

Some aspects of meromorphic functions whose certain differential polynomial share a set.

^aRajeshwari. S, ^aBhuvaneshwari. T, and ^aChethan Kumar S

^aDepartment of Mathematics, School of Engineering, Presidency University, Yelahanka, Bangalore-560064, INDIA

Abstract: This article deals with the most important and interesting topic of Nevanlinna theory that is subject on the unicity problem of the power of a meromorphic functions whose differential polynomial share a set with its derivative. The results which we got in this survey is the improved and generalized results of Meng and Li [10].

Keywords: Meromorphic function; Differential polynomial; Weighted set sharing; Uniqueness.

1. Introduction

In this literature survey, the term used here is accepted value distribution theory of meromorphic functions, equalling the milestone of complex analysis during past decades. It is known that the reader is well-known about elemental notes of the Nevanlinna theory, such as $T(r; \varphi)$, $m(r; \varphi)$, $N(r; \varphi)$, $S(r; \varphi)$ etc...

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This notation can be found in [4, 12] for any two functions of meromorphic which is not constant, φ and ψ , $b \in S(\varphi) \cap S(\psi)$, we say that φ and ψ sharing the value b CM (multiplicities counted), and in case we do not include the multiplicities, then φ and ψ are said to sharing the value b IM (multiplicities ignored).

We indicate by $S(r, \varphi)$ any functions satisfying

$$S(r, \varphi) = o\{T(r, \varphi)\} \text{ as } r \rightarrow \infty$$

probably beyond with a fixed definite measure.

Definition 1.1. [8, 9] For a complex number $b \in \mathbb{C} \cup \{\infty\}$, we indicate by $E_\Omega(b, \varphi)$ the fixed of all ω -points of φ where ω -point with multiplicity m is counted m times if $m \leq \Omega$ and $\Omega + 1$ times if $m > \Omega$. For a complex number $b \in \mathbb{C} \cup \{\infty\}$, thus $E_\Omega(b, \varphi) = E_\Omega(b, \psi)$, then we say that φ and ψ sharing the value with weight Ω . The definition infers that if φ, ψ sharing a value b and weight Ω , then z_0 is a zero of $\varphi - b$ with multiplicity $m (\leq \Omega)$ subject to a zero of $\psi - b$ with multiplicity $n (> \Omega)$, where m is not really equivalent to n . We write φ, ψ sharing (b, Ω) then φ, ψ sharing (b, p) for all integers $p, 0 \leq p \leq \Omega$. And we note that φ, ψ sharing a and b IM or CM if and only if φ, ψ sharing $(b, 0)$ or (b, ∞) respectively.

Definition 1.2. [8] Let I another set of definite factors of $\mathbb{C} \cup \{\infty\}$ and Ω a positive integers or ∞ . Altogether $E_\Omega(I, \Omega)$ the element $\cup_{b \in I} S E_\Omega(b, f)$. Obviously $E_f(I) = E_f(I, \infty)$ as well $\overline{E_f} = (I, 0)$.

In 2019, Meng and Li obtained the following outcomes.

Theorem 1. [10] Let $\varphi(z)$ and $\psi(z)$ be two non-constant meromorphic functions, and let n, d, k be definite integers with $n > 2k + \frac{3k+9}{d}$, $d \geq 2$ and $I = \{b \in \mathbb{C}; b^d = 1\}$, in case $E_{(\varphi^n)(k)}(I, 1) = E_{(\psi^n)(k)}(I, 1)$, subsequently classified among these retains:

1. $\varphi = c_1 e^{cz}$ and $\psi = c_2 e^{-cz}$ for constants c_1, c_2, c other than zero, in order to enable

$$(-1)^{kd} (c_1 c_2)^{nd} (nc)^{2kd} = 1.$$

2. $\varphi = t\psi$ with $t^{nd} = 1, t \in \mathbb{C}$.

Theorem 2. [10]. Let $\varphi(z)$ and $\psi(z)$ be two non-constant mero-morphic functions, and let n, d, k be definite integers with $n > 2k + \frac{8k+14}{d}$, $d \geq 2$ and $I = \{b \in \mathbb{C} : b^d = 1\}$, if $E_{(\varphi^n)(k)}(I, 0) = E_{(\psi^n)(k)}(I, 0)$, subsequently classified among these retains: (1) $\varphi = c_1 e^{cz}$ and $\psi = c_2 e^{-cz}$ for constants c_1, c_2, c other than zero, in order to enable $\varphi = c_1 e^{cz}$ and $\psi = c_2 e^{-cz}$ for constants c_1, c_2, c other than zero, in order to enable $(-1)^{kd}(c_1 c_2)^{nd}(nc)^{2kd} = 1$. (2) $\varphi = t\psi$ with $t^{nd} = 1, t \in \mathbb{C}$.

From this perspective, we accomplished the succeeding conclusion:

Theorem 3. Let $\varphi(z)$ and $\psi(z)$ be two non-constant mero-morphic functions, and let n, d, k be definite integers with $n > 2k + \frac{8k+14}{d}$, $d \geq 2$ and $I = \{b \in \mathbb{C} : b^d = 1\}$, and considering $Q(z) = b_m z^m + b_{m-1} z^{m-1} + \dots + b_1 z + b_0$ be non-zero polynomials where $b_0, b_1, \dots, b_{m-1}, b_m$ are complex constants. If $E_{(\varphi^n)(k)}(I, 1) = E_{(\psi^n)(k)}(I, 1)$, subsequently classified among these retains:

1. $\varphi(z) = \frac{c_1}{\sqrt[n]{c_0}} e^{cz}$ and $\psi(z) = \frac{c_1}{\sqrt[n]{c_0}} e^{-cz}$ constants c_1, c_2, c other than zero, in order to enable $(-1)^k(c_1 c_2)^n(nc)^{2k} = 1$ or $\varphi \equiv h\psi$ for a stable h ensure that $h^d = 1$.

2. φ and ψ persuade the mathematical equation $R(\varphi, \psi) \equiv 0$, whereby.

$$R(\varpi_1, \varpi_2) = \varpi_1^n (b_m \varpi_1 + b_{m-1} \varpi_1^{m-1} + \dots + b_1 \varpi_1 + b_0) - \varpi_2^n (b_m \varpi_2 + b_{m-1} \varpi_2^{m-1} + \dots + b_1 \varpi_2 + b_0).$$

Theorem 4. Let $\varphi(z)$ and $\psi(z)$ be two non-constant mero-morphic functions, and let n, d, k be definite integers with $n > 2k + \frac{6k+2m+8}{2}$, $d \geq 2$ and $I = \{b \in \mathbb{C} : b^d = 1\}$, and considering $Q(z)$ be defined in Theorem 3. If $E_{(\varphi^n Q(\varphi))(k)}(I, 0) = E_{(\psi^n Q(\psi))(k)}(I, 0)$, subsequently conviction of Theorem 3 retains.

2. Some Lemmas

Throughout this, we present few lemmas which will be essential in the development. We propose $\mathcal{R}$ as:

$$\mathcal{R} = \left(\frac{u^{(\alpha)}}{u^{(\beta)}} - \frac{2u^{(\beta)}}{u-1} \right) - \left(\frac{v^{(\alpha)}}{v^{(\beta)}} - \frac{2v^{(\beta)}}{v-1} \right), \text{ where } \alpha = 1, \beta = 1$$

Lemma 1. [2] Let $\mathcal{R}$ be defined at the top. In case $\mathcal{U}$ as well $\mathcal{V}$ sharing (1,1) plus $\mathcal{R} \neq 0$.

