Certified Domination Number in Product of Graphs

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ABSTRACT
A set S of vertices in G = (V, E) is called a dominating set of G if every vertex not in S has at least one neighbour in S. A dominating set S of a graph G is said to be a certified dominating set of G if every vertex in S has either zero or at least two neighbours in $V\backslash S$. The certified domination number, $\gamma_{cer}(G)$of G is defined as the minimum cardinality of certified dominating set of G. In this paper, we study the certified domination number of Cartesian product of some standard graphs.

Keywords: Dominating set, Certified Dominating set, Certified Domination Number, Cartesian product.
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1. Introduction

In this paper, graph G = (V,E) we mean a simple, finite, connected, undirected graph with neither loops nor multiple edges. The order $|V(G)|$ is denoted by n. For graph theoretic terminology we refer to West [7]. The open neighborhood of any vertex v in G is $N(v) = \{x : xv \in E(G)\}$ and closed neighborhood of a vertex v in G is $N[v] = N(v) \cup \{v\}$. The degree of a vertex in the graph G is denoted by deg(v) and the maximum degree (minimum degree) in the graph G is denoted by $\Delta(G)$ ($\delta(G)$). For a set $S \subseteq V(G)$ the open (closed) neighborhood N(S)(N[S]) in G is defined as $N(S) = \cup_{v\epsilon S} N(v)$ ($N[S] = \cup_{v\epsilon S} N[v]$. We write K_n, P_n, and C_n for a complete graph, a path graph, a cycle graph of order n, respectively. The complement of a graph G, denoted by $\overline{G}$, is a graph with the vertex set V (G) such that for every two vertices v and w, $vw \in E(\overline{G})$ if and only if $vw \notin E(\overline{G})$.

The concept of certified domination in graphs was introduced by Dettlaff, Lemanska, Topp, Ziemann and Zylinski[3] and further studied in[2]. It has many application in real life situations. This motivated we to study the certified domination number in corona and Cartesian product of graphs.

In [3], authors studied certified dominaiton number in graphs which is defined as follows: Definition 1.1. Let G = (V, E) be any graph of order n. A subset $S \subseteq V(G)$ is called a Certified dominating set of G if S is a dominating set of G and every vertex in S has either zero or at least two neighbours in $V\backslash S$. The certified domination number defined by $\gamma_{cer}(G)$is the minimum cardinality of certified dominating set in G.

2. Known Results:

Theorem 2.1: [2] For any graph G of order n ≥ 2, every certified dominating set of G contains its extreme vertices.

Theorem 2.2: [2] For any graph G of order n, $1 \leq \gamma_{cer}(G) \leq n$.

Theorem 2.3: [2] For any graph G of order n ≥ 3, $\gamma_{cer}(G)$= 1 if and only if G has a vertex of degree n – 1.

Theorem 2.4: [3] For any Path graph P_n of order n≥1,

$$\gamma_{cer}(G) = \begin{cases} 1 \; if \; n = 1 \; or \; 3 \\ 2 \quad\; if \; n = 2 \\ 4 \quad\; if \; n = 4 \end{cases}$$

$\gamma_{cer}(G) = \left\lceil \frac{n}{3} \right\rceil$if $n \geq 5$.

Theorem 2.5: [3] For the Cycle graph C_n (n≥3), $\gamma_{cer}(G) = \left\lceil \frac{n}{3} \right\rceil$.

3. Cartesian Product of Graphs

The Cartesian graph product $G_1 \times G_2$called graph product of graphs with disjoint vertex sets and edge sets and is the graph with the vertex set $V_1 \times V_2$ and $u = (u_1,\, u_2)$ adjacent with $v = (v_1,\, v_2)$ whenever $[u_1 = v_1$ and u_2 adjacent to $v_2]$ or $[u_2 = v_2$ and u_1 adjacent to $v_1]$.

Theorem 3.1: For n≥3, $\gamma_{cer}(P_2 \times P_n) = \left\lfloor \frac{n+2}{2} \right\rfloor$.

Proof: Let $V(P_2 \times P_n) = \{(u_1,\, v_i), (u_2,\, v_i) \colon 1 \leq i \leq n\}$ be the set of vertices of the first and second row, respectively. We prove this theorem by considering six cases.

