UPPER BOUNDS FOR RUIN PROBABILITY IN A CONTROLLED GENERAL RISK PROCESS WITH RATE OF INTEREST IS A FIRST - ORDER

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Abstract:

In this paper, we study a controlled general risk process where claim is homogeneous Markov chain and rate of interest is a first-order autoregressive process. We assume that claim is homogeneous Markov chain, take a countable number of nonnegative values and rate of interest is a sequence of non-negative random variables what it satifies a first-order autoregressive process. Generalized Lundberg inequalities for ruin probability of this process are derived by the Martingale approach.

Keywords: ruin probability, homogenous Markov chain, controlled risk process, autoregressive process, Supermartingale, Optional Stopping theorem.

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1. Introduction

The ruin problem has been studied by many researchers (J. Grandell (1991), H. U. Gerber (1979), S.D.Promislow (1991)). J. Cai (2002) considered the ruin probabilities in two risk models, with independent premiums and claims and used a first – order autoregressive process to model the rates of in interest. J. Cai and D. C. M. Dickson (2004) built Lundberg inequalities for ruin probabilities in two discrete- time risk process with a Markov chain interest model and independent premiums and claims. J. L. Teugels and B. Sundt (1991, 1995) studied ruin probability under the compound Poisson risk model with the effects of constant rate. H. Yang (1999) given both exponential and non – exponential upper bounds for ruin probabilities in a risk model with constant interest force and independent premiums and claims. L. Xu and R. Wang (2006) given upper bounds for ruin probabilities in a risk model with interest force and independent premiums and claims with Markov chain interest rate.

In addition, many papers studied an insurance model where the risk process can be controlled by proportional reinsurance. The performance criterion is to choose reinsurance control strategies to bound the ruin probability of a discrete-time process with a Markov chain interest. Controlling a risk process is a very active area of research, particularly in the last decade; see (J. Grandell (1991), O. Hernández-Lerma, J. B Lasserre (1996, 1999,2003)), for instance. Nevertheless obtaining explicit optimal solutions is a difficult task in a general setting. Maikol A. Diasparra and Rosaria Romera (2009) obtained generalized Lundberg inequalities for the ruin probabilities in a controlled discrete-time risk process with a Markov chain interest.

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In this article, we extend the model considered by Diasparra and Romera (2009) to introduce homogeneous Markov chain claims and rates of interest as a first-order autoregressive process. Generalized Lundberg inequalities for ruin probability of this process are derived by the Martingale approach.

2. The Model and Basic Assumptions

Let Y_n be the n – th claim payment. The random variable Z_n stands for the length of the n – th period, that is, the time between the ocurrence of the claims Y_{n-1} and Y_n. Let $\{I_n\}_{n\geq 0}$ be the interest rate process. We assume that Y_n, Z_n, I_n are defined on the probability space (Ω, A, P). We consider a discrete – time insurance risk process in with the surplus process $\{U_n\}_{n\geq 1}$ with initial surplus u can be written as

$$U_n = U_{n-1}(1+I_n) + C(b_{n-1}).Z_n - h(b_{n-1}, Y_n), \; for\; n \geq 1 . \tag{2.1}$$

We make several assumptions.

Assumption 2.1. $U_o = u \geq 0$.

Assumption 2.2. $\{I_n\}_{n\geq 0}$ is a sequence of non-negative random variables, where In denotes the rate of interest during the nth period and satisfies

$$I_n = \rho I_{n-1} + \mathrm{W}_n, \tag{2.2}$$

$0 < \rho < 1, I_o = i \geq 0, \{\mathrm{W}_m\}_{n\geq 0}$ is a sequence of independent and identically distributed non-negative continuous random variables with the same distributive function

$$G(z) = P(\omega \in \Omega, \mathrm{W}_o(\omega) \leq z)$$

Assumption 2.3. $\{Z_n\}_{n\geq 0}$ is a sequence of independent and identically distributed non-negative continuous random variables with the same distributive function

$$F(z) = P(\omega \in \Omega; Z_o(\omega) \leq z).$$

With F(0) = 0.