Then,

$$T(\tau, \mathcal{U}) \leq N_2\left(\tau, \frac{1}{u}\right) + N_2(\tau, \mathcal{U}) + N_2\left(\tau, \frac{1}{v}\right) + N_2(\tau, \mathcal{V}) + \frac{1}{2}\bar{N}\left(\tau, \frac{1}{u}\right) + \frac{1}{2}\bar{N}(\tau, \mathcal{U}) + \mathcal{S}(\tau, \mathcal{U}) + \mathcal{S}(\tau, \mathcal{V}).$$

the similar inequality remains for $T(\tau, \mathcal{V})$.

Lemma 2. [2] Let $\mathcal{R}$ be defined at the top. In case $\mathcal{U}$ as well $\mathcal{V}$ sharing (1,0) plus $\mathcal{R} \neq 0$.

Then,

$$T(\tau, \mathcal{U}) \leq N_2\left(\tau, \frac{1}{u}\right) + N_2(\tau, \mathcal{U}) + N_2\left(\tau, \frac{1}{v}\right) + N_2(\tau, \mathcal{V}) + 2\bar{N}(\tau, \mathcal{U}) + \bar{N}\left(\tau, \frac{1}{v}\right) + \bar{N}(\tau, \mathcal{V}) + \mathcal{S}(\tau, \mathcal{U}) + \mathcal{S}(\tau, \mathcal{V}).$$

the similar inequality remains for $T(\tau, \mathcal{V})$.

Lemma 3. [2] Let φ be a non-constant mero-morphic function and n, m, κ be positive integers with $n + m > l$. If z_0 is a pole of φ , as $p + q$ is its multiplicity in some neighbourhood of z_0 .

$$\frac{(\varphi^n Q(\varphi))^{(k)}}{\varphi^{n+m-k}} = \bar{h}_k(z - z_0)^{(p+q-1)k},$$

where $\bar{h}_k$ is regular z_0 and $\bar{h}_k(z_0) \neq 0$, the proof is straight forward and so it is omitted.

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Lemma 5. Let φ be a non-constant mero-morphic function and n, m, κ be positive integers with $n + m > 2k$. Then

$$i. (n + m - 2k)T(\tau, \varphi) + kN(\tau, \varphi) + N\left(\tau, \frac{\varphi^{n+m-k}}{(\varphi^n Q(\varphi))^{(k)}}\right) \leq T\left(\tau, (\varphi^n Q(\varphi))^{(k)}\right) + \mathcal{S}(\tau, \varphi)$$

$$ii. N\left(\tau, \frac{\varphi^{n+m-k}}{(\varphi^n Q(\varphi))^{(k)}}\right) \leq kN(\tau, \varphi) + \mathcal{S}(\tau, \varphi).$$

Proof. We notice that

$$N\left(\tau, (\varphi^n \mathcal{Q}(\varphi))^{(k)}\right) = (n+m)N(\tau, \varphi) + k\bar{N}(\tau, \varphi)$$

$$T\left(\tau, (\varphi^n \mathcal{Q}(\varphi))^{(k)}\right) = (n+m)T(\tau, \varphi)$$

so that $S(\tau, \varphi^n \mathcal{Q}(\varphi)) = \mathcal{S}(\tau, \varphi)$

And by Milloux theorem we bring

$$m\left(\tau, \frac{\varphi^{(j)}}{\varphi}\right) = \mathcal{S}(\tau, \varphi)$$

for $j = 1, 2, \dots, l$. Now

$$(n+m-k)m(\tau, \varphi) = m(\tau, \varphi^{n+m-k})$$

$$\leq m\left(\tau, (\varphi^n \mathcal{Q}(\varphi))^{(k)}\right) + m\left(\tau, \frac{\varphi^{n+m-k}}{(\varphi^n \mathcal{Q}(\varphi))^{(k)}}\right) + \mathcal{S}(\tau, \varphi)$$

$$\leq m\left(\tau, (\varphi^n \mathcal{Q}(\varphi))^{(k)}\right) + T\left(\tau, \frac{\varphi^{n+m-k}}{(\varphi^n \mathcal{Q}(\varphi))^{(k)}}\right) - N\left(\tau, \frac{\varphi^{n+m-k}}{(\varphi^n \mathcal{Q}(\varphi))^{(k)}}\right) + \mathcal{S}(\tau, \varphi).$$

we learn that,

$$T(\tau, \varphi) = T\left(\tau, \frac{1}{\varphi}\right) + O(1),$$

$$\leq m\left(\tau, (\varphi^n \mathcal{Q}(\varphi))^{(k)}\right) + T\left(\tau, \frac{(\varphi^n \mathcal{Q}(\varphi))^{(k)}}{\varphi^{n+m-k}}\right) - N\left(\tau, \frac{\varphi^{n+m-k}}{(\varphi^n \mathcal{Q}(\varphi))^{(k)}}\right) + O(1)$$

$$= m\left(\tau, (\varphi^n \mathcal{Q}(\varphi))^{(k)}\right) + m\left(\tau, \frac{(\varphi^n \mathcal{Q}(\varphi))^{(k)}}{\varphi^{n+m-k}}\right) + N\left(\tau, \frac{(\varphi^n \mathcal{Q}(\varphi))^{(k)}}{\varphi^{n+m-k}}\right) - N\left(\tau, \frac{\varphi^{n+m-k}}{(\varphi^n \mathcal{Q}(\varphi))^{(k)}}\right)$$

$$+ O(1)$$

$$\leq m\left(\tau, (\varphi^n \mathcal{Q}(\varphi))^{(k)}\right) + km(\tau, \varphi) + N\left(\tau, \frac{(\varphi^n \mathcal{Q}(\varphi))^{(k)}}{\varphi^{n+m-k}}\right) - N\left(\tau, \frac{\varphi^{n+m-k}}{(\varphi^n \mathcal{Q}(\varphi))^{(k)}}\right) + \mathcal{S}(\tau, \varphi)$$

By Lemmas 3 & 4, we reach

$$(n+m-k)m(\tau, \varphi)$$

$$\leq m\left(\tau, (\varphi^n \mathcal{Q}(\varphi))^{(k)}\right) + km(\tau, \varphi) + kN(\tau, \varphi) + k\bar{N}(\tau, \varphi) - N\left(\tau, \frac{\varphi^{n+m-k}}{(\varphi^n \mathcal{Q}(\varphi))^{(k)}}\right)$$

$$+ \mathcal{S}(\tau, \varphi)$$

$$= m\left(\tau, (\varphi^n \mathcal{Q}(\varphi))^{(k)}\right) + N\left(\tau, (\varphi^n \mathcal{Q}(\varphi))^{(k)}\right) - (n+m)N(\tau, \varphi) + kT(\tau, \varphi) -$$

$$N\left(\tau, \frac{\varphi^{n+m-k}}{(\varphi^n \mathcal{Q}(\varphi))^{(k)}}\right) + \mathcal{S}(\tau, \varphi).$$

and so

$$(n+m-2k)T(\tau, \varphi) + kN(\tau, \varphi) + N\left(\tau, \frac{\varphi^{n+m-k}}{(\varphi^n \mathcal{Q}(\varphi))^{(k)}}\right) \leq T\left(\tau, (\varphi^n \mathcal{Q}(\varphi))^{(k)}\right) + \mathcal{S}(\tau, \varphi).$$

which is (i).

