Research Article

Characteristic Mode Solution of Complex-Coefficient Complex-Solution Differential Equations

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Abstract: Computation of complex-coefficient complex-solution differential equations is a problem that arises in various domains of science and engineering. This paper aims at applying the Theory of Characteristic Modes (TCM) approach along with the Method of Moments (MoM) in solving these problems with emphasis on procedures for higher differential equations. Several available methods, known in literatures, are available for solving the problem. The complexity of the available methods differs based on the accuracy of the solution. In this paper, the general method is first presented and then a simplified version of it is proposed to solve high order differential equations. Two examples are illustrated, a third and a fourth order complex-coefficients complex-solution differential equations, to show the simplicity of the proposed method. The proposed approach can be also introduced along with other methods to solve these special occurrences differential equations and other boundary value problems.

Keywords: Theory of Characteristic Modes, Method of Moments, Differential Equations.

1. Introduction

In network intrusion detection system (NIDS) research, there are three types of detection approaches misused or signature-

Numerical methods for finding the solutions of complex-coefficient complex-solution differential equations are of great importance and of practical values in various branches of applied mathematics, science, and engineering. Over the years, the subject has been an active research area, and many methods have been proposed for the solution of these problems **(Aksoy U. , 2006)**, **(Aksoy & Celebi, 2010)**, **(Begehr H. , 2005)**, **(Begehr & Hile, 1998)**, **(Begehr, Kumar, Schmersau, & Vanegas, 2005)**, **(Vaitekhovich, 2008)**.

The Characteristic Mode Theory (TCM) for either a conducting body **(Harrington & Mautz, 1985)** or apertures **(Harrington & Mautz, 1971)**, are simple and structured methods that can be used to solve integro-differential complex equations with a good convergence. TCM can fully characterize the radiation and scattering properties of an arbitrary object based only on the object's geometry and material properties. It can be simply described as a numerical procedure based on dividing the general solution of a problem into a set of known solutions with different weights. The orthogonality property of the set function, derived in the procedure, is used to determine the weighted sum and then a general solution is obtained. As dealing with complex quantities, the solution of a complex-coefficient function is complex by nature and can be divided into a real part and an imaginary part. This division once implemented in the differential equation will result into two interconnected equations. Using the Method of Moment (MoM) to solve these equations leads to an eigenvalue problem to be solved under the domain of the problem. Once these eigenmodes and eigenvalues are obtained, a general solution of the problem can be then found. The characteristic mode theory for apertures has been successfully applied to many engineering problems including slots in a conducting plane **(Kabalan, Harrington, Auda, & Mautz, 1987)**, **(El-Hajj, Kabalan, & Harrington, 1992)**, slots in a conducting cylinder **(Kabalan, El-hajj, & Harrington, 1992)**, **(Kabalan, El-Hajj, & Harrington, 1993)**, waveguide problem **(Kabalan & El-Hajj, 1994)**, rectangular apertures **(El-Hajj & Kabalan, 1994)**, and others **(Kabalan & El-Hajj, 1995)**, **(Kabalan, El-Hajj, Khoury, & Rayes, 1998)**.

MoM **(Harrington R. F., 1992)** is a computational method based first on discretizing the equations into small segments. Then, known basis functions with unknown functions are employed to generate a matrix of equations using Galerkin's solution. Galerkin method uses the same functions for expansion and testing. The most delicate part of this method resides in the selection of the basis and testing functions that will affect the accuracy of the solution. The basis functions are used to expand the solutions into sets, and then solving for a null residual error with sets of weighting functions.