Assumption 2.4. $\{Y_n\}_{n\geq 0}$ is an homogeneous Markov chain, such that for any n the values of Y_o are taken from a set of non – negative numbers $G_Y = \{y_1, y_2, ..., y_n, ...\}$ with $Y_o = y_i$ and

$$p_{ij} = P\left[\omega \in \Omega : Y_{n+1}(\omega) = y_j \middle| Y_n(\omega) = y_i\right](n \in N, y_i \in G_Y, y_j \in G_Y),$$

Where $0 \leq p_{ij} \leq 1, \sum_{j=1}^{+\infty} p_{ij} = 1.$

Assumption 2.5. We denote by $C(b)$ the premium left for the insurer if the retention level b is chosen, where $0 < C(b) \leq c, b \in B$.

The process can be controlled by reinsurance, that is, by choosing the retention level (or proportionality factor or risk exposure) $b \in B$ of a reinsurance contract for one period, where $B := [b_{min}, 1]$, $b_{min} \in (0,1]$ will be introduced below. The premium rate c is fixed.

Assumption 2.6. We denote the function $h(b, y)$ with values in $[0, y]$ specifies the fraction of the claim y paid by the insurer, and it also depends on the retention level b at the beginning of the period. Hence $y - h(b, y)$ is the part paid by the reinsurer. The retention level b =1 stands for control action no reinsurance. In this article, we consider the case of proportional reinsurance, which means that

$$h(b,y) = b.y, \text{ with } b \in B. \quad (2.3)$$

Usually, the constant $b_{\min}$ in Assumption 2.5 is chosen by

$$b_{\min} := \min\{b \in (0,1]; C(b) > 0\}. \quad (2.4)$$

Assumption 2.7. We suppose that $\{Y_n\}_{n\geq 0}, \{Z_n\}_{n\geq 0}$ and $\{I_n\}_{n\geq 0}$ are independent.

Assumption 2.8. We consider Markovian control policies $\pi = \{a_n\}_{n\geq 1}$, which at each time n depend only on the current state, that is, $a_n(U_n) := b_n$ for $n \geq 0$. Abusing notation, we will indentify functions $a : X \to B,$ where $X = \square \cup \ell, B$ is the decision space.

Consider an arbitrary initial state $U_o = u \geq 0$ and a control policy $\pi = \{a_n\}_{n\geq 1}$. Then, by iteration of (2.1) and assuming (2.2), it follows that for $n \geq 1$, U_n satisfies

$$U_n = u\prod_{l=1}^{n}(1+I_l) + \sum_{l=1}^{n}\left(C(b_{n-1})Z_l - b_{l-1}.Y_l\prod_{m=l+1}^{n}(1+I_m)\right) \quad (2.5)$$

The ruin probability when using the policy π, given the initial surplus u, and the initial claim $Y_o = y_i$, the initial interest rate $I_o = i_r$ with Assumption 2.1 to 2.8 is defined as

$$\psi^{\pi}(u,y_i,i) = P^{\pi}\left(\bigcup_{k=1}^{\infty}(U_k < 0) \middle| U_o = u, Y_o = y_i, I_o = i\right) \quad (2.6)$$

which we can also express as

$$\psi^{\pi}(u,y_i,i) = P^{\pi}\left(U_k < 0 \text{ for some } k \geq 1 \middle| U_o = u, Y_o = y_i, I_o = i\right) (2.7)$$

Similarly, the ruin probabilities in the finite horizon case with Assumption 2.1 to 2.8, are given by

$$\psi_n^{\pi}(u,y_i,i) = P^{\pi}\left(\bigcup_{k=1}^{n}(U_k < 0) \middle| U_o = u, Y_o = y_i, I_o = i\right) \quad (2.8)$$

Firstly, we have

$$\psi_1^{\pi}(u,y_i,i) \leq \psi_2^{\pi}(u,y_i,i) \leq \ldots \leq \psi_n^{\pi}(u,y_i,i) \leq \ldots, \quad (2.9)$$

and with any $n \in N$,

$$\psi_n^{\pi}(u,y_i,i) \leq 1. \quad (2.10)$$

Thus, from (2.7) and (2.8), we obtain

$$\lim_{n\to\infty}\psi_n^{\pi}(u,y_i,i) = \psi^{\pi}(u,y_i,i).$$

We denote by Π the policy space. A control policy π^* is said to be optimal if for any initial $(Y_o, I_o) = (y_i, i)$, we have