Again with Lemma 3 one can notice that $\frac{\varphi^{n+m-k}}{(\varphi^n \mathcal{Q}(\varphi))^{(k)}}$ is regular at each pole of φ . So poles of $\frac{\varphi^{n+m-k}}{(\varphi^n \mathcal{Q}(\varphi))^{(k)}}$ occur only at the zeros of $(\varphi^n \mathcal{Q}(\varphi))^{(k)}$. We observe by Lemma 4 that a zero of $(\varphi^n \mathcal{Q}(\varphi))^{(k)}$, which is additionally a zero of φ with multiplicity p , is a pole of $\frac{\varphi^{n+m-k}}{(\varphi^n \mathcal{Q}(\varphi))^{(k)}}$ as $(p+q-1)k$ is its multiplicity. Thus

$$N\left(\tau, \frac{\varphi^{n+m-k}}{(\varphi^n \mathcal{Q}(\varphi))^{(k)}} / \varphi = 0\right) = kN(\tau, 0; \varphi) - k\bar{N}(\tau, 0; \varphi).$$

Also

$$N\left(\tau, \frac{\varphi^{n+m-k}}{(\varphi^n \mathcal{Q}(\varphi))^{(k)}} / \varphi \neq 0\right) = N(\tau, 0; (\varphi^n \mathcal{Q}(\varphi))^{(k)} | \varphi \neq 0) \leq N\left(\tau, 0; \frac{(\varphi^n \mathcal{Q}(\varphi))^{(k)}}{\varphi^n \mathcal{Q}(\varphi)}\right)$$

$$\leq T\left(\tau, \frac{(\varphi^n \mathcal{Q}(\varphi))^{(k)}}{\varphi^n \mathcal{Q}(\varphi)}\right) = N\left(\tau, \frac{(\varphi^n \mathcal{Q}(\varphi))^{(k)}}{\varphi^n \mathcal{Q}(\varphi)}\right) + \mathcal{S}(\tau, \varphi)$$

$$= k\bar{N}(\tau, 0; \varphi) + k\bar{N}(\tau, \varphi) + \mathcal{S}(\tau, \varphi).$$

Therefore,

$$\begin{aligned} N\left(\tau, \frac{\varphi^{n+m-k}}{(\varphi^n \mathcal{Q}(\varphi))^{(k)}}\right) &= N\left(\tau, \frac{\varphi^{n+m-k}}{(\varphi^n \mathcal{Q}(\varphi))^{(k)}} / \varphi = 0\right) + N\left(\tau, \frac{\varphi^{n+m-k}}{(\varphi^n \mathcal{Q}(\varphi))^{(k)}} / \varphi \neq 0\right) \\ &\leq k\bar{N}(\tau, 0; \varphi) + k\bar{N}(\tau, \varphi) + \mathcal{S}(\tau, \varphi), \end{aligned}$$

as (ii) follows.

Lemma 6. [13] Let φ be a non-constant mero-morphic function, k be a define integer, afterwards.

$$N_\omega\left(\tau, \frac{1}{\varphi^{(k)}}\right) \leq N_{\omega+k}\left(\tau, \frac{1}{\varphi^{(k)}}\right) + k\bar{N}(\tau, \varphi) + \mathcal{S}(\tau, \varphi).$$

where $N_\omega\left(\tau, \frac{1}{\varphi^{(k)}}\right)$ represents the function counting the zeros of $\varphi^{(k)}$ whereby a zero of multiplicity m is counted m times if $m \leq \omega$ likewise ω times if $m > \omega$. Certainly,

$$N_\omega\left(\tau, \frac{1}{\varphi^{(k)}}\right) = N_1\left(\tau, \frac{1}{\varphi^{(k)}}\right).$$

Lemma 7. [7] Let φ and ψ be two non-constant entire functions, and let n, l be two definite integers with $n > l$, also considering $\mathcal{Q}(z) = \mathfrak{b}_m z^m + \mathfrak{b}_{m-1} z^{m-1} + \dots + \mathfrak{b}_1 z + \mathfrak{b}_0$ be a non-zero polynomials, where $\mathfrak{b}_0, \mathfrak{b}_1, \dots, \mathfrak{b}_{m-1}$ are complex constants. If $(\varphi^n \mathcal{Q}(\varphi))^{(l)} (\psi^n \mathcal{Q}(\psi))^{(l)} \equiv 1$ then $\mathcal{Q}(z)$ reduces to a non-zero monomials, i.e., $\mathcal{Q}(z) = \mathfrak{b}_i z^i \neq 0$ for some $i = 0, 1, 2, \dots, m$. Moreover $\varphi(z) = \frac{d_1}{n+1\sqrt{a_i}} \exp(cz)$ and $\psi(z) = \frac{d_1}{n+1\sqrt{a_i}} \exp(-cz)$ where d_1, d_2, c are three constants to enable $(-1)^l (d_1 d_2)^{(n+l)} \{(n+i)c\}^{2l} = 1$.

2. Proof of Theorems

Proof of Theorem 3. Let

$$\mathcal{U} = ((\varphi^n \mathcal{Q}(\varphi))^{(k)})^d \quad \mathcal{V} = ((\psi^n \mathcal{Q}(\psi))^{(k)})^d. \quad (1)$$

Since, $E_{(\varphi^n \mathcal{Q}(\varphi))^{(k)}}(\mathcal{S}, 1) = E_{(\psi^n \mathcal{Q}(\psi))^{(k)}}(\mathcal{S}, 1)$, we see that $\mathcal{U}$ and $\mathcal{V}$ sharing (1,1). If $\mathcal{R} \neq 0$ then by Lemma 1

$$T(\tau, \mathcal{R}) \leq N_2\left(\tau, \frac{1}{\mathcal{U}}\right) + N_2(\tau, \mathcal{U}) + N_2\left(\tau, \frac{1}{\mathcal{V}}\right) + N_2(\tau, \mathcal{V}) + \frac{1}{2}\bar{N}\left(\tau, \frac{1}{\mathcal{U}}\right) + \mathcal{S}(\tau, \mathcal{U}) + \mathcal{S}(\tau, \mathcal{V}) \quad (2)$$

By Lemma 5, we gain,

$$\begin{aligned} (n+m-2k)T(\tau, \varphi) &\leq T\left(\tau, (\varphi^n \mathcal{Q}(\varphi))^{(k)}\right) + \mathcal{S}(\tau, \varphi) \\ &\leq (k+1)(n+m)T(\tau, \varphi) + \mathcal{S}(\tau, \varphi) \end{aligned} \quad (3)$$

$$\begin{aligned} (n+m-2k)T(\tau, \psi) &\leq T\left(\tau, (\psi^n \mathcal{Q}(\psi))^{(k)}\right) + \mathcal{S}(\tau, \varphi) \\ &\leq (k+1)(n+m)T(\tau, \varphi) + \mathcal{S}(\tau, \varphi) \end{aligned} \quad (4)$$

Since

$$T\left(\tau, ((\varphi^n \mathcal{Q}(\varphi))^{(k)})^d\right) = dT\left(\tau, (\varphi^n \mathcal{Q}(\varphi))^{(k)}\right) + \mathcal{S}\left(\tau, (\varphi^n \mathcal{Q}(\varphi))^{(k)}\right) \quad (5)$$

$$T\left(\tau, ((\psi^n \mathcal{Q}(\psi))^{(k)})^d\right) = dT\left(\tau, (\psi^n \mathcal{Q}(\psi))^{(k)}\right) + \mathcal{S}\left(\tau, (\psi^n \mathcal{Q}(\psi))^{(k)}\right) \quad (6)$$