MoM has been widely in conjunction with other methods for computational purposes. In **(Takei & Ikegaya, 2007)**, MoM is joint with the gradient method to create a novel antenna design method based on representing the antenna structure by small segments. In **(De Menezes, Ajayi, Christopoulos, Sewell, & Borges, 2007)**, it is used with the unscented transform to model the uncertainty in electromagnetic computations. This resulted in the same accuracy when compared to the Monte Carlo approach, but with fewer simulations. MoM is also used in **(Wang, He, Ding, & Chen, 2019)** to analyze the uncertainties of the geometric shape of 3D objects. In **(Xu, Yang, Shen, Zhu, & Huang, 2019)**, an improved MoM is proposed to analyze the scattering properties of multi-object electromagnetic systems while adopting an iterative process to increase the accuracy of the method used. Wavelet packet transform is used in **(Azzam, Malhat, & Zainud-Deen, 2019)** with MoM to reduce the computation time and the storage capacity when dealing with large array elements. In **(Sallam, Vandenbosch, Gielen, & Soliman, 2019)**, MoM is used to find the modal characteristics of plasmonic transmission lines, such as the attenuation and phase constants. In **(Capek, Jelinek, & Gustafsson, 2019)**, MoM is used with Sherman-Morrison-Woodbury identity to investigate the nearest neighbors on a Hamming graph. MoM is also used in **(Rao, 2019)** to solve a time-domain integral equation applied to a wire-grid model of a conducting body. In **(El Misilmani, Kabalan, Abou Shahine, & Al-Husseini, 2015)**, MoM is used to solve real-coefficient real-solution differential equations.

In this paper, TCM is used along MoM to numerically compute the solution of a complex-coefficient complex-solution differential equation. The proposed method is simple and applicable to wide varieties of boundary values problems. In addition, although typical numerical methods involve approximation that introduces error, the proposed method, based on a finite set of basis functions defined over each subdivision, shows direct convergence as the number of divisions increases. The proposed method is done as follows. The problem is first outlined, then a general solution is derived where the unknown function is expanded into a sum of weighted basis functions, for which the weight coefficients are to be found. Galerkin method, which selects testing functions equal to the basis functions, is adopted. The problem then becomes a system of linear equations, which is solved analytically or numerically to find the needed weight coefficients. Two illustrating examples using third-order and fourth-order complex-coefficients complex-solution differential equations are then presented to show the validity of the proposed method.

2. Problem Formulation

MoM starts by expanding the unknown quantity into a set of known functions with unknown coefficients. The resulting equation is then converted into a linear system of equations by enforcing boundary conditions at a number of points. The resulting linear system is then solved analytically for the unknown coefficients. This approach is very simple and quite interesting when applied to differential equation of order less than three, and it is equally applicable for equations of higher order.

To illustrate the proposed method, we will start with some basic mathematical techniques, used for reducing functional equations into matrix equations, and then the solution could be found by matrix inversion. Two simple examples will be used to illustrate the theory without any complicated mathematics.

A general p^{th} order complex-coefficients linear differential equation, defined over a domain *D*, has the form:

$$\mathrm{C_p}\frac{d^p f(x)}{dx^p} + \mathrm{C_{p-1}}\frac{d^{p-1} f(x)}{dx^{p-1}} + \ldots + C_1\frac{df(x)}{dx} + C_0 f(x) = g(x). \tag{1}$$

In (1), p is the order of the differential equation and the coefficients $C_p, C_{p-1},\ldots, C_{1,}C_0$, $f(x)$, and *g(x)* are all complex quantities, with $f(x)$ being the function whose solution is to be determined. Equation (1), called the operator equation, can be written as:

$$L\big(f(x)\big) = g(x) \tag{2}$$

In (2), L is the operator, operating on $f(x)$, and it is given by:

$$L = C_p\frac{d^p\ldots}{dx^p} + C_{n-1}\frac{d^{p-1}\ldots}{dx^{p-1}} + \ldots + C_1\frac{d\ldots}{dx} + C_0 \ldots \tag{3}$$

