Total Neighborhood Magic Labeling: A New Variant of Magic Labeling

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Abstract: In this paper, we introduce a total neighborhood magic labeling, which is a variant of magic labeling. A total neighborhood magic labeling on a graph with v vertices and e edges is a bijection f taking vertices and edges onto the numbers $1,2,\dots,v+e$ with the property that there is a constant k such that at any vertex q, $\sum_{p\in N(q)}(f(p)+f(pq))=k$, where $N(q)$ is the set of neighborhood vertices of p. A graph which admits a total neighborhood magic labeling is called a total neighborhood magic graph.
Here, we study some necessary conditions for the existence of a total neighborhood magic labeling, obtain some non-existence results for star, tree, wheel and bistar graphs and discuss the existence of a total neighborhood magic labeling for $K_{n,n}$ and nC_3.
Keywords: Distance magic labeling, Neighborhood magic labeling, total neighborhood magic labelling.

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1. Introduction

We consider here, all graphs $G=(V,E)$ with vertex set $V(G)$ and edge set $E(G)$ are undirected, finite and simple. We adopt Gross and Yellen [5] for graph theoretic terminology and for number theoretical results, we follow Burton [3]. For acquiring the latest update, we follow a dynamic survey on graph labeling by Gallian [4].

A distance magic labeling of a graph G is a bijection $f: V(G)\to\{1,2,\dots,n\}$ such that $\sum_{p\in N(q)}f(p)=\gamma$, for all $q\in V(G)$, where $N(q)$ is the set of all vertices of $V(G)$ which are adjacent to q. The constant γ is called the *magic constant* of the distance magic labeling of G. A graph which admits a distance magic labeling is called distance magic graph. For any vertex $q\in V(G)$, the neighbor sum $\sum_{p\in N(q)}f(p)$ is called the weight of the vertex $q\in V(G)$, and is denoted by $w(q)$.

The concept of distance magic labeling [6] was introduced and studied by many authors with different names like sigma labeling [8] and 1-vertex magic labeling [1-VML] [7].

The term neighborhood magic labeling is used by B.D.Acharya et al. [1], which is a variant of distance magic labeling in more general way. A graph G is said to be a neighborhood magic graph if there exists an injection $f: V\to R$ satisfying the condition $\sum_{p\in N(q)}f(p)=Q(f)$, for all $q\in V(G)$. The constant $Q(f)$ is called the neighborhood magic index of f and the function f is called neighborhood magic labeling.

Motivated by neighborhood magic labeling and distance magic labeling, we introduce the notion of *total neighborhood magic labeling*. A *total neighborhood magic labeling* of a graph G is a bijection $f: V(G)\cup E(G)\to\{1,2,\dots,|V(G)\cup E(G)|\}$ such that $\sum_{p\in N(q)}(f(p)+f(pq))=k$, for all $q\in V(G)$, where $N(q)$ is the set of all vertices of $V(G)$ which are adjacent to q. The constant k is called the *magic constant* of the total neighborhood magic labeling of G. A graph which admits a total neighborhood magic labeling is called total neighborhood magic graph. For any vertex $q\in V(G)$, the neighbor sum $\sum_{p\in N(q)}(f(p)+f(pq))$ is called the weight of the vertex $q\in V(G)$ and is denoted by $w(q)$.

One can easily verify that cycle C_3 admits a total neighborhood magic labeling with magic constant $k=14$ which is shown in Figure 1.1. The path P_2 is not a total neighborhood magic graph because $f(p)+f(pq)\neq f(q)+f(pq)$ otherwise $f(p)=f(q)$.

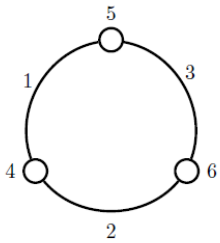

Figure 1.1: C_3

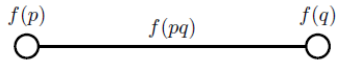

Figure 1.2: P_2

In the next section, we will study some necessary conditions for the existence of total neighborhood magic graphs. In addition, we will determine several classes of graphs which are not total neighborhood magic graphs. At last, we will prove, complete bipartite graph $K_{n,n}$ and nC_3 are total neighborhood magic graphs.

Throughout this paper, minimum and maximum degrees of vertices in $V(G)$ are denoted by δ and Δrespectively.

2.Main Results

We start this section with stating necessary conditions for a total neighborhood magic graph.

Lemma 2.1.

A necessary condition for the existence of a total neighborhood magic labeling f of a graph G is

$$kv = \sum_{\substack{v_i \in V(G) \\ e_i \in E(G)}} \big(d(v_i)f(v_i) + 2f(e_i)\big), \tag{2.1.1}$$

where $d(v_i)$is the degree of vertex v_i and $v$is the number of vertices of G.