It is easy to see that

$$\mathcal{S}\left(\tau, ((\varphi^n \mathcal{Q}(\varphi))^{(k)})^d\right) = \mathcal{S}\left(\tau, (\varphi^n \mathcal{Q}(\varphi))^{(k)}\right) + \mathcal{S}(\tau, \varphi) \quad (7)$$

$$\mathcal{S}\left(\tau, ((\psi^n \mathcal{Q}(\psi))^{(k)})^d\right) = \mathcal{S}\left(\tau, (\psi^n \mathcal{Q}(\psi))^{(k)}\right) + \mathcal{S}(\tau, \psi) \quad (8)$$

On the other hand

$$N_2(\tau, \mathcal{U}) = N_2\left(\tau, \left((\varphi^n \mathcal{Q}(\varphi))^{(k)}\right)^d\right) = 2\bar{N}(\tau, \varphi) \quad (9)$$

$$N_2(\tau, \mathcal{V}) = N_2\left(\tau, \left((\psi^n \mathcal{Q}(\psi))^{(k)}\right)^d\right) = 2\bar{N}(\tau, \psi) \quad (10)$$

$$\frac{1}{2}\bar{N}(\tau, \mathcal{U}) = \frac{1}{2}\bar{N}(\tau, \varphi) \quad (11)$$

By Lemma 6, we took

$$\begin{aligned} N_2\left(\tau, \frac{1}{\mathcal{U}}\right) &= N_1\left(\tau, \frac{1}{\left((\varphi^n \mathcal{Q}(\varphi))^{(k)}\right)^d}\right) = 2\bar{N}\left(\tau, \frac{1}{(\varphi^n \mathcal{Q}(\varphi))^{(k)}}\right) \\ &\leq 2N_{k+1}\left(\tau, \frac{1}{(\varphi^n \mathcal{Q}(\varphi))}\right) + 2k\bar{N}(\tau, \varphi^n \mathcal{Q}(\varphi)) + \mathcal{S}(\tau, \varphi) \\ &\leq 2(k+1)\bar{N}\left(\tau, \frac{1}{\varphi}\right) + mN\left(\tau, \frac{1}{\varphi}\right) + 2k\bar{N}(\tau, \varphi) + \mathcal{S}(\tau, \varphi). \end{aligned} \quad (12)$$

$$\begin{aligned} \frac{1}{2}\bar{N}\left(\tau, \frac{1}{\mathcal{U}}\right) &= \frac{1}{2}\bar{N}\left(\tau, \frac{1}{\left((\varphi^n \mathcal{Q}(\varphi))^{(k)}\right)^d}\right) = \frac{1}{2}\bar{N}\left(\tau, \frac{1}{(\varphi^n \mathcal{Q}(\varphi))^{(k)}}\right) \\ &\leq \frac{1}{2}N_{k+1}\left(\tau, \frac{1}{\varphi^n \mathcal{Q}(\varphi)}\right) + \frac{k}{2}\bar{N}\left(\tau, \frac{1}{\varphi^n \mathcal{Q}(\varphi)}\right) + \mathcal{S}(\tau, \varphi^n) \\ &\leq \frac{k+1}{2}\bar{N}\left(\tau, \frac{1}{\varphi}\right) + mN\left(\tau, \frac{1}{\varphi}\right) + \frac{k}{2}\bar{N}(\tau, \varphi) + \mathcal{S}(\tau, \varphi). \end{aligned} \quad (13)$$

And

$$\begin{aligned} N_2\left(\tau, \frac{1}{\mathcal{V}}\right) &= 2\bar{N}\left(\tau, \frac{1}{\left((\psi^n \mathcal{Q}(\psi))^{(k)}\right)^d}\right) \leq 2\left(\bar{N}\left(\tau, \frac{1}{\psi^{n+m-k}}\right)\right) + N\left(\tau, \frac{\psi^{n+m-k}}{(\psi^n \mathcal{Q}(\psi))^{(k)}}\right) \\ &\leq 2\left(\bar{N}\left(\tau, \frac{1}{\psi}\right)\right) + N\left(\tau, \frac{\psi^{n+m-k}}{(\psi^n \mathcal{Q}(\psi))^{(k)}}\right). \end{aligned} \quad (14)$$

$$N_2\left(\tau, \frac{1}{\mathcal{U}}\right) \leq 2\left(\bar{N}\left(\tau, \frac{1}{\varphi}\right)\right) + N\left(\tau, \frac{\psi^{n+m-k}}{(\psi^n \mathcal{Q}(\psi))^{(k)}}\right). \quad (15)$$

Combining (2), (9), (10), (11), (12), (13) and (14), we deduct

$$\begin{aligned} T\left(\tau, \left((\varphi^n \mathcal{Q}(\varphi))^{(k)}\right)^d\right) &\leq \frac{5k+4m+5}{2}\bar{N}\left(\tau, \frac{1}{\varphi}\right) + \frac{5k+5}{2}\bar{N}(\tau, \varphi) + 2\bar{N}(\tau, \psi) + 2\bar{N}\left(\tau, \frac{1}{\psi}\right) \\ &\quad + 2N\left(\tau, \frac{\psi^{n+m-k}}{(\psi^n \mathcal{Q}(\psi))^{(k)}}\right) + \mathcal{S}(\tau, \varphi) + \mathcal{S}(\tau, \psi) \\ &\leq (3k+2m+3)T(\tau, \varphi) + 4T(\tau, \psi) + 2k\bar{N}(\tau, \varphi) \\ &\quad + 2N\left(\tau, \frac{\psi^{n+m-k}}{(\psi^n \mathcal{Q}(\psi))^{(k)}}\right) + \mathcal{S}(\tau, \varphi) + \mathcal{S}(\tau, \psi). \end{aligned} \quad (16)$$

Similarly, we have

$$\begin{aligned} T\left(\tau, \left((\varphi^n \mathcal{Q}(\varphi))^{(k)}\right)^d\right) &\leq (4k+2m+3)T(\tau, \psi) + 4T(\tau, \varphi) + 2k\bar{N}(\tau, \psi) \\ &\quad + 2N\left(\tau, \frac{\psi^{n+m-k}}{(\psi^n \mathcal{Q}(\psi))^{(k)}}\right) + \mathcal{S}(\tau, \varphi) + \mathcal{S}(\tau, \psi). \end{aligned} \quad (17)$$

By Lemma 5, we took

$$(n + m - 2k)d T(\tau, \varphi) + kdN(\tau, \varphi) + d N\left(\tau, \frac{\varphi^{n+m-k}}{(\varphi^n Q(\varphi))^{(k)}}\right) \leq dT\left(\tau, (\varphi^n Q(\varphi))^{(k)}\right) + \mathcal{S}(\tau, \varphi). \quad (18)$$

$$(n + m - 2k)d T(\tau, \psi) + kdN(\tau, \psi) + d N\left(\tau, \frac{\psi^{n+m-k}}{(\psi^n Q(\psi))^{(k)}}\right) \leq dT\left(\tau, (\psi^n Q(\psi))^{(k)}\right) + \mathcal{S}(\tau, \psi). \quad (19)$$