As $f(x)$ is usually a complex-valued solution, and the coefficients $C_p, C_{p-1},\ldots, C_{1,}C_0$ are complex, the operator in (2) can be defined, according to TCM theory, as the sum of a real operator ***G*** and an imaginary operator ***B***, as

$$L\big(f(x)\big) = \boldsymbol{G}\big(f(x), f^*(x)\big) + j\boldsymbol{B}(f(x), f^*(x)) \tag{4}$$

where

$$G(f(x), f^*(x)) = L\big(f(x)\big) + L^*\big(f^*(x)\big) \tag{5}$$

$$B\big(f(x), f^*(x)\big) = L\big(f(x)\big) - L^*\big(f^*(x)\big)$$

with

$$G\big(f(x), f^*(x)\big) = The\ Real\ part\ of\ g(x) = Re(g(x)) \tag{6}$$

and

$$B\big(f(x), f^*(x)\big) = The\ imaginary\ part\ of\ g(x) = Im(g(x)) \tag{7}$$

These operators are determined from the given differential equation. In the next step, the domain of solution is divided into N segments of equal length. The function $f(x)$ is then defined over the nth interval of definition as follows:

$$f(x) \cong \sum_{n=1}^{N} V_n f_n(x); \qquad n = 1, 2, \ldots, N \tag{8}$$

In (8), $f_n(x)$ is the nth expansion function, V_n is its unknown coefficient to be determined, and N is the total number of expansion functions. Substituting (8) into (4), the following is obtained:

$$\mathrm{L}\big(\mathrm{f(x)}\big) = \mathrm{C_p} \sum_{\mathrm{n=1}}^{\mathrm{N}} \mathrm{V_n} \frac{\mathrm{d^p(f_i(x))}}{\mathrm{dx^p}} + \mathrm{C_{p-1}} \sum_{\mathrm{n=1}}^{\mathrm{N}} \mathrm{V_{n-1}} \frac{\mathrm{d^{p-1}(f_{i-1}(x))}}{\mathrm{dx^{p-1}}} + \cdots \tag{9}$$

Taking the complex conjugate of (9):

$$L^*\big(f(x)\big) = C_p{}^* \sum_{n=1}^{N} V_n{}^* \frac{d^p(f_i(x))}{dx^p} + C_{p-1}^* \sum_{n=1}^{N} V_{n-1}^* \frac{d^{p-1}(f_{i-1}(x))}{dx^{p-1}} + \cdots \tag{10}$$

Denoting $O(.)$ any of the two operators defined in (5) and h the right side of equations (6) and (7), the problem now is to solve the equation:

$$O(f) = h \Rightarrow O(f) - h = 0 \tag{11}$$

In (11), $O(.)$ is a linear integro-differential operator, $f(x)$ is an unknown complex function to be solved, and $h(x)$ is a known complex function. To find $f(x)$, it should be approximated by a sum of weighted basis or expansion functions as follows:

$$f(x) \cong \sum_{n=1}^{N} V_n f_{n(x)} = \sum_{n=1}^{N} a_n f_n(x) + j \sum_{n=1}^{N} b_n f_n(x) \qquad n = 1, 2, \ldots, N \tag{12}$$

In (4), a_n are the unknown coefficients of the real part, b_n are the unknown coefficients of the imaginary part of f_n, and N is the total number of expansion functions. Op_{real} and Op_{imag} are the real and imaginary parts of the linear integro-differential operator, respectively. Since the operator O is linear, the differential equation could be written as:

$$Op_{real}[\sum_{n=1}^{N} a_n f_n(x) + j \sum_{n=1}^{N} b_n f_n(x)]] + jOp_{imag}[\sum_{n=1}^{N} a_n f_n(x) + j \sum_{n=1}^{N} b_n f_n(x)] = g_R + jg_I \tag{13}$$

After expanding the equation, two residual equations can be formed and given by:

$$R\ (real) = g_R - Real\{left\ side\} \tag{14}$$

$$R\ (imag) = g_I - Imag\{left\ side\}$$

The next step is to take the inner product of equation (14) with the testing function denoted by w_m. The testing function should be equal to f_n as imposed by the method of Galerkin. The inner product is defined as:

$$\langle C, D \rangle = \int\limits_{\substack{domain of \\ solution}} C^*(x) \cdot D(x) dx \tag{15}$$

