H -Supplemented Sub Modules through Non-Cosingular Modules

Tolesa DEKEBA Bekele*

Department of Mathematics, College of Natural and Computational Science,

Dire Dawa University,Dire Dawa, Ethiopia

E-mail: toleyoobii2020@gmail.com

ORCID ID: 0000-0003-1841-612X

Abstract: The classes of H-supplemented sub modules are a very nice generalization of lifting modules which have been studied approximately recently. In this paper, we would like to address some general and specific characterizations and properties of H-supplemented and γ-H-supplemented sub modules. Suppose M be a module over a commutative ring R, then M is called γ-H-supplemented if and only if for every sub-module N of M there is a direct summand D of M such that $M = N + F$ implies $M = D + F$ for every submodule F of M with M/F non-cosingular. Also we demonstrate that M is γ-H-supplemented if and only if for every submodule N of M there exists a direct summand D of M such that $(N + D)/N \ll_{\gamma} M/N$ and $(N + D)/D \ll_{\gamma} M/D$. In addition, we prove that if every δ-cosingular R-module is semisimple, then $\overline{Z}(M)$ is a direct summand of M for every R-module M if and only if $\overline{Z}_{\delta}(M)$ is a direct summand of M for every R-module M.
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1. Introduction and Preliminaries

Let R be a commutative ring,and let $M$and Nbe an R-modules, then N is said to be submodule of M, if $N \leq M$. A submodule N of M is said to be small in Mif $N + K \neq M$ for every proper submodule $K$of$M$, and denoted by$N \ll M$, as given in [1],[2],[7], [10], [16], and [19]. Also as generalized by Zhou in [19] a sub-module $N$of $M$is said to be δ-small in M (denoted by$N \ll_{\delta} M$) provided $M \neq N + K$ for any proper submodule $K$of $M$with M/Ksingular. In his manuscript, Zhou presented the general properties and a certain useful properties of δ-small submodules of a module. A module M is called small if it is a small submodule of some module, equivalently, M is a small submodule of its injective hull. A submodule N of M is called coclosed if N/K is small in M/K, then $N = K$. The important concept in module theory which is closely associated to smallness is lifting modules. A module $M$is said to be lifting, if every submodule $N$of Mcontains a direct summand $D$of Msuch that$N/D \ll M/D$. A numeral of consequences regarding to lifting modules have been performed in the literature of recent years and many generalizations of the theories of lifting modules have been introduced and studied by several authors.

In [14], the authors established that a module M is called H-supplemented in case for every submodule N of$M$, there exists a direct summand D of M such that $M = N + F$ if and only if $M = D + F$ for every sub module F of M. In [14] different definitions, unusual properties and being a generalization of lifting modules, all directed many researchers to study and investigate H-supplemented modules were demonstrated. Then several authors had tried to consider the H-supplemented. Also in [11], for a ring R and an R-module M such that every (simple) cosingular R-module is M-projective. In addition, the authors proved that every simple cosingular module is M-projective if and only if for $N \leq T \leq M$, at any time T/N is simple cosingular, and then N is a direct summand of T.Again in [11], they proved that every simple cosingular right R-module is projective if and only if R is a right GV -ring. In their manuscript it is also shown that for a perfect ring R, every cosingular R-module is projective if and only if R is a right GV -ring.In [4] the authors demonstrated the conceptions of $E - H$ -supplemented characterizations of modules and a similar property for a module M by bearing in mind Hom$R(N, M)$ instead of S where N is several module.

In [10], the authors deliberated several general properties of H-supplemented modules such as homomorphic images and direct summands of these modules. Then [7], the authors presented various equivalent conditions for a module to be H-supplemented that shows that this class of modules is closely related to the concept of small submodules. In fact in , the authors demonstrated that a module M is H-supplemented if and only if for every submodule N of M there is a direct summand D of M such that$((N + D)/N) \ll M/N$and $((N + D)/D) \ll M/D$. In addition the author refer the readers to [8],[9],[10],[11],[14],[15],and [19].