$$\psi^{\pi^*}(u,y_i,i) \leq \psi^{\pi}(u,y_i,i) \text{ for all } \pi \in \Pi.$$

3. Upper Bounds For Ruin Probability by the Martingale Approacch

We now construct upper bounds for ruin probabilities is the martingale approach. To this end, let $V_n = U_n \prod_{i=1}^{n} (1+I_i)^{-1}$ with $n \geq 1$, be the so-called discounted risk process. The ruin probabilities φ_n^π in (2.8) associated to the $\{V_n, n = 1, 2, ...\}$ are

$$\psi_n^\pi(u_o, y_i, i) = P^\pi\left(\bigcup_{k=1}^{n}\left(V_k < 0 \middle| U_o = u_o, Y_o = y_i, I_o = i\right)\right).$$

In the classical risk model, process $\left\{e^{-R_o U_n}\right\}_{n\geq 1}$ is a martingale. However, for our model (2.5), there is no constant $r > 0$ such that $\left\{e^{-rU_n}\right\}_{n\geq 1}$ is a martingale. Still, there exits a constant $r > 0$ such that $\left\{e^{-rV_n}\right\}_{n\geq 1}$ is a supermartingale, which allows us to derive probability inequalities by the optional stopping theorem. Such a constant is defined in the following Lemmas.

Lemma 3.1. Let model (2.5) satisfy assumptions 2.1 to 2.8. Assume that for each $y_i \in G_Y = \{y_1, y_2, ..., y_n, ...\}$, $bE^\pi\left(Y_1 \middle| Y_o = y_i\right) < C(b)E^\pi(Z_1)$ and $P^\pi\left(bY_1 - C(b)Z_1 > 0 \middle| Y_o = y_i\right) > 0$ then there exists a constant $R_o = R_o(b)$ satisfying

$$E^\pi\left[e^{-R_o[C(b)Z_1 - bY_1]} \middle| Y_o = y_i\right] = 1 \qquad (2.11)$$

Proof.
Define

$$f_i(t) = E^\pi\left[e^{-t[C(b)Z_1 - bY_1]} \middle| Y_o = y_i\right] - 1, t \in (0; +\infty)$$

We have

$$f_i(0) = -E^\pi\left[C(b)Z_1 - bY_1 \middle| Y_o = y_i\right] = -C(b)E^\pi(Z_1) + bE^\pi\left(Y_1 \middle| Y_o = y_i\right) < 0$$ (by ndependence).(2.12)

and the second derivative is

$$f_i''(t) = E^\pi\left[\left[C(b)Z_1 - bY_1\right]^2 e^{-t[C(b)Z_1 - bY_1]} \middle| Y_o = y_i\right] > 0$$

This implies that

$f_i(t)$ is a convex function with $f_i(0) = 0$ (2.13)

By $P^\pi\left(bY_1 - C(b)Z_1 > 0 \middle| Y_o = y_i\right) > 0$, we can find some constant $\delta > 0$ such that

$$P^\pi\left(bY_1 - C(b)Z_1 > \delta > 0 \middle| Y_o = y_i\right) > 0.$$

Then, we get

$$f_i(t) = E^\pi\left[e^{-t[C(b)Z_1 - bY_1]} \middle| Y_o = y_i\right] - 1$$
$$\geq E^\pi\left(\left\{e^{-t[C(b)Z_1 - bY_1]} \middle| Y_o = y_i\right\}.1_{\{bY_1 - C(b)Z_1 > \delta | Y_o = y_i\}}\right) - 1$$
$$\geq e^{t\delta} P^\pi\left(bY_1 - C(b)Z_1 > \delta \middle| Y_o = y_i\right) - 1.$$

This implies that $\lim_{t\to+\infty} \mathrm{f}_i(t) = +\infty$ (2.14)

From (2.12), (2.13) and (2.14) there exists a unique positive constant R_i satisfying $f_i(R_i)=0$.

Let: $R_o=\inf\left\{R_i>0: E^{\pi}\left[e^{-R_o[C(b)Z_1-bY_1]}\middle|Y_o=y_i\right]=1\right\}$ then R_o satisfying (2.11).

Lemma 3.2. Let model (2.5) satisfy assumptions 2.1 to 2.8.