Case (i). Let n=2. Consider the set $S = \{(u_1,\, v_1), (u_1,\, v_2)\}$. Clearly the set S is a minimum dominating set of $P_2 \times P_n$and each vertices in S has exactly two neighbours in $V(P_2 \times P_n) - S$. Hence, $\gamma_{cer}(P_2 \times P_n) = |S|$= 2 =$\left\lfloor \frac{n+2}{2} \right\rfloor$.

Case (ii). Let n=3. Consider the set $S_1 = \{(u_1,\, v_1), (u_2,\, v_3)\}$. Clearly the set S_1 is a minimum dominating set of $P_2 \times P_n$and each vertices in S_1 has exactly two neighbours in $V(P_2 \times P_n) - S_1$. Hence, $\gamma_{cer}(P_2 \times P_n) = |\, S_1|$= 2 =$\left\lfloor \frac{n+2}{2} \right\rfloor$.

Case (iii). Let n be even and $n \equiv 0(mod\; 4)$. Consider the set $S_2 = \Big\{(u_1,\, v_n), (u_1,\, v_i) \colon i = 4p - 1, 1 \leq p \leq \frac{n}{4}$, $(u_2,\, v_i) \colon i = 4p - 3,\; 1 \leq p \leq \frac{n}{4}\Big\}$.Clearly, the set $S_2$is a minimum dominating set of $P_2 \times P_n$and$|N(u) \cap S_2| \geq 2$ for every $u \in V(P_2 \times P_n) - S_2$. Hence, $\gamma_{cer}(P_2 \times P_n) = |\, S_2| = \left\lfloor \frac{n+2}{2} \right\rfloor$.

Case (iv). Let n be even and $n \not\equiv 0(mod\; 4)$. Consider the set $S_3 = \Big\{(u_1,\, v_i) \colon i = 4p - 1, 1 \leq p \leq \left\lceil \frac{n-3}{4} \right\rceil$, $(u_2,\, v_i) \colon i = 4p - 3,\; 1 \leq p \leq \left\lceil \frac{n-1}{4} \right\rceil\Big\}$.Clearly, the set $S_3$is a minimum dominating set of $P_2 \times P_n$ and$|N(u) \cap S_3| \geq 2$ for every $u \in V(P_2 \times P_n) - S_3$. Hence, $\gamma_{cer}(P_2 \times P_n) = |\, S_3| = \left\lfloor \frac{n+2}{2} \right\rfloor$.

Case (v). Let n be odd and $n \equiv 1(mod\; 4)$. Consider the set $S_4 = \Big\{(u_1,\, v_i) \colon i = 4p - 1, 1 \leq p \leq \left\lceil \frac{n-2}{4} \right\rceil$, $(u_2,\, v_i) \colon i = 4p - 3,\; 1 \leq p \leq \left\lceil \frac{n-1}{4} \right\rceil\Big\}$.Clearly, the set S_4 is a minimum dominating set of $P_2 \times P_n$ and$|N(u) \cap S_4| \geq 2$ for every $u \in V(P_2 \times P_n) - S_4$. Hence, $\gamma_{cer}(P_2 \times P_n) = |\, S_4| = \left\lfloor \frac{n+2}{2} \right\rfloor$.

Case (vi). Let n be odd and $n \not\equiv 1(mod\; 4)$. Consider the set $S_5 = \Big\{(u_1,\, v_i) \colon i = 4p - 1, 1 \leq p \leq \left\lceil \frac{n}{4} \right\rceil$, $(u_2,\, v_i) \colon i = 4p - 3,\; 1 \leq p \leq \left\lceil \frac{n-2}{4} \right\rceil\Big\}$.Clearly, the set S_5 is a minimum dominating set of $P_2 \times P_n$ and$|N(u) \cap S_5| \geq 2$ for every $u \in V(P_2 \times P_n) - S_5$. Hence, $\gamma_{cer}(P_2 \times P_n) = |\, S_5| = \left\lfloor \frac{n+2}{2} \right\rfloor$.

Theorem 3.2: For n≥3, $\gamma_{cer}(P_3 \times P_n) = \left\lfloor \frac{3n+4}{4} \right\rfloor$.