In [13], the authors considered the concepts of H-supplemented modules via preradicals. If τ specifies a preradical, a module M, τ-H-supplemented provided for every submodule N of M, there is a direct summand D of M such that $((N+D)/N) \subseteq \tau(M/N)and((N+D)/D) \subseteq \tau(M/D$.Also, in they demonstrated that, if$\overline{Z}(M) = 0$ or$\overline{Z}(M) = M$), then M is called a cosingular (non-cosingular) module. For more discussion, the author referee the readers to [15], [17], and [18]. Let M be a module over a commutative ring R. According to [16], M is called non cosingular provided that$\overline{Z}(M) = M$ or$\overline{Z}(M) = 0$, where $\overline{Z}(M) = \{Kerf \mid f : M \to U\}$ in which U is an arbitrary small right R-module (see also [3], and [4]). Let R be a ring. By [13], R is said to be generalized V-ring (just GV-ring) provided every simple singular right R-module is injective. Also, R is right GV if and only if every simple cosingular right R-module is projective. Let M be an R-module where R is a ring. Let $K \leq M$, then we say K is t-small in M, denoted by $K \ll_t M$, if the inclusion $\overline{Z}^2(M) \subseteq K + N$ implies that$\overline{Z}^2(M) \subseteq N$. We call M, t-small; provided M is a t-small submodule of a module L (see [4] and [6]).

This paper is structured as follows; In Section 2, the author presents a new generalization of the perception of small submodules γ-small submodules. In this part various general properties of γ-small submodules are established. Also, their relation between γ-small submodules and small submodules are considered. In Section 3, the author shall introduce a generalization of H-supplemented modules. A module M is γ-H-supplemented for every submodule N of M there is a direct summand D of M such that $M = N + F$ if and only if $M = D + F$, for every submodule F of M with M/F non cosingular. In addition, the author delivers an equivalent condition for this definition influencing the close relation of γ-H-supplemented modules to the concept of $\gamma -$ small submodules.

2. Properties of γ-Small Sub modules

In this section, the author delivered the definition of a new generalization of smallsubmodules.

Definition 2.1.[3] Let N be a submodule of M. Then N is said to be γ-small in M, denoted by $N \ll_\gamma M$ if$M = N + F$ with M/F is non cosingular implies $M = F$.

That means $M \neq N + F$ for every proper submodule F of M with M/F non cosingular. Every small submodule of a module is γ-small in that module.

Proposition 2.2. [3] Let M be an R-module. Let $A \leq B \leq M$. Then $B \ll_\gamma M$ if and only if $A \ll_\gamma M$ and$B/A \ll_\gamma M/A$.

Proof: Suppose that $B \ll_\gamma M$ and let U be a submodule of M such that $M = A + U$ with M/U non cosingular. Since $A \leq B$, then $M = B + U$. Being Ba γ-small submodule of M implies $M = U$. Thus $A \ll_\gamma M$. Let us assume that$M/A = B/A + L/A$, for some submodule L of M and $\frac{(M/A)}{(L/A)} \cong M/L$ is non cosingular. Then $M = B + L$ combining with $B \ll_\gamma M$ yields that $M = L$.

Conversely, suppose that $A \ll_\gamma M$ and$\frac{B}{A} \ll_\gamma \frac{M}{A}$. To prove that $B \ll_\gamma M$, suppose $M = B + U$ with $M/_U$ non cosingular. Therefore$\frac{M}{A} = \frac{B}{A} + \frac{U+A}{A}$.

Note that $\frac{M/A}{(U+A)/A} \cong M/U + A$ is non cosingular. Since $B/_A \ll_\gamma M/_A$, then $M/A = (U+A)/A$ this implies that $M = U + A$. Since $A \ll_\gamma M$ and M/U is noncosingular we conclude that $M = U$.

Therefore, it follows that $B \ll_\gamma M$.

Theorem 2.2.[6]Let M be an R-module. Let A, B be submodules of M with $A \leq B$. If $A \ll_\gamma M$, then $A \ll_\gamma M$.