Assume that for each $y_i\in G_Y=\{y_1,y_2,...,y_n,...\}$,

$$P^{\pi}\left(\left[bY_1-C(b)Z_1\right](1+I_1)^{-1}>0\middle|Y_o=y_i,I_o=i\right)>0$$

and $E^{\pi}\left(-\left[C(b)Z_1-bY_1\right](1+I_1)^{-1}\middle|Y_o=y_i,I_o=i\right)<0$ (2.15)

there exits $\rho_{ir}>0$ satisfying that

$$E^{\pi}\left(e^{-\rho_{ik}[C(b)Z_1-bY_1](1+I_1)^{-1}}\middle|Y_o=y_i,I_o=i\right)=1 \quad (2.16)$$

Then

$$R_1=\min\rho_{ir}\geq R_o \quad (2.17)$$

And, furthermore, for all $y_i\in G_Y=\{y_1,y_2,...,y_n,...\}$

$$E^{\pi}\left(e^{-R_1[C(b)Z_1-bY_1](1+I_1)^{-1}}\middle|Y_o=y_i,I_o=i\right)\leq 1 \quad (2.18)$$

Proof

For each $y_i\in G_Y=\{y_1,y_2,...,y_n,...\}$, let

$$l_{ir}(t)=E^{\pi}\left(e^{-r[C(b)Z_1-bY_1](1+I_1)^{-1}}\middle|Y_o=y_i,I_o=i\right), \text{for t > 0.}$$

Then the first derivative of $l_{ir}(t)$ tại t = 0 is

$$l_{ir}'(0)=E^{\pi}\left(-\left[C(b)Z_1-bY_1\right](1+I_1)^{-1}\middle|Y_o=y_i,I_o=i\right)<0$$

and the second derivative is

$$l_{ir}''(t)=E^{\pi}\left(\left(\left[C(b)Z_1-bY_1\right](1+I_1)^{-1}\right)^2 e^{-r[C(b)Z_1-bY_1]\left[(1+I_1)^{-1}\right]^2}\middle|Y_o=y_i,I_o=i\right)>0$$

This shows that $l_{ir}(t)$ is a convex function. From (2.15) implies that $\lim_{t\to+\infty}\mathrm{f}_{\mathrm{ir}}(t)=+\infty$.

Let ρ_{ir} be the unique positive root of the equation $l_{ir}(t)=0$ on $(0;+\infty)$.

Further, if $0<\rho<\rho_{ir}$. However,

$$E^{\pi}\left(e^{-R_o[C(b)Z_1-bY_1](1+I_1)^{-1}}\middle|Y_o=y_i,I_o=i\right)=\sum_{i,j}p_{ij}E\left[e^{-R_o\left[C(b)Z_1-by_j\right](1+I_1)^{-1}}\right]$$

$$\text{(by Jensen's inequality)} \leq E\left[e^{-R_o[C(b)Z_1-bY_1]}\middle|Y_o=y_i\right]^{(1+I_1)^{-1}}$$

Consequentlty, by Lemma 3.1, we have $E\left[e^{-R_o[C(b)Z_1-bY_1]}\middle|Y_o=y_i\right]=1$. Hence,

$$E^{\pi}\left(e^{-R_o[C(b)Z_1-bY_1](1+I_1)^{-1}}\middle|Y_o=y_i,I_o=i_r\right)\leq 1.$$

This implies that $l_{\mathrm{ir}}(R_o)\leq 0$. Moreover, $R_o\leq\rho_{\mathrm{ir}}$ for i, r and so

$R_1:=\min_{i,r}\rho_{ir}\geq R_o$.

Thus, (2.13) holds. In addition $R_1 \le \rho_{ir}$ for all i, r, wich implies that $l_{ir}(R_1) \le 0$. This yields (2.14).

Theorem 3.1. Under the hypetheses of Lemma 3.1 and Lemma 3.2, for all $y_i \in G_Y = \{y_1, y_2, ..., y_n, ...\}$ and $u \ge 0$ then

$$\psi^{\pi}(u, y_i, i_r) \le e^{-R_1 u}. \qquad (2.19)$$

Proof

With $V_k = U_k \prod_{l=1}^{k} (1+I_l)^{-1}$ satisfies that

$$V_k = u + \sum_{l=1}^{k} \left(\left(C(b) Z_1 - bY_l \right) \prod_{t=1}^{l} (1+I_t)^{-1} \right) \qquad (2.20)$$

