\omega), G\Delta(\omega), F\Delta(\omega)). \tag{9}$$

Clearly, Δ is a mild solution of the problem (1)-(2) iff the equation $\Delta = (T_1 + T_2)(\Delta)$ has a solution $\Delta \in B_r$.

Therefore, the existence solution of the problem (1)-(2) is equivalent to determining a positive constant r_0, such that $T_1 + T_2$ has a fixed point on B_{r_0}.

The proof has been divided into four steps.

Step 1.$\| T_1\Delta + T_2\Delta \| \leq r_0$ whenever $\Delta \in B_{r_0}$.

For every $\Delta \in B_{r_0}$, we have

$$\| (T_1\Delta)(\omega) + (T_2\Delta)(\omega) \| \leq \| \Delta_0 \| + L(\| \Delta \| + (\omega)((N_1^k + N_1^h) \| \Delta \| + N_2^k + N_2^h)) + \frac{1-\nu}{\beta(\nu)}(M_1(\| \Delta \| + (C^g + C^f)\omega \| \Delta \|)) + \frac{1-\nu}{\beta(\nu)} M_2 + \frac{\nu}{\beta(\nu)}(M_1(\| \Delta \| + (C^g + C^f)\omega \| \Delta \|))({}^{AB}I_{0+}^{\nu})(\omega) + \frac{\nu}{\beta(\nu)} M_2({}^{AB}I_{0+}^{\nu})(\omega)$$

$$\leq \| \Delta_0 \| + U \| \Delta \| + U((N^k + N^h)\omega + \frac{1-\nu}{\beta(\nu)}(1 + (C^g + C^f)\omega)$$

$$+\frac{I^{\nu}}{\beta(\nu)\Gamma(\nu)}(1+(C^{g}+C^{f})\omega)) \parallel \Delta \parallel +U((N^{k}+N^{h})\omega+\frac{1-\nu}{\beta(\nu)}(1+(C^{g}+C^{f})\omega)$$
$$+\frac{I^{\nu}}{\beta(\nu)\Gamma(\nu)}(1+(C^{g}+C^{f})\omega))$$
$$\leq r_0(1-2U)+2Ur_0$$
$$\leq r_0.$$

Hence, $\parallel T_1\Delta+T_2\Delta \parallel\leq r_0$ for every $\Delta\in B_{r_0}$.

Step 2. T_1 is contraction on B_{r_0}.

If, for any $\Delta,\Phi\in B_{r_0}$, according to (A5) and (8), we have

$$\parallel (T_1\Delta)(\omega)-(T_1\Phi)(\omega)\parallel\leq\parallel \Delta_0-\Phi_0\parallel +\iota\parallel \Delta_0-\Phi_0\parallel +L\parallel \Delta-\Phi\parallel +L(N^{k}+N^{h})\omega\parallel \Delta-\Phi\parallel$$
$$\leq\parallel \Delta_0-\Phi_0\parallel (1+\iota+L\parallel \Delta-\Phi\parallel +L(N^{k}+N^{h})\omega\parallel \Delta-\Phi\parallel)$$
$$\leq R\parallel \Delta_0-\Phi_0\parallel,$$

which implies that $\parallel T_1\Delta-T_1\Phi\parallel\leq R\parallel \Delta_0-\Phi_0\parallel$. Since $R<1$, where $R=1+\iota+L\parallel \Delta-\Phi\parallel +L(N^{k}+N^{h})\omega\parallel \Delta-\Phi\parallel$, therefore T_1 is a contraction.

Step 3. T_2 is completely continuous operator.

Now, we will prove that T_2 is continuous on B_{r_0}. For any $\Delta_n,\Delta\in B_{r_0}, n=1,2,\ldots$ with $\lim_{n\to\infty}\parallel \Delta_n-\Delta\parallel=0$, we get $\lim_{n\to\infty}\Delta_n(\omega)=\Delta(\omega)$, for $\omega\in[0,1]$. Thus, by (A1), we have

$$\lim_{n\to\infty} B(\omega,\Delta_n(\omega),G\Delta_n(\omega),F\Delta_n(\omega))=B(\omega,\Delta(\omega),G\Delta(\omega),F\Delta(\omega)),$$

for $\omega\in[0,1]$.