Proof:

We can see that, the sum of all total neighborhood magic weights of all vertices in G is kv. And in the right hand side, the sum counts the label of vertex $v_i$exactly $d(v_i)$ times and label of edges e_i exactly two times. Hence the equation holds.

■

Further, equation (2.1.1) contains each label once and each vertex label $f(v_i)$ an additional $(d_i - 1)$times, where d_i is the degree of vertex $v_i$and each edge label $f(e_i)$ an additional one time. So equation (2.1.1) becomes,

$$kv = \sigma_1^{v+e} + \sum\big((d_i - 1)f(v_i) + f(e_i)\big). \tag{2.1.2}$$

Theorem 2.2

If total neighborhood magic labeling is exist for 2-regular graph then magic constant of the graph is

$$k = 2(2v + 1).$$

Proof:

For 2-regular graph, $d(p) = 2$, so by equation (2.1.1), $k = 2(2v + 1)$.

■

A total neighborhood magic labeling for C_3 is given in Figure 1.1 with $k = 14$.

Lemma 2.3

If G is a total neighborhood magic graph of v-vertices and e-edges with maximum degree Δ and minimum degree δ, then

$$\Delta(2\Delta + 1) \leq \delta(2v + 2e - 2\delta + 1). \quad (2.3.1)$$

Proof: We assume that G is a total neighborhood magic graph.

Let $d(v_i) = \Delta$(maximum) and $d(v_j) = \delta$ (minimum), for some $v_i, v_j \in V(G)$.

So,

$$1 + 2 + \cdots + 2\Delta \leq w(v_i) \leq (v + e) + (v + e - 1) + \cdots + (v + e - 2\Delta + 1) \quad (2.3.2)$$

and

$$1 + 2 + \cdots + 2\delta \leq w(v_j) \leq (v + e) + (v + e - 1) + \cdots + (v + e - 2\delta + 1). \quad (2.3.3)$$

Since G is a total neighborhood magic graph, $w(v_i) = w(v_j)$.By equations (2.3.2) and (2.3.3)

$$1 + 2 + \cdots + 2\Delta \leq w(v_i) = w(v_j) \leq (v + e) + (v + e - 1) + \cdots + (v + e - 2\delta + 1).$$

Thus,

$$\Delta(2\Delta + 1) \leq \delta(2v + 2e - 2\delta + 1).$$

Hence the proof.

■

From the above theorem it follows that, if $\Delta(2\Delta + 1) > \delta(2v + 2e - 2\delta + 1)$ then there does not exist total neighborhood magic labeling.

Theorem 2.4

Let G be a tree with v vertices and e edges. If Δ is an even integer and $v \leq \frac{\Delta^2}{2} + \left\lceil \frac{\Delta}{4} \right\rceil$, then G is not a total neighborhood magic graph.

Proof: Let G be a tree. Since $e = v - 1$ and $\delta = 1$, $\delta(2v + 2e - 2\delta + 1) = 4v - 3$,

One can see that, $\Delta(2\Delta + 1)$ is always greater than $4v - 3$ when $v \leq \frac{\Delta^2}{2} + \left\lceil \frac{\Delta}{4} \right\rceil$.

Hence by Lemma 2.3, G is not a total neighborhood magic graph.

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Theorem 2.5

Let G be a tree with v vertices and e edges and Δ is an odd integer.

(1) For $\Delta = 4i - 1, i \in N$, if $v \leq \frac{(\Delta+1)(2\Delta-3)}{4} + \left\lceil \frac{\Delta}{2} \right\rceil$, then G is not a total neighborhood magic graph.

(2) For $\Delta = 4i + 1, i \in N$, if $v \leq \frac{(\Delta-1)(2\Delta+1)}{4} + \left\lceil \frac{\Delta}{2} \right\rceil$, then G is not a total neighborhood magic graph.

Proof: Proof is similar to Theorem 2.4.

■

From Theorem 2.4 and Theorem 2.5, it follows that every star is not a total neighborhood magic graph.

Theorem 2.6

For $n > 7$, the wheel graph W_n is not a total neighborhood magic graph.

Proof: Let $G = W_n$ be a wheel graph with $v = |V(G)| = n + 1$ and $e = |E(G)| = 2n$.

Now, $\Delta(2\Delta + 1) = n(2n + 1)$ and $\delta(2v + 2e - 2\delta + 1) = 9(2n - 1)$.

And it is easy to see that $n(2n + 1) > 9(2n - 1)$ for $n > 7$.

By Lemma 2.3, W_n is not a total neighborhood magic graph for $n > 7$.

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Lemma 2.7

If atleast one of the vertex of graph G has atleast two pendant vertices then G is not a total neighborhood magic graph.

Proof: Let G be a total neighborhood magic graph under total neighborhood magic labeling f with magic constant k and let us asssume that for some vertex v_i of G has two pendent vertices say, u_1 and u_2.