Proof: Let $V(P_3 \times P_n) = \{(u_1,\, v_i), (u_2,\, v_i), (u_3,\, v_i) \,:\, 1 \leq i \leq n\}$ be the set of vertices of the first and second row, third row respectively. We prove this theorem by considering five cases.

Case (i). Let n=3. Consider the set $S = \{(u_1,\, v_1), (u_2,\, v_2), (u_3,\, v_3)$. Clearly that S is a minimum dominating set of $P_3 \times P_n$and each vertices in S has exactly two neighbours in $V(P_3 \times P_n) - S$. Hence, $\gamma_{cer}(P_3 \times P_n) = |S|$= 3 =$\left\lfloor \frac{3n+4}{2} \right\rfloor$.

Case (ii). Let $n \equiv 0(mod\; 4)$. Consider the set $S_1 = \Big\{(u_1,\, v_i), (u_3,\, v_i) \colon i = 4p - 1, 1 \leq p \leq \left\lceil \frac{n-1}{4} \right\rceil$, $(u_2,\, v_i) \colon i = 4p - 3,\; 1 \leq p \leq \left\lceil \frac{n-3}{4} \right\rceil\Big\}$. Clearly, the set S_1 is a minimum dominating set of $P_3 \times P_n$ and$|N(u) \cap S_1| \geq 2$ for every $u \in V(P_3 \times P_n) - S_1$. Hence, $\gamma_{cer}(P_3 \times P_n) = |\, S_1| = \left\lfloor \frac{3n+4}{4} \right\rfloor$.

Case (iii). Let $n \equiv 1(mod\ 4)$. Consider the set $S_2 = \left\{(u_1, v_i), (u_3, v_i): i = 4p-1, 1 \le p \le \left\lceil \frac{n-2}{4} \right\rceil, (u_2, v_i): i = 4p-3,\ 1 \le p \le \left\lceil \frac{n}{4} \right\rceil \right\}$. Clearly, the set S_2 is a minimum dominating set of $P_3 \times P_n$ and $|N(u) \cap S_2| \ge 2$ for every $u \in V(P_3 \times P_n) - S_2$. Hence, $\gamma_{cer}(P_3 \times P_n) = |S_2| = \left\lfloor \frac{3n+4}{4} \right\rfloor$.

Case (iv). Let $n \equiv 2(mod\ 4)$. Consider the set $S_3 = \left\{(u_1, v_i), (u_3, v_i): i = 4p-1, 1 \le p \le \left\lceil \frac{n-3}{4} \right\rceil, (u_2, v_i): i = 4p-3,\ 1 \le p \le \left\lceil \frac{n-1}{4} \right\rceil, (u_2, v_n) \right\}$. Clearly, the set S_3 is a minimum dominating set of $P_3 \times P_n$ and $|N(u) \cap S_3| \ge 2$ for every $u \in V(P_3 \times P_n) - S_3$. Hence, $\gamma_{cer}(P_3 \times P_n) = |S_3| = \left\lfloor \frac{3n+4}{4} \right\rfloor$.

Case (v). Let $n \equiv 3(mod\ 4)$. Consider the set $S_4 = \left\{(u_1, v_i), (u_3, v_i): i = 4p-1, 1 \le p \le \left\lceil \frac{n-4}{4} \right\rceil, (u_2, v_i): i = 4p-3,\ 1 \le p \le \left\lceil \frac{n-2}{4} \right\rceil, (u_2, v_n), (u_2, v_{n-1}) \right\}$. Clearly, the set S_4 is a minimum dominating set of $P_3 \times P_n$ and $|N(u) \cap S_4| \ge 2$ for every $u \in V(P_3 \times P_n) - S_4$. Hence, $\gamma_{cer}(P_3 \times P_n) = |S_4| = \left\lfloor \frac{3n+4}{4} \right\rfloor$.

Theorem 3.3: For $n \ge 4$, $\gamma_{cer}(P_3 \times P_n) = \begin{cases} n+1 & \text{if } n = 5, 6, 9. \\ n & \text{if otherwise.} \end{cases}$

Proof: Let $V(P_4 \times P_n) = \{(u_1, v_i), (u_2, v_i), (u_3, v_i), (u_4, v_i) : 1 \le i \le n\}$ be the set of vertices of the first, second, third and fourth row respectively. We prove this theorem by considering four cases.