From (16), (17), (18) and (19), we have

$$\begin{aligned} & (n + m - 2k)d T(\tau, \varphi) + kdN(\tau, \varphi) + d N\left(\tau, \frac{\varphi^{n+m-k}}{(\varphi^n Q(\varphi))^{(k)}}\right) + (n + m - 2k)d T(\tau, \psi) + kdN(\tau, \psi) \\ & \quad + d N\left(\tau, \frac{\psi^{n+m-k}}{(\psi^n Q(\psi))^{(k)}}\right) \\ & \leq (4k + 2m + 3)T(\tau, \varphi) + 4T(\tau, \psi) + 2k\bar{N}(\tau, \varphi) + 2N\left(\tau, \frac{\psi^{n+m-k}}{(\psi^n Q(\psi))^{(k)}}\right) \\ & \quad + (4k + 2m + 3)T(\tau, \psi) + 4T(\tau, \varphi) + 2k\bar{N}(\tau, \psi) \\ & \quad + 2N\left(\tau, \frac{\psi^{n+m-k}}{(\psi^n Q(\psi))^{(k)}}\right) + \mathcal{S}(\tau, \varphi) + \mathcal{S}(\tau, \psi). \end{aligned} \quad (20)$$

Since $d \geq 2$

$$dN\left(\tau, \frac{\varphi^{n+m-k}}{(\varphi^n Q(\varphi))^{(k)}}\right) \geq 2N\left(\tau, \frac{\varphi^{n+m-k}}{(\varphi^n Q(\varphi))^{(k)}}\right) \quad (21)$$

$$dN\left(\tau, \frac{\psi^{n+m-k}}{(\psi^n Q(\psi))^{(k)}}\right) \geq 2N\left(\tau, \frac{\psi^{n+m-k}}{(\psi^n Q(\psi))^{(k)}}\right) \quad (22)$$

$$kdN(\tau, \varphi) \geq 2k\bar{N}(\tau, \varphi) \quad (23)$$

$$kdN(\tau, \psi) \geq 2k\bar{N}(\tau, \psi) \quad (24)$$

Therefore

$$\begin{aligned} & (nd + md - 2kd - 4k - 2m - 7)T(\tau, \varphi) + (nd + md - 2kd - 4k - 2m - 7)T(\tau, \psi) \\ & \leq \mathcal{S}(\tau, \varphi) + \mathcal{S}(\tau, \psi). \end{aligned} \quad (25)$$

which contradicts $n + m > 2k + \frac{6k+2m+7}{d}$. Hence $\mathcal{R} \equiv 0$. By integrating, we must

$$\frac{1}{v-1} = \frac{\mathcal{A}^*}{u-1} + \mathcal{B}^*. \quad (26)$$

Where $\mathcal{A}^* \neq 0$ as well as $\mathcal{B}^*$ are consistence. Thus

$$\mathcal{V} = \frac{(\mathcal{B}^*+1)\mathcal{U}+(\mathcal{A}^*-\mathcal{B}^*-1)}{\mathcal{B}^*\mathcal{U}+(\mathcal{A}^*-\mathcal{B}^*)} \quad (27)$$

and

$$\mathcal{U} = \frac{(\mathcal{B}^*-\mathcal{A}^*)\mathcal{V}+(\mathcal{A}^*-\mathcal{B}^*-1)}{\mathcal{B}^*\mathcal{V}-(\mathcal{B}^*+1)} \quad (28)$$

further in our view can have three sub cases:

Sub case 1: $\mathcal{B}^* \neq 0, -1$. Then from (28) we may

$$\bar{N}\left(\tau, \frac{1}{\nu - \frac{\mathcal{B}^*+1}{\mathcal{B}^*}}\right) = \bar{N}(\tau, \mathcal{U}). \quad (29)$$

By of Nevanlinna second elementary theorem and (14), can take

$$\begin{aligned} T(\tau, \nu) &\leq \bar{N}(\tau, \nu) + \bar{N}\left(\tau, \frac{1}{\nu}\right) + \bar{N}\left(\tau, \frac{1}{\nu - \frac{\mathcal{B}^*+1}{\mathcal{B}^*}}\right) + \mathcal{S}(\tau, \nu) \\ &\leq \bar{N}(\tau, \nu) + \bar{N}_2\left(\tau, \frac{1}{\nu}\right) + \bar{N}(\tau, \mathcal{U}) + \mathcal{S}(\tau, \nu) \\ &\leq \bar{N}(\tau, \psi) + 2\left(\bar{N}\left(\tau, \frac{1}{\psi}\right) + N\left(\tau, \frac{\psi^{n+m-k}}{(\psi^n \mathcal{Q}(\psi))^{(k)}}\right)\right) + \bar{N}(\tau, \varphi) + \mathcal{S}(\tau, \psi). \end{aligned} \quad (30)$$

if $\mathcal{A}^* - \mathcal{B}^* \neq 0$, as a result of (27) that

$$\bar{N}\left(\tau, \frac{1}{\mathcal{U} - \frac{\mathcal{B}^*+1-\mathcal{A}^*}{\mathcal{B}^*+1}}\right) = \bar{N}\left(\tau, \frac{1}{\nu}\right). \quad (31)$$

Again by Nevanlinna second elementary theorem, also (15),

$$\begin{aligned} T(\tau, \mathcal{U}) &\leq \bar{N}(\tau, \mathcal{U}) + \bar{N}\left(\tau, \frac{1}{\mathcal{U}}\right) + \bar{N}\left(\tau, \frac{1}{\mathcal{U} - \frac{\mathcal{B}^*+1-\mathcal{A}^*}{\mathcal{B}^*+1}}\right) + \mathcal{S}(\tau, \mathcal{U}) \\ &\leq \bar{N}(\tau, \mathcal{U}) + \bar{N}_2\left(\tau, \frac{1}{\mathcal{U}}\right) + \bar{N}(\tau, \nu) + \mathcal{S}(\tau, \mathcal{U}) \\ &\leq \bar{N}(\tau, \varphi) + 2\left(\bar{N}\left(\tau, \frac{1}{\varphi}\right) + N\left(\tau, \frac{\varphi^{n+m-k}}{(\varphi^n \mathcal{Q}(\varphi))^{(k)}}\right)\right) \\ &\quad + N_{k+1}\left(\tau, \frac{1}{\psi^n \mathcal{Q}(\psi)}\right) + k\bar{N}(\tau, \psi) + \mathcal{S}(\tau, \psi) \\ &\leq \bar{N}(\tau, \varphi) + 2\left(\bar{N}\left(\tau, \frac{1}{\varphi}\right) + N\left(\tau, \frac{\varphi^{n+m-k}}{(\varphi^n \mathcal{Q}(\varphi))^{(k)}}\right)\right) + (k+1)N\left(\tau, \frac{1}{\psi}\right) \\ &\quad + mN\left(\tau, \frac{1}{\psi}\right) + k\bar{N}(\tau, \psi) + \mathcal{S}(\tau, \psi). \end{aligned} \quad (32)$$