Applying (15) to (12), we obtain:

$$< w_m, (O(f_n) - h) > = 0; \quad m = 1, 2, \ldots, N \tag{16}$$

Since V_n is a constant, it could be taken outside of the inner product and the inner product could be then written as:

$$\sum_{n=1}^{N} V_n < w_m, O(f_n) > = < w_m, h >; m = 1, 2, \ldots, N \tag{17}$$

This could be written in matrix form as follows:

$$[Z][V] = [H] \tag{18}$$

with matrix Z defined as:

$$[Z] = \begin{bmatrix} < f_1, O(f_1) > & \cdots & < f_1, O(f_n) > \\ \vdots & \ddots & \vdots \\ < f_m, O(f_1) > & \cdots & < f_M, O(f_n) > \end{bmatrix} \tag{19}$$

and vectors I and V as follows:

$$[V] = [v_1, v_2 \ \ \dots \ \ v_N]^T \tag{20}$$

$$[H] = [< h, w_1 > \ < h, w_2 > \ \ \dots \ \ < h, w_M >]^T \tag{21}$$

It is then obvious that the vector of unknown coefficients I could be written as:

$$[V] = [Z]^{-1}[H] \tag{22}$$

Combining the real and imaginary parts into one equation, the following is obtained:

$$\begin{bmatrix} Z_R & -Z_I \\ Z_I & Z_R \end{bmatrix} \begin{bmatrix} \sum_{n=1}^{N} a_n \\ \sum_{n=1}^{N} b_n \end{bmatrix} = \begin{bmatrix} V_R \\ V_I \end{bmatrix} \tag{23}$$

Once the coefficients of the column matrix V (a_n and b_n) are obtained from both equations, the solution of the differential equation is then obtained. It is here to note the following:

1. f_n must satisfy the boundary conditions of the problem. For this, it is important to carefully select this function.
2. The number of divisions N required to obtain the exact solution increases as the order of the equation increases.

3. Third Order Example Of a Complex Function Complex Coefficient Different Equation

In this section, the proposed method is applied to a third-order differential complex coefficient example. The required steps are detailed, and the complete solution is provided. Let a 3rd order differential complex coefficient equation be written as:

$$j\frac{d^3 f(x)}{dx^3} + (2+j)\frac{d^2 f(x)}{dx^2} + (1+j)\frac{df(x)}{dx} + 2jf(x) = -2jx^3 - (1+3j)x^2 - (12+6j)x + (2-8j) \tag{24}$$

with $f(0) = 0, f'(0) = 1 + j, and\ f''(0) = -2j.$

The steps of the solution of this equation are outlined in the following:

1. Identify the real and the imaginary part of the operator:

$$\begin{gathered} L(f) = j\big(f'''_R + jf'''_I\big) + (2+j)\big(f''_R + jf''_I\big) + (1+j)\big(f'_R + jf'_I\big) + 2j(f_R + jf_I) \\ Op_{real} = -f'''_I + 2f''_R - f''_I + f'_R - f'_I - 2f_I \\ Op_{imag} = f'''_R + 2f''_I + f''_R + f'_I + f'_R + 2f_R \end{gathered} \tag{25}$$

2. Identify the real and the imaginary part of the right-side function:

$$\begin{gathered} Real\{g(x)\} = -x^2 - 12x + 2 \\ Imag\{g(x)\} = -2x^3 - 3x^2 - 6x - 8 \end{gathered} \tag{26}$$

3. Equate the real and the imaginary parts from both sides:

$$\begin{gathered} Op_{real} = Real\{g(x)\} \\ Op_{imag} = Imag\{g(x)\} \end{gathered} \tag{27}$$