Let $S_n = e^{-R_1 V_n}$. Then

$$S_{n+1} = S_n e^{-R_1 \left(C(b) Z_{n+1} - bY_{n+1} \right) \prod_{t=1}^{n+1} (1+I_t)^{-1}}.$$

Thus, for any $n \ge 1$,

$$E^{\pi}\left[S_{n+1} \middle| Y_1, ..., Y_n, Z_1, ..., Z_n, I_1, ..., I_n \right]$$

$$= S_n E^{\pi}\left[e^{-R_1 \left(C(b) Z_{n+1} - bY_{n+1} \right) \prod_{t=1}^{n+1} (1+I_t)^{-1}} \middle| Y_1, ..., Y_n, Z_1, ..., Z_n, I_1, ..., I_n \right]$$

$$= S_n E^{\pi}\left[e^{-R_1 \left(C(b) Z_{n+1} - bY_{n+1} \right) \prod_{t=1}^{n+1} (1+I_t)^{-1}} \middle| Y_1, ..., Y_n, I_1, ..., I_n \right]$$

$$= S_n E^{\pi}\left[e^{-R_1 \left(C(b) Z_{n+1} - bY_{n+1} \right) \prod_{t=1}^{n+1} (1+I_t)^{-1}} \middle| Y_n, I_1, ..., I_n \right]$$

From $0 \le \prod_{t=1}^{n} \left(1+I_t \right)^{-1} \le 1$ and Jensen's inequality implies

$$S_n E^{\pi}\left[e^{-R_1 \left(C(b) Z_{n+1} - bY_{n+1} \right) \prod_{t=1}^{n+1} (1+I_t)^{-1}} \middle| Y_n, I_1, ..., I_n \right] \le S_n E^{\pi}\left[e^{-R_1 \left(C(b) Z_{n+1} - bY_{n+1} \right) (1+I_{n+1})^{-1}} \middle| Y_n, I_1, ..., I_n \right]^{\prod_{t=1}^{n} (1+I_t)^{-1}}$$

In addition,

$$E^{\pi}\left[e^{-R_1 \left(C(b) Z_{n+1} - bY_{n+1} \right) (1+I_{n+1})^{-1}} \middle| Y_n, I_1, ..., I_n \right] = E^{\pi}\left[e^{-R_1 \left(C(b) Z_{n+1} - bY_{n+1} \right) (1+I_{n+1})^{-1}} \middle| Y_n, I_n \right]$$

$$= E^{\pi}\left[e^{-R_1 \left(C(b) Z_1 - bY_1 \right) (1+I_1)^{-1}} \middle| Y_o, I_o \right] \le 1.$$

Thus, we have

$$E^{\pi}\left[S_{n+1} \middle| Y_1, ..., Y_n, Z_1, ..., Z_n, I_1, ..., I_n \right] \le S_n$$

This implies that $\{S_n\}_{n \ge 1}$ is a supermartingale.

Let $T_i = \min\{n : V_n < 0 | I_o = i\}$ where V_n is given by (2.20). Then T_i is a stopping time and $n \wedge T_i = \min\{n, T_i\}$ is a finite stopping time. Thus, by the optional stopping theorem for martingale, we get

$$E^{\pi}\left(S_{n\wedge T_i}\right) \le E^{\pi}\left(S_o\right) = e^{-R_1 u}.$$

Hence,

$$e^{-R_1 u} \ge E^{\pi}\left(S_{n\wedge T_i}\right) \ge E^{\pi}\left(\left(S_{n\wedge T_i}\right).1_{(T_i\le n)}\right) \ge E^{\pi}\left(\left(S_{T_i}\right).1_{(T_i \le n)}\right)$$

$$= E^{\pi}\left(e^{-R_1 V_{T_i}}.1_{(T_i\le n)}\right) \ge E^{\pi}\left(1_{(T_i\le n)}\right) \ge \psi_n^{\pi}(u, y_i, i_r). \qquad (2.21)$$

where (2.21) follows because $V_{T_i} < 0$. Thus, by letting $n \to +\infty$ in (2.19) we obtain.

4. Conclusion

We studied a controlled general risk process where claim is homogeneous Markov chain and rate of interest is a first-order autoregressive process. Using Lemma 3.1 and Lemma 3.2, Theorem 3.1 provide a upper bounds for probability $\psi^{\pi}(u, y_i, i)$ by the Martingale approach.

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