Proof: Suppose that $A \ll_\gamma B$. Let $M = A + U$, such that M/U is non cosingular. Since $B = B \cap M = B \cap (A + U) = A + (B \cap U)$, we have $B/B \cap U \cong \big((B+U)\big)/U = M/U$which implies $B/B \cap U$ is non cosingular. By $A \ll_\gamma B$ we conclude that $B = B \cap U$.

Therefore $M = U$. Hence the result.

Theorem 2.3.[5]Let M be an R-module. Let $f: M \to M'$be an epimorphism such that $A \ll_\gamma M$, then $A \ll_\gamma M'$.

Proof: Suppose that $A \ll_\gamma M$ and $f(A) + Y = M'$ for a submodule Y of M' such that $M'/_Y$ is non cosingular. $M/f^{-1}(Y)$a homomorphic image of M/Yimplies$M/f^{-1}(Y)$is non cosingular.

Hence $M = f^{-1}(Y)$. It is easy to verify that$M' = Y$.

Theorem 2.4. [5]Let M be an R-module. Let $M = M_1 \oplus M_2$ be an R-module and let $A_1 \leq M_1$ and $A_2 \leq M_2$. Then $A_1 \oplus A_2 \ll_\gamma M_1 \oplus M_2$if and only if $A_1 \ll_\gamma M_1$ and $A_2 \ll_\gamma M_2$.

Proof: Suppose that $A_1 \oplus A_2 \ll_\gamma M_1 \oplus M_2$. Let $f: M_1 \oplus M_2 \to M_1$ be the projection on M_1. Since, $A_1 \oplus A_2 \ll_\gamma M_1 \oplus M_2$, then $f(A_1 \oplus A_2) \ll_\gamma f(M_1 \oplus M_2) \Longrightarrow A_1 \ll_\gamma M_1$. Similarly $A_2 \ll_\gamma M_2$.

Conversely, suppose that $A_1 \ll_\gamma M_1$ and $A_2 \ll_\gamma M_2$. Let $A_1 + A_2 + F = M_1 + M_2$ with $(M_1 + M_2)/F$ non cosingular.

Therefore$(M_1 + M_2)/A_2 + F$ as a homomorphic image of$(M_1 + M_2)/F$, is non cosingular. Since $A_1 \ll_\gamma M_1 + M_2$by (2), we determine that $A_2 + F = M_1 + M_2$. Now $A_2 \ll_\gamma M_1 + M_2 \Longrightarrow F = M_1 + M_2$ as required.

Proposition 2.5.[4]Let M be a module such that for every $N \leq M$, there exists a direct summand K of M such that $M = N + K$ and $N \cap K$ is cosingular. If M is projective, then M satisfies H.

Proof: Suppose that $N \leq M$. By hypothesis, there exists a direct summand M_2 of M such that $M = N + M_2$ and $N \cap M_2$ is cosingular. Let $M = M_1 \oplus M_2$. Since M_1 is M_2-projective, there exists a submodule A of N such that $M = A \oplus M_2$. Then by modular low, $N = A \oplus (N \cap M_2)$. Then, it is vibrant that $(N + A)/A$ and $(N + A)/N$ are cosingular. Hence M satisfies H.

Theorem 2.6.[5]Let M be an R-module and $A \leq B$. If B is a supplement submodule in M and $A \ll_\gamma M$, then $A \ll_\gamma B$.

Proof: Suppose that, $A \ll_\gamma M$ and B be a supplement sub modules of B' in M. Then $M = B + B'$and $B \cap B' \ll B$. To show that $A \ll_\gamma B$, let $B = A + U$ such that B/U is non cosingular. Then $M = B + B' = A + U + B'$. Since $M/(U + B') = (A + U + B')/(U + B) \cong A/(A \cap (U + B'))$and $A/(A \cap (U + B'))$is a homomorphic image of$A/(A \cap U) \cong B/U$, then it will be non cosingular. Since$A \ll_\gamma M, M = U + B'$. Now being $B \cap B'$a small submodule of B implies $B = B \cap M = B \cap (U + B) = U + (B \cap B') = U$.It follows that $A \ll_\gamma B$. The following delivers a characterization of a module M such that every submodule of M is γ -small in M.