From (18), (19), (30) and (32), we get

$$\begin{aligned} (n+m-2k)dT(\tau, \varphi) + kdN(\tau, \varphi) + dN\left(\tau, \frac{\varphi^{n+m-k}}{(\varphi^n \mathcal{Q}(\varphi))^{(k)}}\right) + (n+m-2k)dT(\tau, \psi) \\ + kdN(\tau, \psi) + dN\left(\tau, \frac{\psi^{n+m-k}}{(\psi^n \mathcal{Q}(\psi))^{(k)}}\right) \\ \leq \bar{N}(\tau, \varphi) + 2\left(\bar{N}\left(\tau, \frac{1}{\varphi}\right) + N\left(\tau, \frac{\varphi^{n+m-k}}{(\varphi^n \mathcal{Q}(\varphi))^{(k)}}\right)\right) \end{aligned}$$

$$+(k+1)\bar{N}\left(\tau, \frac{1}{\psi}\right) + mN\left(\tau, \frac{1}{\psi}\right) + k\bar{N}(\tau, \psi) \quad (33)$$

$$+ \bar{N}(\tau, \psi) + 2\left(\bar{N}\left(\tau, \frac{1}{\psi}\right) + N\left(\tau, \frac{\psi^{n+m-k}}{(\psi^n \mathcal{Q}(\psi))^{(k)}}\right)\right) \\ + \bar{N}(\tau, \varphi) + 2N\left(\tau, \frac{1}{\psi}\right) + \mathcal{S}(\tau, \varphi) + \mathcal{S}(\tau, \psi).$$

Since $d \geq 2$

$$dN\left(\tau, \frac{\varphi^{n+m-k}}{(\varphi^n \mathcal{Q}(\varphi))^{(k)}}\right) \geq 2N\left(\tau, \frac{\varphi^{n+m-k}}{(\varphi^n \mathcal{Q}(\varphi))^{(k)}}\right). \quad (34)$$

$$dN\left(\tau, \frac{\psi^{n+m-k}}{(\psi^n \mathcal{Q}(\psi))^{(k)}}\right) \geq 2N\left(\tau, \frac{\psi^{n+m-k}}{(\psi^n \mathcal{Q}(\psi))^{(k)}}\right). \quad (35)$$

$$kdN(\tau, \varphi) \geq 2k\bar{N}(\tau, \varphi) \quad (36)$$

$$kdN(\tau, \psi) \geq 2k\bar{N}(\tau, \psi) \quad (37)$$

Therefore

$$(nd + md - 2kd - 4k - 2m - 7)T(\tau, \varphi) + (nd + md - 2kd - 4k - 2m - 7)T(\tau, \psi) \\ \leq \mathcal{S}(\tau, \varphi) + \mathcal{S}(\tau, \psi). \quad (38)$$

which contradicts $n + m > 2k + \frac{6k+2m+7}{d}$. Moreover $\mathcal{A}^* - \mathcal{B}^* - 1 \equiv 0$. Then by (27)

$$\bar{N}\left(\tau, \frac{1}{u + \frac{1}{\mathcal{B}^*}}\right) = \bar{N}(\tau, \mathcal{V}). \quad (39)$$

Again by Nevanlinna second elementary theorem, also (15)

$$T(\tau, \mathcal{U}) \leq \bar{N}(\tau, \mathcal{U}) + \bar{N}\left(\tau, \frac{1}{\mathcal{U}}\right) + \bar{N}\left(\tau, \frac{1}{\mathcal{U} + \frac{1}{\mathcal{B}^*}}\right) + \mathcal{S}(\tau, \mathcal{U}) \\ \leq \bar{N}(\tau, \mathcal{U}) + \bar{N}_2\left(\tau, \frac{1}{\mathcal{U}}\right) + \bar{N}(\tau, \mathcal{V}) + \mathcal{S}(\tau, \mathcal{U}) \\ \leq \bar{N}(\tau, \varphi) + 2\left(\bar{N}\left(\tau, \frac{1}{\varphi}\right) + N\left(\tau, \frac{\varphi^{n+m-k}}{(\varphi^n \mathcal{Q}(\varphi))^{(k)}}\right)\right) + \bar{N}(\tau, \psi) + \mathcal{S}(\tau, \varphi). \quad (40)$$

Combine (18), (19), (30) and (40), Therefore

$$(nd + md - 2kd - 2)T(\tau, \varphi) + (nd + md - 2kd - 2)T(\tau, \psi) \leq \mathcal{S}(\tau, \varphi) + \mathcal{S}(\tau, \psi). \quad (41)$$

which violates our given assumption.

Sub case 2: $\mathcal{B}^* = -1$. Then

$$\mathcal{V} = \frac{\mathcal{A}^*}{\mathcal{A}^* + 1 - \mathcal{U}} \quad (42)$$

and

$$\mathcal{U} = \frac{(1 + \mathcal{A}^*)\mathcal{V} - \mathcal{A}^*}{\mathcal{V}}. \quad (43)$$

If $\mathcal{A}^* + 1 \neq 0$, we obtain

$$\bar{N}\left(\tau, \frac{1}{u - \mathcal{A}^* - 1}\right) = \bar{N}(\tau, \mathcal{V}). \quad (44)$$

$$\bar{N}\left(\tau, \frac{1}{v - \frac{\mathcal{A}^*}{\mathcal{A}^* + 1}}\right) = \bar{N}\left(\tau, \frac{1}{u}\right). \quad (45)$$

By similar argument we can obtain a contradiction. Therefore $\mathcal{A}^* + 1 = 0$, then $\mathcal{UV} \equiv 1$ that is

$$\left((\varphi^n \mathcal{Q}(\varphi))^{(k)}\right)^d \left((\psi^n \mathcal{Q}(\psi))^{(k)}\right)^d = 1.$$

By Lemma 7, we have $\varphi(z) = \frac{d_1}{n+1\sqrt{a_1}} e^{cz}$ and $\psi(z) = \frac{d_2}{n+1\sqrt{a_1}} e^{-cz}$ where the constants d, d_2, c enables $(-1)^k (d_1 d_2)^{(n+i)} \{(n+i)c^{2k} = 1$.

Sub case 3: $\mathcal{B}^* = 0$. Then (27) and (28) gives $\mathcal{V} = \frac{u + \mathcal{A}^* - 1}{\mathcal{A}^*}$ and $\mathcal{U} = \mathcal{A}^* \mathcal{V} + 1 - \mathcal{A}^*$. If $\mathcal{A}^* - 1 \neq 0$, then

$$\bar{N}\left(\tau, \frac{1}{u - \mathcal{A}^* - 1}\right) = \bar{N}\left(\tau, \frac{1}{v}\right). \quad (46)$$

$$\bar{N}\left(\tau, \frac{1}{v - \frac{1 - \mathcal{A}^*}{\mathcal{A}^*}}\right) = \bar{N}\left(\tau, \frac{1}{u}\right). \quad (47)$$

ProceedingsimilarlyasinSubcase1, wecometoacontradiction. Therefore $\mathcal{A}^* - 1 = 0$, then $\mathcal{U} \equiv \mathcal{V}$, that is, $\left((\varphi^n \mathcal{Q}(\varphi))^{(k)}\right)^d = \left((\psi^n \mathcal{Q}(\psi))^{(k)}\right)^d$, we have

$$\varphi^n (a_m \varphi^m + \dots + a_0) = \psi^n (a_m \psi^m + \dots + a_0) \quad (48)$$

considering $\bar{h} = \frac{\varphi}{\psi}$. If $\bar{h}$ is stable, using $\varphi = \psi \bar{h}$ in (48) we perform

$$a_m \psi^{n+m} (\bar{h}^{n+m} - 1) + a_{m-1} \psi^{n+m-1} (\bar{h}^{n+m} - 1) + \dots + a_0 \psi^n (\bar{h}^n - 1) = 0$$

which indicates $\bar{h}^d = 1$, when $d = (n + m, \dots, n + m - i, \dots, n)$, $a_{m-i} \neq 0$ for few $i = 0, 1, \dots, m$. Thus $\varphi(z) = \psi(z) \bar{h}$ for a constant $\bar{h}$ such a way that $\bar{h}^d = 1$ where $d = (n + m, \dots, n + m - i, \dots, n)$, $a_{m-i} \neq 0$ for few $i = 0, 1, \dots, m$. If $\bar{h}$ is not a stable, then we understand by (48) that φ and ψ satisfy the algebraic equation $R(\varphi, \psi) \equiv 0$, where $R(\varphi_1 \varphi_2) = \varphi_1^n (a_m \varphi_1^m + a_{m-1} \varphi_1^{m-1} + \dots + a_1 \varphi_1 + a_0) - \varphi_2^n (a_m \varphi_2^m + a_{m-1} \varphi_2^{m-1} + \dots + a_1 \varphi_2 + a_0)$.