So we conclude that

$$\sup_{\omega\in[0,1]}\parallel B(\omega,\Delta_n(\omega),G\Delta_n(\omega),F\Delta_n(\omega))-B(\omega,\Delta(\omega),G\Delta(\omega),F\Delta(\omega))\parallel\longrightarrow 0 \text{ as } n\longrightarrow\infty.$$

On other hand, for $t\in[0,1]$

$$\parallel (T_2\Delta_n)(\omega)-(T_2\Delta)(\omega)\parallel$$
$$\leq\frac{1-\nu}{\beta(\nu)}\sup_{\omega\in[0,1]}\parallel B(\omega,\Delta_n(\omega),G\Delta_n(\omega),F\Delta_n(\omega))-B(\omega,\Delta(\omega),G\Delta(\omega),F\Delta(\omega))\parallel$$
$$+\frac{1^{\nu}}{\beta(\nu)\Gamma(\nu)}\sup_{\omega\in[0,1]}\parallel B(\omega,\Delta_n(\omega),G\Delta_n(\omega),F\Delta_n(\omega))-B(\omega,\Delta(\omega),G\Delta(\omega),F\Delta(\omega))\parallel$$
$$\leq(\frac{1-\nu}{\beta(\nu)}-\frac{1^{\nu}}{\beta(\nu)\Gamma(\nu)})\sup_{\omega\in[0,1]}\parallel B(\omega,\Delta_n(\omega),G\Delta_n(\omega),F\Delta_n(\omega))-B(\omega,\Delta(\omega),G\Delta(\omega),F\Delta(\omega))\parallel$$
$$\longrightarrow 0 \text{ as } n\longrightarrow\infty.$$

T_2 is continuous on B_{r_0}.

Next, we have to prove that $T_2\Delta,\Delta\in B_{r_0}$ is relatively compact for which we prove it is uniformly bounded and equicontinuous.

For any $\Delta\in B_{r_0}$, we have $\parallel T_2\Delta\parallel\leq r_0$, which means that $(T_2\Delta)(\omega),\Delta\in B_{r_0}$ is uniformly bounded. Next, we verify that $(T_2\Delta)(\omega),\Delta\in B_{r_0}$ is a equicontinuous. For any $\Delta\in B_{r_0}$ and $0\leq\omega_1\leq\omega_2\leq\omega$, we get

$$\parallel (T_2\Delta)(\omega_2)-(T_2\Delta)(\omega_1)\parallel\leq\frac{1-\nu}{\beta(\nu)}q(\omega_2-\omega_1)+\frac{\nu}{\beta(\nu)}q(\omega_2-\omega_1)\frac{(\omega_2-\omega_1)^{\nu}}{\nu\Gamma(\nu)}$$
$$\leq(\frac{1-\nu}{\beta(\nu)}-\frac{(\omega_2-\omega_1)^{\nu}}{\beta(\nu)\Gamma(\nu)})q(\omega_2-\omega_1)$$
$$\longrightarrow 0 \text{ as } \omega_2\longrightarrow\omega_1,$$

which $\Rightarrow T_2$ is a equicontinuous on B_{r_0}.Thus, T_2 is relatively compact and hence T_2 is completely continuous.