Proposition 2.7. Let M be a simple supplemented module. The following are equivalent:

(1) Every submodule of M is γ-small in M;
(2) None of nonzero homomorphic images of M is non cosingular;
(3) $\overline{Z}(M) \ll M$.

Proof: (1) ⇒ (2): Suppose that every submodule of M is γ -small in M. Consider a submodule F of M such that M/F is noncosingular. Since $M = M + F$ and $M \ll_\gamma M$, then $M = F$.

(2)⇒ (3): Suppose that $F$is a proper sub module of M. Then$\overline{Z}(M/F) \neq M/F$. Therefore, $\overline{Z}(M) + F \neq M$ and which implies that$\overline{Z}(M) \ll M$.

(3)⇒ (2): Suppose M be amply supplemented and $\overline{Z}(M) \ll M$. Suppose that M/F be a non cosingular homomorphic image of M. Then$M/F = \overline{Z}(M/F) = \overline{Z}^2(M/F) = \frac{\overline{Z}^2(M)+F}{F}$. Since $\overline{Z}(M)$ is a cosingular module, then$\overline{Z}^2(M) = 0$. Therefore$M/F = 0$.

Proposition 2.8. Suppose M be a module and $N \leq M$. Let N is non cosingular, then $N \ll M$ if and only if $N \ll_\gamma M$:

Proof: Let $N \ll_\gamma M$ and N be non cosingular. Let $N + F = M$. Then, we have$N/N \cap F \cong M/F$is non cosingular. Hence $M = F$. It follows that $N \ll M$. Hence the result.

Lemma 2.9. Let N be a proper submodule of M withM/Nbe a non cosingular. Let $x \in M/N$such that $Rx + N = M$. Then there is a maximal submodule K of M with M/K non cosingular and $x \notin K$.

Proof: Set $A = \{L \leq M/N \subseteq L,\ M/L$ is non cosingular, $x \notin L\}$. Then $A = \emptyset$ since $N \in A$. Suppose $\{L_\alpha\}$ is a chain in A. Then we prove that A has a maximal element. Obviously, $\cup L_\alpha$ is a submodule of M and $N \subseteq\cup L_\alpha$. It is clear that $x \notin \cup L_\alpha$.

Note that$M/\cup L_\alpha$is non cosingular as well as M/L_α for each α. Hence A has a maximal element say$K$.

Now, Let $K \subset T \subseteq M$ for a submodule T which properly contains K. Then, since K is the maximal element of A, $T \notin A$. Hence $x \in T$. Thus, $M = Rx + N \subseteq T$.Therefore, it shows that K is a maximal submodule of M.

Theorem 2.10. Suppose M be a module. Then $\gamma(M) = \cap\{N \leq_{max} M : M/N$ is non cosingular$\}$.

Proof: Let N be a maximal submodule of M with ${}^{M}/_{N}$ non cosingular. Let $K \ll_{\gamma} M$. Consider the submodule $N + K$ of M. Suppose that $N + K = M$, then $M = N$ as $K \ll_{\gamma} M$, which is a contradiction. Hence $N + K = N \Rightarrow K \subseteq N$. Therefore $\sum_{K \ll_{\gamma} M} K \subseteq N$. Then $\sum_{K \ll_{\gamma} M} K \subseteq \cap\{N | N \leq_{max} M \text{ and } M/N$ is non cosingular$\}$.

For the other side of presence, let $x \in \{N: N \leq_{max} M \text{ and } {}^{M}/_{N}$ is non cosingular$\} = P$.

Suppose that $xR + L = M$ with ${}^{M}/_{L}$ be a non cosingular. If $L \neq M$, then by lemma 2.9, there is a maximal submodule K' of M with ${}^{M}/_{K'}$, non cosingular and $x \notin K$. But $x \in P \Rightarrow x \in K$, a contraction.

Therefore $L = M$. So $xR \ll_{\gamma} M \Rightarrow x \in \sum_{K \ll_{\gamma} M} K$. Therefore, it follows that $P \subseteq \sum_{K \ll_{\gamma} M} K$.