4. Choose the same basis and weighting functions that satisfy the boundary conditions of the problem. It is to note that $f_n(x)$ must not be equal to zero at the derivatives so that all the coefficients of the differential equation enter in the solution. In this case, we select:

$$f_n(x) = w_m(x) = x - x^{n+1} \tag{28}$$

5. Calculate the inner product of (25) with the weighting function, and apply the real and imaginary parts of the operator on the basis function:

$$< Op_{real}, w_m > = < Real\{g(x)\}, w_m >$$

$$< -f_{nI}''' + 2f_{nR}'' - f_{nI}'' + f_{nR}' - f_{nI}' - 2f_{nI}, w_m > = < -x^2 - 12x + 2, w_m > \tag{29}$$

After substituting (13) into the left-hand side of (29), the real operator becomes:

$$-\sum_{n=1}^{N} b_n f_n'''(x) + 2\sum_{n=1}^{N} a_n f_n''(x) - \sum_{n=1}^{N} b_n f_n''(x) + \sum_{n=1}^{N} a_n f_n'(x) - \sum_{n=1}^{N} b_n f_n'(x) - 2\sum_{n=1}^{N} b_n f_n(x) \tag{30}$$

Rearranging (30), the following is obtained:

$$\sum_{n=1}^{N} a_n[2 f_n''(x) + f_n'(x)] - \sum_{n=1}^{N} b_n[f_n'''(x) + f_n''(x) + f_n'(x) + 2f_n(x)] = \sum_{n=1}^{N} a_n Op_1 - \sum_{n=1}^{N} b_n Op_2 \tag{31}$$

Substituting (31) into (29), we obtain:

$$\sum_{n=1}^{N} a_n < Op_1, w_m > - \sum_{n=1}^{N} b_n < Op_2, w_m > = < -x^2 - 12x + 2, w_m > \tag{32}$$

In a matrix format, (32) can be written as:

$$\sum_{n=1}^{N} a_n Z_1 - \sum_{n=1}^{N} b_n Z_2 = V_R \tag{33}$$

The same procedure is repeated for the imaginary case to obtain the extended imaginary operator:

$$\sum_{n=1}^{N} a_n f_n'''(x) + 2\sum_{n=1}^{N} b_n f_n''(x) + \sum_{n=1}^{N} a_n f_n''(x) + \sum_{n=1}^{N} b_n f_n'(x) + \sum_{n=1}^{N} a_n f_n'(x) + 2\sum_{n=1}^{N} a_n f_n(x)$$

$$= \sum_{n=1}^{N} a_n[f_n'''(x) + f_n''(x) + f_n'(x) + 2f_n(x)] + \sum_{n=1}^{N} b_n[2f_n''(x) + f_n'(x)]$$

$$= \sum_{n=1}^{N} a_n Op_2 + \sum_{n=1}^{N} b_n Op_1 \tag{34}$$

Taking the inner product (16) of:

$$< (\sum_{n=1}^{N} a_n Op_2 + \sum_{n=1}^{N} b_n Op_1), w_m > = < -2x^3 - 3x^2 - 6x - 8, w_m > \tag{35}$$

the following matrix equation is obtained:

$$\sum_{n=1}^{N} a_n Z_2 + \sum_{n=1}^{N} b_n Z_1 = V_I \tag{36}$$

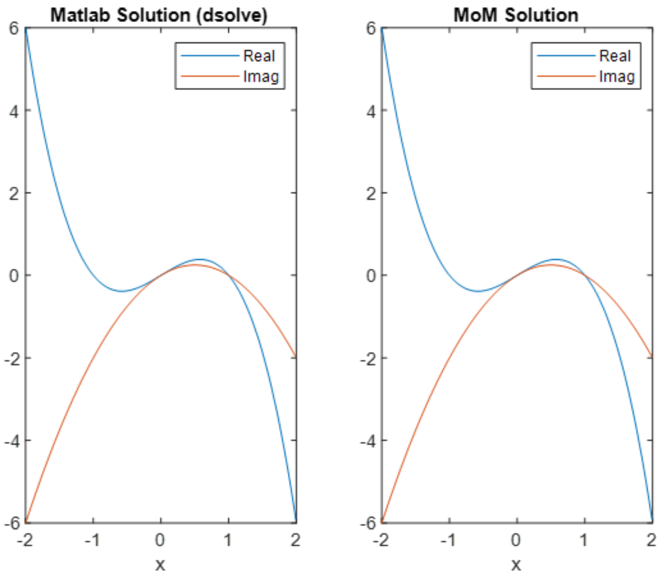
6. Solve the system of the form $A.x = b$

$$\begin{bmatrix} Z_1 & -Z_2 \\ Z_2 & Z_1 \end{bmatrix} \begin{bmatrix} \sum_{n=1}^{N} a_n \\ \sum_{n=1}^{N} b_n \end{bmatrix} = \begin{bmatrix} V_R \\ V_I \end{bmatrix} \tag{37}$$

7. Finally, create the function $f(x) = \sum_{n=1}^{N} a_n f_n(x) + j\sum_{n=1}^{N} b_n f_n(x)$ and plot it.

Applying the proposed method on the 3rd order example, Fig. 1 shows a comparison between the solution of the same example obtained using *dsolve* Matlab function and that obtained with the proposed MoM method in this paper. As can be seen, MoM solution is in excellent analogy with that obtained using Matlab. The number of subsections N on the interval [-2 2] is chosen to be 12. For higher values of N, no changes in the graph were obtained meaning a convergence of the solution.

Figure 1: Comparison between MoM solution and Matlab dsolve function solution of the 3rd order Example



4.Fourth Order Example of a Complex Function Complex Coefficient Different Equation

To further validate and illustrate the proposed method, a fourth-order complex coefficient differential equation is considered.

$$3j\frac{d^4f(x)}{dx^4} + (2+j)\frac{d^3f(x)}{dx^3} + j\frac{d^2f(x)}{dx^2} - \frac{df(x)}{dx} - 2f(x)$$
$$= -4jx^6 + (2-2j)x^5 - (55+2j)x^4 - (242-454j)x^3 - (2295+55j)x^2 - (32+298j)x - (59-18j) \quad (38)$$

with $f(0) = 0, f'(0) =, f''(0) = -8 + j, and\ f'''(0) = 6 + 6j.$

The steps of the solution of this equation are outlined in the following:

1. Identify the real and the imaginary part of the operator:

$$L(f) = 3j(f_R^{IV} + jf_I^{IV}) + (2+j)(f_R''' + jf_I''') + j(f_R'' + jf_I'') - (f_R' + jf_I') - 2(f_R + jf_I)$$
$$Op_{real} = -3f_I^{IV} + 2f_R''' - f_I''' - f_I'' - f_R' - 2f_R$$
$$Op_{imag} = 3f_R^{IV} + 2f_I''' + f_R''' + f_R'' - f_I' - 2f_I \quad (39)$$

2. Identify the real and the imaginary part of the right-side function:

$$Real\{g(x)\} = 2x^5 - 55x^4 - 242x^3 - 2295x^2 - 32x - 59 \quad (40)$$
$$Imag\{g(x)\} = -4x^6 - 2x^5 - 2x^4 + 454x^3 - 55x^2 - 298x - 18$$

3. Equate the real and the imaginary parts from both sides:

$$Op_{real} = Real\{g(x)\}$$
$$(41)$$
$$Op_{imag} = Imag\{g(x)\}$$

4. Select $f_n(x)$:

$$f_n(x) = w_m(x) = x - x^{n+1} \quad (42)$$

5. Calculate the inner product of (39) with the weighting function and apply the real and imaginary parts of the operator on the basis function:

$$< Op_{real}, w_m > = < Real\{g(x)\}, w_m >$$
$$< -3f_{nI}^{IV} + 2f_{nR}''' - f_{nI}''' - f_{nI}'' - f_{nR}' - 2f_{nR}, w_m > = < 2x^5 - 55x^4 - 242x^3 - 2295x^2 - 32x - 59, w_m > \quad (43)$$