Proof of Theorem 4. considering

$$\mathcal{U} = \left((\varphi^n \mathcal{Q}(\varphi))^{(k)}\right)^d \quad \mathcal{V} = \left((\psi^n \mathcal{Q}(\psi))^{(k)}\right)^d \quad (49)$$

since $E_{(\varphi^n)^{(k)}}(\mathcal{S}, 0) = E_{(\psi^n)^{(k)}}(\mathcal{S}, 0)$, we see that $\mathcal{U}$ and $\mathcal{V}$ sharing $(1, 0)$. If $\mathcal{R} \neq 0$ then by Lemma 1

$$\begin{aligned} T(\tau, \mathcal{U}) &\leq N_2\left(\tau, \frac{1}{u}\right) + N_2(\tau, \mathcal{U}) + N_2\left(\tau, \frac{1}{v}\right) + N_2(\tau, \mathcal{V}) + \frac{1}{2} \bar{N}\left(\tau, \frac{1}{u}\right) + \bar{N}_2\left(\tau, \frac{1}{u}\right) \\ &\quad + 2\bar{N}(\tau, \mathcal{U}) + \bar{N}\left(\tau, \frac{1}{v}\right) + \bar{N}(\tau, \mathcal{V}) \mathcal{S}(\tau, \mathcal{U}) + \mathcal{S}(\tau, \mathcal{V}) \end{aligned} \quad (50)$$

Make use of Lemma 5, we receive,

$$\begin{aligned} (n + m - 2k)T(\tau, \varphi) &\leq T\left(\tau, (\varphi^n \mathcal{Q}(\varphi))^{(k)}\right) + \mathcal{S}(\tau, \varphi) \\ &\leq (k + 1)(n + m)T(\tau, \varphi) + \mathcal{S}(\tau, \varphi). \end{aligned} \quad (51)$$

$$(n+m-2k)T(\tau, \psi) \leq T\left(\tau, (\psi^n \mathcal{Q}(\psi))^{(k)}\right) + \mathcal{S}(\tau, \psi) \quad (52)$$

$$\leq (k+1)(n+m)T(\tau, \psi) + \mathcal{S}(\tau, \psi).$$

Since

$$T\left(\tau, \left((\varphi^n \mathcal{Q}(\varphi))^{(k)}\right)^d\right) = dT\left(\tau, (\varphi^n \mathcal{Q}(\varphi))^{(k)}\right) + \mathcal{S}\left(\tau, (\varphi^n \mathcal{Q}(\varphi))^{(k)}\right) \quad (53)$$

$$T\left(\tau, \left((\psi^n \mathcal{Q}(\psi))^{(k)}\right)^d\right) = dT\left(\tau, (\psi^n \mathcal{Q}(\psi))^{(k)}\right) + \mathcal{S}\left(\tau, (\psi^n \mathcal{Q}(\psi))^{(k)}\right) \quad (54)$$

It is easy to see that

$$\mathcal{S}\left(\tau, \left((\varphi^n \mathcal{Q}(\varphi))^{(k)}\right)^d\right) = \mathcal{S}\left(\tau, (\varphi^n \mathcal{Q}(\varphi))^{(k)}\right) + \mathcal{S}(\tau, \varphi) \quad (55)$$

$$\mathcal{S}\left(\tau, \left((\psi^n \mathcal{Q}(\psi))^{(k)}\right)^d\right) = \mathcal{S}\left(\tau, (\psi^n \mathcal{Q}(\psi))^{(k)}\right) + \mathcal{S}(\tau, \psi) \quad (56)$$

On the other hand

$$N_2(\tau, \mathcal{U}) = N_2\left(\tau, \left((\varphi^n \mathcal{Q}(\varphi))^{(k)}\right)^d\right) = 2\bar{N}(\tau, \varphi) \quad (57)$$

$$N_2(\tau, \mathcal{V}) = N_2\left(\tau, \left((\psi^n \mathcal{Q}(\psi))^{(k)}\right)^d\right) = 2\bar{N}(\tau, \psi) \quad (58)$$

$$2\bar{N}(\tau, \mathcal{U}) = 2\bar{N}(\tau, \varphi) \quad (59)$$

$$2\bar{N}(\tau, \mathcal{V}) = 2\bar{N}(\tau, \psi) \quad (60)$$

By Lemma 6, we have

$$\begin{aligned} N_2\left(\tau, \frac{1}{\mathcal{U}}\right) &= N_1\left(\tau, \frac{1}{\left((\varphi^n \mathcal{Q}(\varphi))^{(k)}\right)^d}\right) = 2\bar{N}\left(\tau, \frac{1}{(\varphi^n \mathcal{Q}(\varphi))^{(k)}}\right) \\ &\leq 2N_{k+1}\left(\tau, \frac{1}{(\varphi^n \mathcal{Q}(\varphi))}\right) + 2k\bar{N}(\tau, \varphi^n \mathcal{Q}(\varphi)) + \mathcal{S}(\tau, \varphi) \\ &\leq 2(k+1)\bar{N}\left(\tau, \frac{1}{\varphi}\right) + mN\left(\tau, \frac{1}{\varphi}\right) + 2k\bar{N}(\tau, \varphi) + \mathcal{S}(\tau, \varphi). \end{aligned} \quad (61)$$

$$\begin{aligned} 2\bar{N}\left(\tau, \frac{1}{\mathcal{U}}\right) &= 2\bar{N}\left(\tau, \frac{1}{\left((\varphi^n \mathcal{Q}(\varphi))^{(k)}\right)^d}\right) = 2\bar{N}\left(\tau, \frac{1}{(\varphi^n \mathcal{Q}(\varphi))^{(k)}}\right) \\ &\leq 2N_{k+1}\left(\tau, \frac{1}{\varphi^n \mathcal{Q}(\varphi)}\right) + 2k\bar{N}\left(\tau, \frac{1}{\varphi^n \mathcal{Q}(\varphi)}\right) + \mathcal{S}(\tau, \varphi^n \mathcal{Q}(\varphi)) \\ &\leq 2(k+1)\bar{N}\left(\tau, \frac{1}{\varphi}\right) + mN\left(\tau, \frac{1}{\varphi}\right) + 2k\bar{N}(\tau, \varphi) + \mathcal{S}(\tau, \varphi). \end{aligned} \quad (62)$$