Remark 2.11. Let R be a ring and M be a right R-module. If SN denotes the class of simple non-cosingular right R-modules, then $\gamma(M) = RejM(SN) = \cap\{I : R/I$ is simple injective$\}$.

Proposition 2.12. Let R be a ring. Then $\gamma(R_R)$ is the largest γ -small right ideal of R.

Proof: Let $\gamma(R_R) + I = R$ where ${}^{R}/_{I}$, is non cosingular. Then there is a maximal right ideal I_0 of R such that $I \subseteq I_0$. Note that ${}^{R}/_{I_0}$ is noncosingular as well as ${}^{R}/_{I}$. Then we conclude that $\gamma(R_R) \subseteq I_0 \Longrightarrow I_0 = R$, a contradiction. Therefore $I = R$, as required.

A ring R is said to be a right V-ring (GV -ring), in case every simple (singular) right R-module is injective. It follows from ([15], Proposition 2.5) that R is a right V -ring if and only if every right R-module is noncosingular.

Proposition 2.13. [13] Let R be a ring. Then every simple right R-module is small (cosingular) if and only if $\gamma(R_R) = R$. In particular, if R is a right GV-ring and $\gamma(RR) = R$, then R is a semisimple ring.

Proof: Let R be a ring such that all simple right R-modules are small. It follows that there does not exist a simple injective right R-module combining with the definition of $\gamma(R_R)$ imply $\gamma(R_R) = R$.

Conversely, let $\gamma(R_R) = R$. Then we will demonstrate that every simple right R-module is small. Let M be a simple right R-module which is not small. Then, M is injective. Since M is simple, there is a maximal right ideal I of R such that $M \cong R/I$. Since R/I is simple injective, we conclude that $\gamma(R_R \nsubseteq R$ that is a contradiction. It follows that every simple right R-module is small (cosingular).

For the concluding, if R is a right GV-ring and $\gamma(R_R) = R$, then each simple right R-module is projective. Therefore, R is semisimple. Let R be a commutative domain which is not a field. Then every finitely generated R-module is small and hence cosingular. Therefore, every simple R-module is small showing that $\gamma(R) = R$.

3. γ-H-Supplemented Modules

In this section, the author present recollect that a module M is called H-supplemented in case for every submodule N of M, there is a direct summand D of M such that $M = N + F$ if and only if $M = D + F$ for every submodule F of M. Let us present a generalization of H-supplemented modules where we deliberate the class of non cosingular modules instead of the class of all modules.

Definition 3.1.[3] Let M be a module. Then M is said to be γ-H-supplemented, delivered for every submodule N of M there is a direct summand D of M such that $M = N + F$ if and only if $M = D + F$ for every submodule F of M with M/F non cosingular.

Note that for a non cosingular module, two notions H-supplemented and γ-H-supplemented coincide.

The following provides an equivalent condition for a module to be γ-H-supplemented.

Lemma 3.2. Let M be a module. Then M is γ -H-supplemented if and only if for every sub-module N of M there exists a direct summand D of M such that $(N + D)/N \ll_{\gamma} M/N$ and $(N + D)/D \ll_{\gamma} M/D$

Proof: Suppose that M be γ-H-supplemented and $N \leq M$. Then there is a direct summand D of M such that $M = N + F$ if and only if $M = D + F$, for every submodule F of M such that M/F is non cosingular. Suppose that $(N + D)/N + F/N = M/N$ for a submodule F of M containing N with M/F non cosingular. Then $D + F = M$. Then by hypothesis $N + F = M \Longrightarrow F = M$. Hence the result.

To verify the second γ -small case, let $(N + D)/D + Y/D = M/D$, where M/Y is non-cosingular. Then $N + Y = M$. Being M a γ -H-supplemented module implies $D + Y = M$. Therefore, $Y = M$. Conversely, suppose that

$N + F = M$ with M/F non cosingular. Then$((N + D))/D + ((F + D))/D = M/D$. Note that $M/(F + D)$is non cosingular as well as M/F is non cosingular. Hence $F + D = M$, since$(N + D)/D \ll_\gamma M/D$.