Case (i). Let $n = 5$ or 9. Consider the set $S = \left\{(u_1, v_i), : i = 4p-1, 1 \le p \le \left\lceil \frac{n-3}{4} \right\rceil, (u_3, v_i): i = 4p-3,\ 1 \le p \le \left\lceil \frac{n}{4} \right\rceil \right\}$. Clearly, the set S is a minimum dominating set of $P_4 \times P_n$ and $|N(u) \cap S| \ge 2$ for every $u \in V(P_4 \times P_n) - S$. Hence, $\gamma_{cer}(P_4 \times P_n) = |S| = n+1$.

Case (ii). Let $n = 6$. Consider the set $S_1 = \left\{(u_1, v_i), : i = 4p-2, 1 \le p \le \left\lceil \frac{n}{4} \right\rceil, (u_2, v_4), (u_3, v_1), (u_3, v_6), (u_4, v_3), (u_4, v_5) \right\}$. Clearly, the set S_1 is a minimum dominating set of $P_4 \times P_n$ and $|N(u) \cap S_1| \ge 2$ for every $u \in V(P_4 \times P_n) - S_1$. Hence, $\gamma_{cer}(P_4 \times P_n) = |S_1| = n+1$.

Case (iii). Let $n \equiv 0(mod\ 4)$. Consider the set $S_2 = \left\{(u_1, v_i), : i = 4p-2, 1 \le p \le \frac{n}{4}, (u_4, v_i): i = 4p-1,\ 1 \le p \le \frac{n}{4} \right\}$. Clearly, the set S_2 is a minimum dominating set of $P_4 \times P_n$ and $|N(u) \cap S_2| \ge 2$ for every $u \in V(P_4 \times P_n) - S_2$. Hence, $\gamma_{cer}(P_4 \times P_n) = |S_2| = n$.

Case (iv). Let $n \not\equiv 0(mod\ 4)$ and $n \ne 5,6,9$. Now we split $P_4 \times P_n$ into k number of $P_4 \times P_4$ and $P_4 \times P_3$ blocks B_i in $1 \le i \le k$ such that k is maximum. Also, assume, $|V(B_i)| \ge |V(B_{i+1})|$ and $V(B_i) \cap V(B_{i+1}) = \emptyset$. Let us consider the vertices of $P_4 \times P_4$ as $V(P_4 \times P_4) = \{(u_i, v_j), 1 \le i = j \le 4\}$ and the vertices $P_4 \times P_3$ as $V(P_4 \times P_3) = \{(p_i, q_j), 1 \le i \le 4, 1 \le j \le 3\}$. Let $S = \{(u_3, v_1), (u_1, v_2), (u_2, v_4), (u_4, v_3)\}$ is a minimum certified dominating set of each B_i in $P_4 \times P_4$. We consider the following three sub-cases:

Sub-case (i): Blocks B_i contains only one copy of $P_4 \times P_3$.
Let $U = \{(p_1, q_1), (p_3, q_3), (p_4, q_1)\}$ be the set of vertices belongs to $P_4 \times P_3$ $block$. Then, the set $S \cup U$ is the minimum certified domination set of $P_4 \times P_n$ and so $\gamma_{cer}(P_4 \times P_n) = n$.
Sub-case (ii): Blocks B_i contains two copies of $P_4 \times P_3$, say (B_i, B_{i+1}).
Let $L = \{(p_1, q_1), (p_3, q_3), (p_4, q_1)\}$ be the set of vertices belongs to B_i and let $M = \{(p_1, q_1), (p_2, q_3), (p_4, q_2)\}$ be the set of vertices belongs to B_{i+1}. Then the set $S \cup L \cup M$ is the minimum certified domination set of $P_4 \times P_n$ and so $\gamma_{cer}(P_4 \times P_n) = n$.
Sub-case (iii): Blocks B_i contains two copies of $P_4 \times P_3$, say (B_i, B_{i+1}, B_{i+2}).
Let $N = \{(p_1, q_2), (p_3, q_3), (p_4, q_1)\}$ be the set of vertices belongs to B_i and B_{i+2} and let $O = \{(p_1, q_1), (p_2, q_3), (p_4, q_2)\}$ be the set of vertices belongs to B_{i+1}. Then the set $S \cup N \cup O$ is the minimum certified domination set of $P_4 \times P_n$ and so $\gamma_{cer}(P_4 \times P_n) = n$.