After substituting (13) into the left-hand side of (43), the real operator becomes:

$$-3\sum_{n=1}^{N} b_n f_n^{IV}(x) + 2\sum_{n=1}^{N} a_n f_n'''(x) - \sum_{n=1}^{N} b_n f_n'''(x) - \sum_{n=1}^{N} b_n f_n''(x) - \sum_{n=1}^{N} a_n f_n'(x) - 2\sum_{n=1}^{N} a_n f_n(x) \quad (44)$$

Rearranging (44), the following is obtained:

$$\sum_{n=1}^{N} a_n[2 f_n'''(x) - f_n'(x) - 2f_n(x)] - \sum_{n=1}^{N} b_n[3f_n^{IV}(x) + f_n'''(x) + f_n''(x)]+$$

$$= \sum_{n=1}^{N} a_n\, Op_1 - \sum_{n=1}^{N} b_n\, Op_2 \quad (45)$$

Substituting (45) into (43), we obtain:

$$\sum_{n=1}^{N} a_n < Op_1, w_m > - \sum_{n=1}^{N} b_n < Op_2, w_m > = < 2x^5 - 55x^4 - 242x^3 - 2295x^2 - 32x - 59, w_m > \quad (46)$$

In a matrix format, (46) can be written as:

$$\sum_{n=1}^{N} a_n\, Z_1 - \sum_{n=1}^{N} b_n\, Z_2 = V_R \quad (47)$$

The same procedure is repeated for the imaginary case to obtain the extended imaginary operator:

$$< 3f_R^{IV} + 2f_I''' + f_R''' + f_R'' - f_I' - 2f_I, w_m > = < -4x^6 - 2x^5 - 2x^4 + 454x^3 - 55x^2 - 298x - 18, w_m >$$

$$\sum_{n=1}^{N} a_n[3f_n^{IV}(x) + f_n'''(x) + f_n''(x)] + \sum_{n=1}^{N} b_n[2f_n'''(x) - f_n'(x) - 2f_n(x)]$$

$$= \sum_{n=1}^{N} a_n\, Op_2 + \sum_{n=1}^{N} b_n\, Op_1 \quad (48)$$

Taking the inner product (16) of:

$$< (\sum_{n=1}^{N} a_n\, Op_2 + \sum_{n=1}^{N} b_n\, Op_1), w_m > = < -4x^6 - 2x^5 - 2x^4 + 454x^3 - 55x^2 - 298x - 18, w_m > \quad (49)$$

the following matrix equation is obtained:

$$\sum_{n=1}^{N} a_n\, Z_2 + \sum_{n=1}^{N} b_n\, Z_1 = V_I \quad (50)$$

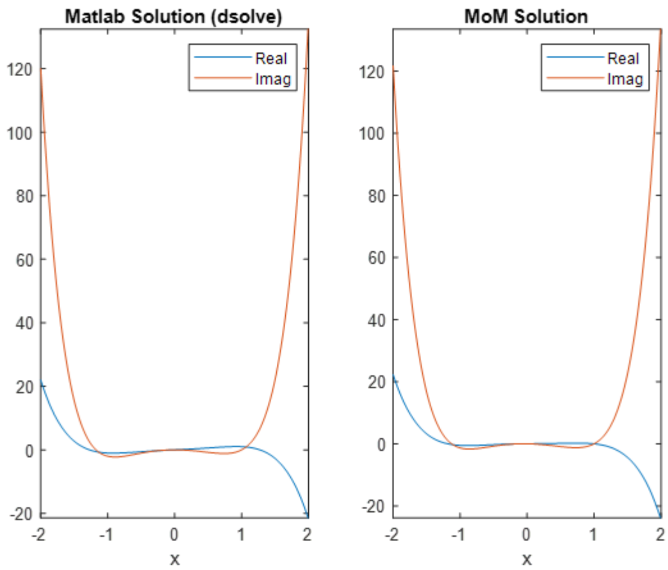
6. Solve the system of the form $A.x = b$