Suppose that $D + Y = M$ for a submodule Y of M such that M/Y is non cosingular. Then $(N + D)/N + (N + Y)/N = M/NandM/(N + Y)$ as a homomorphic image of M/Y is non cosingular. Being $(N + D)/N$ a γ-small submodule of M/N linking with last equality implies $N + Y = M$.

Theorem 3.3. Let M be an indecomposable module. Then M is γ-H-supplemented if and only if for every proper sub module N of M, we have $N \ll_\gamma M$ or$M/N \ll_\gamma M/N$.

Proof. Let M be indecomposable and γ -H-supplemented. Consider an arbitrary proper submodule N of M. Then there is a direct summand D of M such that $(N + D)/N \ll_\gamma M/N$ and$(N + D)/D \ll_\gamma M/D$. Suppose $D = 0$. Then clearly$N \ll_\gamma M$. Then, $D = M$ implies$M/N \ll_\gamma M/N$.

Conversely, suppose that $N < M$. If $N \ll_\gamma M$, then $(N + 0)/N \ll_\gamma M/N$ and $(N + 0)/0 \ll_\gamma M/0$. Otherwise,$(N + M)/M \ll_\gamma M/M$ and$(N + M)/N \ll_\gamma M/N$. If M is amply supplemented and indecomposable, then M is γ -H-supplemented if and only if for every submodule N of M we have $N \ll_\gamma M$ or$\overline{Z}\,(M/N) \ll M/N$.

Next the author present an example of γ-H-supplemented and non-γ -H-supplemented modules.

Example 3.4. Let $M = Q$ as an Z-module. If M is γ-H-supplemented, then for every proper submodule N of M, we conclude that $N \ll_\gamma M$ or $M/N \ll_\gamma M/N$ by Proposition 3.3 M is non cosingular.

So that $N \ll M$ or$M/N \ll_\gamma M/N$. Second case will not happen. It follows that every submodule of M must be small in M, that is a contradiction. Therefore, M is not γ-H-supplemented.

Proposition 3.5. Let M be a module and N a projection invariant submodule of M. If M is γ -H-supplemented, then M/N is also γ-H-supplemented.

Proof. Suppose thatK/N be an arbitrary submodule ofM/N. Then there exists a direct summand D of M such that $M = K + F$ if and only if $M = D + F$ for every submodule F of M such that$M/F$is non-cosingular.

Now, put $M = D \oplus D'$. Since N is a projection invariant sub module of M , we accomplished that$(N + D)/N \oplus (N + D)/N = M/N$. Now, suppose that $K/N + Y/N = M/N$ for a submodule Y/N of M/N with M/Y noncosingular. Then $K + Y = M$ and by hypothesis $M = D + Y$. Undoubtedly now $M/N = (D + N)/N + Y/N$.

Now for the other implication, let $M/N = (D + N)/N + T/N$withM/Tnon cosingular. Hence $M = D + T$and again by assumption $M = K + T$. Obviously $M/N = K/N + T/N$. It is known that a module $M$is said to be distributive in case the lattice of submodules of $M$is distributive, i.e. for each submodules $N, K, L$of $M$the equalities $(N \cap L) + (N \cap K) = N \cap (L + K)$and $N + (K \cap L) = (N + K) \cap (N + L)$hold.

Definition 3.6.[4] Let M and N be modules. Let $f \in HomR(N, M)$. Then M is called $f\ - H$ -supplemented (or H -supplemented relative to f) if there exists a direct summand D of M such that $(Imf + D)/Imf$ is small in M/Imf and $(Imf + D)/D$ is small in M/D. This is equivalent to saying that $Imf\beta D$ in M.

Corollary 3.7. Every homomorphic image of a distributive γ-H-supplemented module is γ-H-supplemented.

Corollary 3.8. Every direct summand of a weak duo γ -H-supplemented module is γ-Hsupplemented.