Theorem 3.4: For $n \ge 2$, $\gamma_{cer}(C_3 \times P_n) = \begin{cases} \left\lceil \frac{3n}{4} \right\rceil + 1 & if\ n \equiv 0(mod\ 4) \\ \left\lceil \frac{3n}{4} \right\rceil & otherwise \end{cases}$

Proof: Let $V(C_3 \times P_n) = \{(u_1, v_i), (u_2, v_i), (u_3, v_i) : 1 \le i \le n\}$ be the set of vertices of the first, second and third row respectively. We prove this theorem by considering six cases.

Case (i). Let n=2. Consider the set $S = \{(u_1, v_2), (u_2, v_1)\}$. Clearly the set S is a minimum dominating set of $C_3 \times P_n$ and each vertices in S has exactly two neighbours in $V(C_3 \times P_n) - S$. Hence, $\gamma_{cer}(C_3 \times P_n) = |S| = 3 = \left\lceil \frac{3n}{4} \right\rceil$.

Case (ii). Let n=3. Consider the set $S_1 = \{(u_1, v_3), (u_2, v_2), (u_3, v_1)\}$. Clearly the set S_1 is a minimum dominating set of $C_3 \times P_n$ and $|N(u) \cap S_1| \geq 2$ for every $u \in V(C_3 \times P_n) - S_1$. Hence, $\gamma_{cer}(C_3 \times P_n) = |S_1|= 3 =\left\lceil\frac{3n}{4}\right\rceil$.

Case (iii). Let $n \equiv 0(mod\ 4)$. Consider the set $S_2 = \left\{(u_1, v_i), : i = 4p-3, 1 \leq p \leq \left\lceil\frac{n-3}{4}\right\rceil, (u_2, v_i), (u_3, v_i) : i = 4p-1,\ 1 \leq p \leq \left\lceil\frac{n-3}{4}\right\rceil\right\}$. Clearly, the set S_2 is a minimum dominating set of $C_3 \times P_n$ and $|N(u) \cap S_2| \geq 2$ for every $u \in V(C_3 \times P_n) - S_2$. Hence, $\gamma_{cer}(C_3 \times P_n) = |S_2| = \left\lceil\frac{3n}{4}\right\rceil + 1$.

Case (iv). Let $n \equiv 1(mod\ 4)$. Consider the set $S_3 = \left\{(u_1, v_i), : i = 4p-3, 1 \leq p \leq \left\lceil\frac{n}{4}\right\rceil, (u_2, v_i), (u_3, v_i) : i = 4p-1,\ 1 \leq p \leq \left\lceil\frac{n-2}{4}\right\rceil\right\}$. Clearly, the set S_3 is a minimum dominating set of $C_3 \times P_n$ and $|N(u) \cap S_3| \geq 2$ for every $u \in V(C_3 \times P_n) - S_3$. Hence, $\gamma_{cer}(C_3 \times P_n) = |S_3| = \left\lceil\frac{3n}{4}\right\rceil$.

Case (v). Let $n \equiv 2(mod\ 4)$. Consider the set $S_4 = \left\{(u_1, v_i), : i = 4p-3, 1 \leq p \leq \left\lceil\frac{n-1}{4}\right\rceil, (u_2, v_i), (u_3, v_i) : i = 4p-1,\ 1 \leq p \leq \left\lceil\frac{n-3}{4}\right\rceil\right\}$. Clearly, the set S_4 is a minimum dominating set of $C_3 \times P_n$ and $|N(u) \cap S_4| \geq 2$ for every $u \in V(C_3 \times P_n) - S_4$. Hence, $\gamma_{cer}(C_3 \times P_n) = |S_4| = \left\lceil\frac{3n}{4}\right\rceil$.