$$\begin{bmatrix} Z_1 & -Z_2 \\ Z_2 & Z_1 \end{bmatrix} \begin{bmatrix} \sum_{n=1}^{N} a_n \\ \sum_{n=1}^{N} b_n \end{bmatrix} = \begin{bmatrix} V_R \\ V_I \end{bmatrix} \quad (51)$$

7. Finally, create the function $f(x) = \sum_{n=1}^{N} a_n f_n(x) + j\sum_{n=1}^{N} b_n f_n(x)$ and plot it.

Applying the proposed method on the 4rd order example, Fig. 2 shows a comparison between the solution of the same example obtained using *dsolve* Matlab function and that obtained with the proposed MoM method. As can be seen, MoM solution is also in excellent analogy with that obtained using Matlab, as obtained in the previous example. The number of subsections N on the interval [-2 2] is also chosen to be 12. For higher values of N, no changes in the graph were obtained meaning a convergence of the solution.

Figure 2: Comparison between MoM solution and Matlab dsolve function solution of the 4th order example



5.Convergence of the Solution

This section shows the convergence of the solution of $f(x)$ in a least square sense over the domain of its definition. Let

$$f^N(x) = \sum_{n=1}^{N} U_n f_n(x) \tag{52}$$

be the N-term modal expansion of $f(x)$ to be the quality toward which it converges, then:

$$f(x) = \lim_{N\to\infty} \sum_{n=1}^{N} U_n f_n(x) \tag{53}$$

If $S_n = \lim\limits_{N\to\infty} U_n$ (54)

then, $f(x) = \sum_{n=1}^{N} S_n f_n(x)$ (55)

The difference between (53) and (55) is called the residual R_N, and it can be written as:

$$R_N = \sum_{n=1}^{N} U_n f_n(x) - \sum_{n=1}^{N} S_n f_n(x) \tag{56}$$

The convergence of $f(x)$ in a least square sense over the domain of definition is equivalent to the requirement that the square of the norm of the residual $\|R_N\|^2$ in (57) to be minimum.

$$\|R_N\|^2 = \int_{Domain\ of\ definition} R_N^* \cdot R_N \, dx \tag{57}$$

Substituting (56) into (57) and using the first orthogonality relationship of each of the defined real and imaginary operators, $\|R_N\|^2$ can be written as:

$$\|R_N\|^2 = \epsilon[\sum_{n=1}^{N} |S_n - V_n|^2 + \sum_{n=N+1}^{\infty} |S_n|^2] \tag{58}$$

The quantity (58) is minimum when

$S_n = V_n$ for n = 1, 2,…, N. (59)

It is now evident from (54) that V_n is given by (59). Therefore, the modal coefficients V_n minimize (58) so that the convergence of the modal solution of $f(x)$ is in a least squares sense over the domain of definition.

6.Conclusion

In this paper, the Method of Moments along with the Characteristic Mode Theory is used to solve complex-coefficients complex-solution differential equations. The functions to be found are expanded into a sum of unknown coefficients multiplied by a basis function. Then, the inner product of the differential equation with a weighting function equal to the basis function, as imposed by the method of Galerkin, is performed. Finally, the unknown coefficients are calculated by a simple matrix inversion. The sum of the basis function multiplied by the corresponding coefficients result in the solution of the unknown function. Two examples using third-order and fourth-order complex-coefficients complex-solution differential equations were illustrated with detailed solution to show the validity of the proposed method.

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