Following these corollaries, the author presents the following theorem;

Theorem 3.9. Let $M = M_1 \oplus M_2$be a distributive module. Then M is γ -H-supplemented module if and only if $M_1$and $M_2$are γ -H-supplemented.

Proof Suppose that M_1 and M_2 be γ-H-supplemented and $N \leq M$. Set $N_1 = N \cap M_1$ and$N_2 = N \cap M_2$. Then $N = N_1 + N_2$. Now, there are direct summands D_i of M_i for $i = 1,2$, such that $(N_i + D_i)/N_i \ll_\gamma M_i/N_i$and $(N_i + D)/D_i \ll_\gamma M_i/D_i$.

Now we will show that $(N + D)/N \ll_\gamma M/N$and$(N + D)/D \ll_\gamma M/D$,where $D = D_1 \oplus D_2$ which is a direct summand of M. Suppose that$(N + D)/N \ll_\gamma F/N = M/N$for a submodule $F$of Mcontaining $N$with M/Fnon cosingular.

Then $D + F = M$. This follows that $D_1 + (F \cap M_1) = M_1$. Now $(N_1 + D_1)/N_1 \ll_\gamma (F \cap M_1/N_1 = M_1/N_1$ and$M_1/F \cap M_1 \cong D_1/F \cap D_1$as a direct summand of $D/(F \cap D) \cong M/F$is anon cosingular module.

Therefore, $F \cap\ M_1 = M_1$ this implies that M is contained in F.

Now consider again the equality $D + F = M$. Therefore$D_1 + (F \cap M_2) = M_2$. Since $(N_2 + D_2) + F \cap M_2/N_2 = M_2/N_2and(N_2 + D_2)/N_2 \ll_\gamma M_2/N_2$and also$M_2/F \cap M_2 \cong D_2/F \cap D_2$, since a direct summand of $F/F \cap D \cong M/F$is non cosingular, we determine that $F \cap M_2 = M_2$.

So that M_2 is contained in Fwhich indicates that $F = M$. For the other γ-small case, let $(N + D)/D + T/D = M/D$, provided that $T/D \leq M/D$and$M/T$is non cosingular.

Now $N + T = M$ and hence$N_1 + (T \cap M_1) = M_1$. Being $(M_1 + D_1)/D_1$a γ-small submodule ofM_1/D_1combining with the fact that$M_1/(T \cap M_1) \cong N_1/(T \cap N_1)$as a direct summand of$N/N \cap T \cong M/T$is non cosingular and the latest impartiality imply that $T \cap M_1 = M_1$ and therefore$M_1 \subseteq T$. By the equivalent procedure, T will contain M_2. Hence $T = M$ as required. Now that $M$is γ-H-supplemented.

4. Conclusions

In general, in the classes of H-supplemented sub modules, we would addressed some general and specific characterizations and properties of $\boldsymbol{H}$-supplemented and $\boldsymbol{\gamma}$-$\boldsymbol{H}$-supplemented sub modules. Suppose $\boldsymbol{M}$ be a module over a commutative ring $\boldsymbol{R}$, then $\boldsymbol{M}$ is called $\boldsymbol{\gamma}$-$\boldsymbol{H}$-supplemented if and only if for every sub-module $\boldsymbol{N}$ of $\boldsymbol{M}$ there is a direct summand $\boldsymbol{D}$ of $\boldsymbol{M}$ such that $\boldsymbol{M = N + F}$ implies $\boldsymbol{M = D + F}$ for every submodule $\boldsymbol{F}$ of $\boldsymbol{M}$ with $\boldsymbol{M/F}$non-cosingular. Also we demonstrate that $\boldsymbol{M}$ is $\boldsymbol{\gamma}$-$\boldsymbol{H}$-supplemented if and only if for every submodule $\boldsymbol{N}$ of $\boldsymbol{M}$ there exists a direct summand $\boldsymbol{D}$ of $\boldsymbol{M}$ such that $(\boldsymbol{N + D})/\boldsymbol{N} \ll_\gamma \boldsymbol{M/N}$ and$(\boldsymbol{N + D})/\boldsymbol{D} \ll_\gamma \boldsymbol{M/D}$. In addition, we prove that if every δ-cosingular $\boldsymbol{R}$-module is semisimple, then $\overline{\boldsymbol{Z}}(\boldsymbol{M})$ is a direct summand of $\boldsymbol{M}$ for every R-module M if and only if $\overline{\boldsymbol{Z}}_\delta(\boldsymbol{M})$ is a direct summand of $\boldsymbol{M}$ for every $\boldsymbol{R}$-module $\boldsymbol{M}$.