Case (v). Let $n \equiv 3(mod\ 4)$. Consider the set $S_5 = \left\{(u_1, v_i), : i = 4p-3, 1 \leq p \leq \left\lceil\frac{n-2}{4}\right\rceil, (u_2, v_i), (u_3, v_i) : i = 4p-1,\ 1 \leq p \leq \left\lceil\frac{n}{4}\right\rceil\right\}$. Clearly, the set S_5 is a minimum dominating set of $C_3 \times P_n$ and $|N(u) \cap S_5| \geq 2$ for every $u \in V(C_3 \times P_n) - S_5$. Hence, $\gamma_{cer}(C_3 \times P_n) = |S_5| = \left\lceil\frac{3n}{4}\right\rceil$.

Theorem 3.5: For n≥2, $\gamma_{cer}(C_4 \times P_n) = n$.
Proof: Let $V(C_4 \times P_n) = \{(u_1, v_i), (u_2, v_i), (u_3, v_i), (u_3, v_i) : 1 \leq i \leq n\}$ be the set of vertices of the first, second, third and fourth row respectively. We prove this theorem by considering two cases.
Case (i). Let n be even. Consider the set $S = \left\{(u_2, v_i), : i = 2p-1, 1 \leq p \leq \left\lceil\frac{n-1}{2}\right\rceil, (u_4, v_i) : i = 2p,\ 1 \leq p \leq \frac{n}{2}\right\}$. Clearly the set S is a minimum dominating set of $C_4 \times P_n$ and $|N(u) \cap S| \geq 2$ for every $u \in V(C_4 \times P_n) - S$. Hence, $\gamma_{cer}(C_4 \times P_n) = |S| = n$.

Case (ii). Let n be odd. Consider the set$S_1 = \left\{(u_2, v_i), : i = 2p-1, 1 \leq p \leq \left\lceil\frac{n}{2}\right\rceil, (u_4, v_i) : i = 2p,\ 1 \leq p \leq \left\lceil\frac{n}{2}\right\rceil\right\}$. Clearly the set S_1 is a minimum dominating set of $C_4 \times P_n$ and $|N(u) \cap S_1| \geq 2$ for every $u \in V(C_4 \times P_n) - S_1$. Hence, $\gamma_{cer}(C_4 \times P_n) = |S_1| = n$.

Theorem 3.6: For n≥2, $\gamma_{cer}(C_3 \times C_n) = \left\lceil\frac{3n}{4}\right\rceil$.
Proof: Let $V(C_3 \times C_n) = \{(u_1, v_i), (u_2, v_i), (u_3, v_i) : 1 \leq i \leq n\}$ be the set of vertices of the first, second, third row respectively. We prove this theorem by considering two cases.
Case (i). Let n=2. Consider the set $S = \{(u_1, v_1), (u_2, v_1), (u_3, v_3)\}$. Clearly the set S is a minimum dominating set of $C_3 \times C_n$. Hence, $\gamma_{cer}(C_3 \times C_n) = |S|= 3=\left\lceil\frac{3n}{4}\right\rceil$.

Case (ii). Let $n \equiv 0(mod\ 4)$. Consider the set $S_1 = \left\{(u_1, v_i): i = 4p+1, 1 \leq p \leq \left\lceil\frac{n-3}{4}\right\rceil, n \geq 8, (u_2, v_i), : i = 4p-2, 1 \leq p \leq \frac{n-2}{4}\right\}$. Clearly, the set S_1 is a minimum dominating set of $C_3 \times C_n$ andevery vertex in S_1 has greater than two neighbours in $V(C_3 \times C_n) - S_1$. Therefore, that S_1 is a minimum certified dominating set of $C_3 \times C_n$ and hence, $\gamma_{cer}(C_3 \times C_n) = |S_1|= \left\lceil\frac{3n}{4}\right\rceil$.

Results 3.7:

(i) For n≥2, $\gamma_{cer}(C_4 \times C_n) = n$.

(ii) For n≥5, $\gamma_{cer}(C_3 \times C_n) = \begin{cases} n & if\ n \equiv 0(mod\ 5) \\ n+2 & if\ n \equiv 3(mod\ 5) \\ n+1 & Otherwise \end{cases}$

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