Step 4. To conclude, the existence of the fixed point of the operator $T=T_1+T_2$, it sufficient to show that the set $\epsilon=\{\Phi\in X:\Phi=\lambda T_1(\frac{\Phi}{\lambda})+\lambda T_2(\Phi)\}$ is bounded. Let $\lambda\in(0,1)$, then for each $\omega\in J$,

$$\Delta(\omega) = \lambda T_1(\frac{\Delta}{\lambda}) + \lambda T_2(\Delta)(\omega),$$

from (A1)–(A7), we have

$$\| \Delta(\omega) \| \leq \lambda \| \Delta_0 \| - \lambda \| A(0, \frac{u}{\lambda}(0), 0, 0) \| + \lambda \| A(\omega, \frac{u}{\lambda}(\omega), \frac{Ku}{\lambda}(\omega), \frac{Hu}{\lambda}(\omega)) \|$$

$$+\lambda \| \quad {}^{AB}I_{0+}^{\nu} B(\omega, \Delta(\omega), G\Delta(\omega), F\Delta(\omega)) \|$$

$$\leq \| \phi \| + L(\| \Delta \| + (N^k + N^h)\omega\psi(\| x \|) + (N^k + N^h)\omega) + (\frac{\nu}{\beta(\nu)} + \frac{1^\nu}{\beta(\nu)\Gamma(\nu)})(M(\| \Delta \|$$

$$+(C^g + C^f)\omega\psi(\| \Delta \|) + (C^g + C^f)\omega)) + (\frac{\nu}{\beta(\nu)} + \frac{1^\nu}{\beta(\nu)\Gamma(\nu)})M.$$

Put $\mu(\omega) = \max\{|\Delta(s)|: 0 \leq s \leq \omega\}, \omega \in J$. Then $\| \Delta \| \leq \mu(\omega)$ for all $\omega \in J$, and we have

$$\mu(\omega) \leq \| \phi \| + L\mu(s) + L(N^k + N^h)\omega\psi(\mu(s)) + L(N^k + N^h)\omega + \vartheta^*\mu(s)$$

$$+\vartheta^*(C^g + C^f)\omega\psi(\mu(s)) + \vartheta^*(C^g + C^f)\omega + \vartheta^*$$

$$\leq \| \phi \| + \vartheta^* + (L + \vartheta^*)\mu(s) + L(N^k + N^h)\omega\psi(\mu(s)) + \vartheta^*(C^g + C^f)\omega\psi(\mu(s))$$

$$+L(N^k + N^h)\omega + \vartheta^*(C^g + C^f)\omega$$

$$(1 - L - \vartheta^*)\mu(\omega) \leq \| \phi \| + \vartheta^* + (L(N^k + N^h)\omega + \vartheta^*(C^g + C^f)\omega)(\psi(\mu(s)) + 1).$$

Consequently, if $\| \Delta \|_\infty = \sup \| \Delta(\omega) \|: 0 \leq \omega \leq 1$. Then above inequality becomes

$$(1 - L - \vartheta^*) \| \Delta \|_\infty \leq \| \phi \| + \vartheta^* + (L(N^k + N^h)\omega + \vartheta^*(C^g + C^f)\omega)(\psi(\| \Delta \|_\infty) + 1).$$

i.e.

$$\frac{(1-L-\vartheta^*)\|\Delta\|_\infty}{\|\phi\| + \vartheta^* + (L(N^k+N^h)\omega + \vartheta^*(C^g+C^f)\omega)(\psi\|\Delta\|_\infty + 1)} \leq 1.$$

Then by (A7), there is an M_* such that $\| \Delta \|_\infty \neq M_*$. Consider $U = \{\Delta \in C([0,1], X): \| \Delta \|_\infty \leq M_*\}$, then in U there is no $\Delta \in \partial U$ such that $\Delta = \lambda T(\Delta)$ where $\lambda \in (0,1)$. We states that T has a fixed point Δ in U, which implies that Δ is a solution of (1)-(2), and the proof is completed.

4. Conclusion

The existence and uniqueness of solutions to the nonlinear term of fractional VFIDE with neutral and Atangana-Baleanu derivative in the Caputo sense were investigated in this work. Our findings expand and bring together many of the literary findings. This article contributed in particular to the growth of the fractional calculus with a generic formulation of a FD in respect of another function, in the FDE.The topic examined in this manuscript can be expanded to a greater extent by use of a generic formulation of the Hilfer FD. In addition, we focus on nonlinear fractional systems for VFIDE with nonlocal conditions.

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