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References

Calısıcı, Hamza, and Ali Pancar. "Finitely⊕− Supplemented Modules." International Journal of Pure and Applied Mathematics 21, no. 1 (2005): 51-57.

Çalişici, H., Pancar, A. ⊕-Cofinitely Supplemented Modules. Czech Math J **54,** 1083–1088 (2004). https://doi.org/10.1007/s10587-004-6453-1

Hamzekolaee, A.R.M. An approach to H-supplemented modules via noncosingular modules. Ann Univ Ferrara (2021). https://doi.org/10.1007/s11565-021-00356-8

Hamzekolaee, Ali Reza Moniri, Abdullah Harmanci, Yahya Talebi, and BurcuÜngör. "A new approach to H-supplemented modules via homomorphisms." Turkish Journal of Mathematics 42, no. 4 (2018): 1941-1955.

Hosseinpour, Mehrab, and A. R. Moniri Hamzekolaee. "A Kind of F-inverse Split Modules." Journal of Algebraic Systems 7, no. 2 (2020): 167-178.

Hosseinpour, Mehrab, and Ali Reza Moniri Hamzekolaee. "γ-Small Submodules and γ-Lifting Modules." East West Math 22, no. 01 (2020): 52-63.

Kosan, M.T., Keskin Tütüncü, D.: "H-supplemented duo modules." J. Algebra Appl. 6(6), 965–971 (2007)

Kuratomi, Yosuke. "H-supplemented modules and generalizations of quasi-discrete modules." Communications in Algebra 44, no. 7 (2016): 2747-2759.

Lomp, Christian. "On dual Goldie dimension." PhD diss., ProQuest Dissertations & Theses,, 1996.

Mohamed, S.H., Müller, B.J.: "Continuous and Discrete Modules. London Mathematical Society Lecture Note Series Book 147". Cambridge University Press, Cambridge (1990)

Talebi, Y., Moniri Hamzekolaee, A.R., Hosseinpour, M., Harmanci, A., Ungor, B.: Rings for which\every cosingular module is projective. Hacet. J. Math. Stat. **48**(4), 973–984 (2019)

Talebi, Y., M. Hosseinpour, and AR Moniri Hamzekolaee. "Modules for which Every Non-cosingular Submodule Is A Summand." Bulletin of the Iranian Mathematical Society 43, no. 3 (2017).

Talebi, Y., Tribak, R., Moniri Hamzekolaee, A.R.: On H-cofinitely supplemented modules. Bull. Iranian Math. Soc. **30**(2), 325–346 (2013)

Talebi, Y., Moniri Hamzekolaee, A.R., Keskin Tütüncü, D.: "H-supplemented modules with respect to a preradical. Algebra Discrete Math". 12(1), 116–131 (2011)

Talebi, Y., Vanaja, N.: "The torsion theory cogenerated by M-small modules". Commun. Algebra **30**(3), 1449–1460 (2002)

Tribak, Rachid. "H-Supplemented modules with small radical." East West Math 11, no. 02 (2009): 211-221.

Wang, Y., Wu, D.: "On H-supplemented modules." Commun. Algebra **40**(10), 3679–3689 (2012)

Zhou, Y. Generalizations of Perfect, Semiperfect, and Semiregular Rings. Algebra Colloq. **7,** 305–318 (2000). https://doi.org/10.1007/s10011-000-0305-9

Ziane, M., El Moussaouy, A. Modules in which every surjective endomorphism has a μμ-small kernel. *Ann Univ Ferrara* **66,** 325–337 (2020). https://doi.org/10.1007/s11565-020-00347-1