$$\bar{N}\left(\tau, \frac{1}{\mathcal{V}}\right) = \bar{N}\left(\tau, \frac{1}{\left((\psi^n \mathcal{Q}(\psi))^{(k)}\right)^d}\right) = \bar{N}\left(\tau, \frac{1}{(\psi^n \mathcal{Q}(\psi))^{(k)}}\right) \quad (63)$$

$$\begin{aligned} &\leq N_{k+1}\left(\tau, \frac{1}{(\psi^n \mathcal{Q}(\psi))}\right) + k\bar{N}(\tau, \psi^n \mathcal{Q}(\psi)) + \mathcal{S}(\tau, \psi^n \mathcal{Q}(\psi)) \\ &\leq 2(k+1)\bar{N}\left(\tau, \frac{1}{\psi}\right) + mN\left(\tau, \frac{1}{\psi}\right) + 2k\bar{N}(\tau, \psi) + \mathcal{S}(\tau, \varphi). \end{aligned}$$

$$\begin{aligned} N_2\left(\tau, \frac{1}{\mathcal{V}}\right) &= 2\bar{N}\left(\tau, \frac{1}{(\psi^n \mathcal{Q}(\psi))^{(k)}}\right) \\ &\leq 2\left(\bar{N}\left(\tau, \frac{1}{\psi^{n+m-k}}\right)\right) + N\left(\tau, \frac{\psi^{n+m-k}}{(\psi^n \mathcal{Q}(\psi))^{(k)}}\right) \end{aligned} \quad (64)$$

$$\leq 2 \left(\bar{N} \left(\tau, \frac{1}{\psi} \right) \right) + N \left(\tau, \frac{\psi^{n+m-k}}{(\psi^n \mathcal{Q}(\psi))^{(k)}} \right).$$

Combining (50), (53), (57), to (64), we deduce

$$\begin{aligned} T \left(\tau, \left((\varphi^n \mathcal{Q}(\varphi))^{(k)} \right)^d \right) &\leq (4k + 2m + 4) \bar{N} \left(\tau, \frac{1}{\varphi} \right) + (4k + 4) \bar{N}(\tau, \varphi) + (k + m + 3) \bar{N} \left(\tau, \frac{1}{\psi} \right) \\ &\quad + (k + 2) \bar{N}(\tau, \psi) + 2N \left(\tau, \frac{\psi^{n+m-k}}{(\psi^n \mathcal{Q}(\psi))^{(k)}} \right) + \mathcal{S}(\tau, \varphi) + \mathcal{S}(\tau, \psi) \\ &\leq (6k + 2m + 8)T(\tau, \varphi) + (2k + m + 5)T(\tau, \psi) + 2k \bar{N}(\tau, \varphi) \\ &\quad + 2N \left(\tau, \frac{\psi^{n+m-k}}{(\psi^n \mathcal{Q}(\psi))^{(k)}} \right) + \mathcal{S}(\tau, \varphi) + \mathcal{S}(\tau, \psi). \end{aligned} \quad (65)$$

Likewise, we have

$$\begin{aligned} T \left(\tau, \left((\varphi^n \mathcal{Q}(\varphi))^{(k)} \right)^d \right) &\leq (6k + 2m + 8)T(\tau, \psi) + (2k + m + 5)T(\tau, \varphi) + 2k \bar{N}(\tau, \psi) \\ &\quad + 2N \left(\tau, \frac{\psi^{n+m-k}}{(\psi^n \mathcal{Q}(\psi))^{(k)}} \right) + \mathcal{S}(\tau, \varphi) + \mathcal{S}(\tau, \psi). \end{aligned} \quad (66)$$

By Lemma 5, we have

$$(n + m - 2k)dT(\tau, \varphi) + kdN(\tau, \varphi) + dN \left(\tau, \frac{\varphi^{n+m-k}}{(\varphi^n \mathcal{Q}(\varphi))^{(k)}} \right) \leq dT \left(\tau, (\varphi^n \mathcal{Q}(\varphi))^{(k)} \right) + \mathcal{S}(\tau, \varphi). \quad (67)$$

$$(n + m - 2k)dT(\tau, \psi) + kdN(\tau, \psi) + dN \left(\tau, \frac{\psi^{n+m-k}}{(\psi^n \mathcal{Q}(\psi))^{(k)}} \right) \leq dT \left(\tau, (\psi^n \mathcal{Q}(\psi))^{(k)} \right) + \mathcal{S}(\tau, \psi). \quad (68)$$

From (65), (66), (67) and (68), we have

$$\begin{aligned} &(n + m - 2k)dT(\tau, \varphi) + kdN(\tau, \varphi) + dN \left(\tau, \frac{\varphi^{n+m-k}}{(\varphi^n \mathcal{Q}(\varphi))^{(k)}} \right) + (n + m - 2k)dT(\tau, \psi) + kdN(\tau, \psi) + \\ &\quad dN \left(\tau, \frac{\psi^{n+m-k}}{(\psi^n \mathcal{Q}(\psi))^{(k)}} \right) \\ &\leq (6k + 2m + 8)T(\tau, \varphi) + (2k + m + 5)T(\tau, \psi) + 2k \bar{N}(\tau, \varphi) + 2N \left(\tau, \frac{\psi^{n+m-k}}{(\psi^n \mathcal{Q}(\psi))^{(k)}} \right) \\ &\quad + (6k + 2m + 8)T(\tau, \psi) + 4T(\tau, \psi) + (2k + m + 5)T(\tau, \varphi) + 2k \bar{N}(\tau, \psi) \\ &\quad + 2N \left(\tau, \frac{\psi^{n+m-k}}{(\psi^n \mathcal{Q}(\psi))^{(k)}} \right) + \mathcal{S}(\tau, \varphi) + \mathcal{S}(\tau, \psi). \end{aligned} \quad (69)$$

Since $d \geq 2$

$$dN \left(\tau, \frac{\varphi^{n+m-k}}{(\varphi^n \mathcal{Q}(\varphi))^{(k)}} \right) \geq 2N \left(\tau, \frac{\varphi^{n+m-k}}{(\varphi^n \mathcal{Q}(\varphi))^{(k)}} \right) \quad (70)$$

$$dN\left(\tau, \frac{\psi^{n+m-k}}{(\psi^n \varrho(\psi))^{(k)}}\right) \geq 2N\left(\tau, \frac{\psi^{n+m-k}}{(\psi^n \varrho(\psi))^{(k)}}\right) \quad (71)$$

$$kdN(\tau, \varphi) \geq 2k\bar{N}(\tau, \varphi) \quad (72)$$

$$kdN(\tau, \psi) \geq 2k\bar{N}(\tau, \psi) \quad (73)$$

Therefore

$$\begin{aligned} (nd + md - 2kd - 4k - 2m - 7)T(\tau, \varphi) + (nd + md - 2kd - 4k - 2m - 7)T(\tau, \psi) \\ \leq \mathcal{S}(\tau, \varphi) + \mathcal{S}(\tau, \psi). \end{aligned} \quad (74)$$

which contradicts $n + m > 2k + \frac{6k+2m+7}{d}$. Hence $\mathcal{R} \equiv 0$. Similar to the argument in Theorem 3 we see that Theorem 4 holds.

Conclusion. The subject of the paper is the power of a meromorphic function whose certain differential polynomial sharing a set with its derivative. The tool we used in our discussion is classical valued distribution theory of meromorphic functions, which is a part of the milestone of complex analysis during last century. Here we introduce the idea of set sharing problems in different aspects and its relevant concepts like uniqueness of differential polynomials.

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