So we have $w(u_1) = w(u_2)$, which implies that $f(v_i) + f(v_i u_1) = f(v_i) + f(v_i u_2)$ and hence $f(v_i u_1) = f(v_i u_2)$, which contradicts to our hypothesis.

Hence the proof.

■

From Theorem 2.4 and Theorem 2.5 we cannot say about the bistar graphs whether they are total neighborhood magic graphs. But from Lemma 2.7, we can state the following Theorem for bistar graph.

Theorem 2.8

The Bi-star graph $B(m, n)$; $m, n \geq 2$ is not a total neighborhood magic graph.

Proof:

In bi-star graph, apex vertex hasat least two pendant vertices, so by Lemma 2.7, bi-star graph is not total neighborhood magic graph. ■

Now, we check the *duality* of the total neighborhood magic labeling. Given a labeling f, its *dual* labeling is a bijection$f'\colon V(G) \cup E(G) \to \{1, 2, \dots, |V(G) \cup E(G)|\}$defined by

$$f'(p) = v + e + 1 - f(p), \text{for any vertex } p$$

$$f'(pq) = v + e + 1 - f(pq), \text{for any vertex } pq$$

Theorem 2.9

The dual of a total neighborhood magic labeling for a graph G with magic constant k is a total neighborhood magic labeling with magic constant $k' = 2r(v + e + 1) - k$ if and only if G is ar-regualr graph.

Proof: Let f be a total neighborhood magic labeling with magic constant k and let f' be a dual of a total neighborhood magic labeling f. Thus by definition of dual of a labeling,

$$k' = \sum_{p \in N(q)} \big(f'(p) + f'(pq)\big)$$

$$= \sum_{p \in N(q)} \big((v + e + 1) - f(p) + (v + e + 1) - f(pq)\big)$$

$$= 2 \sum_{p \in N(q)} (v + e + 1) - \sum_{p \in N(q)} (f(p) + f(pq))$$

$$= 2 \sum_{p \in N(q)} (v + e + 1) - k$$

$$= 2r(v + e + 1) - k, where\ r\ is\ the\ degree\ of\ vertex\ p.$$

Clearly k' is constant if and only if r is constant.

Hence, the dual of a total neighborhood magic labeling for a graph G with magic constant $k$is a total neighborhood magic labeling with magic constant $k' = 2r(v + e + 1) - k$ if and only if G is ar-regualr graph.

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An illustration of the dual of a total neighborhood magic labeling is given in Figure 2.1.

Figure 2.1: Dual labeling for C_3 with $k = k' = 14$

Theorem 2.10

For even $n \neq 2$, $K_{n,n}$has a total neighborhood magic labeling with magic constant$\frac{n}{2}(3n^2 + 2n + 2)$.

Proof: Let $G = K_{n,n}$ be a graph with

$$V(G) = \{u_{i,}v_i / 1 \le i \le n\},\ \ v = |V(G)| = 2n$$

$$E(G) = \{u_i v_j\ /\ 1 \le i \le n, 1 \le j \le n\},\ \ e = |E(G)| = n^2 \text{ and } v + e = n^2 + 2n.$$

Define a bijection $f: V(G) \cup E(G) \to \{1,2,\dots,|V(G) \cup E(G)|\}$ as follows:

$$f(u_i) = \begin{cases} n^2 + 2i - 1; & i \text{ is odd} \\ n^2 + 2i; & i \text{ is even} \end{cases}$$

$$f(v_i) = \begin{cases} n^2 + 2i; & i \text{ is odd} \\ n^2 + 2i - 1; & i \text{ is even} \end{cases}$$

To label vertices we have used integers $n^2+1, n^2+2, \dots, n^2+2n$ from the set $\{1,2,\dots,n^2, n^2+1, n^2+2,\dots,n^2+2n\}$. Thus, remaining integers are $1,2,\dots,n^2$.We will used them for edge labeling. We represent edge labeling by an $n \times n$ matrix

$$\begin{bmatrix} a_{11} & a_{12}\cdots & a_{1n} \\ \vdots & \vdots\ \ \cdots & \vdots \\ a_{n1} & a_{n2}\cdots & a_{nn} \end{bmatrix}$$

on the set $\{1,2,\dots,n^2\}$, where $a_{ij} = f(u_i v_j)$.

For a total neighborhood magic graph, we have to prove that, all row-sums and column-sums of a matrix must be equal. That is, represented matrix must be RC magic square of order n on the numbers $\{1,2,\dots,n^2\}$. There is a standard construction in [2] for magic square of all even orders with magic square constant, $h = \frac{n(n^2+1)}{2}$. Now, we calculate the weight of each vertex of G as follows;

For each u_i and v_j, $N(u_i) = \{v_1, v_2, \dots, v_n\}$ and $N(v_j) = \{u_1, u_2, \dots, u_n\}$.

Therefore,

$$\sum_{p \in N(v_j)} \left(f(p) + f(pv_j)\right) = \sum_{i=1}^{n} \left(f(u_i) + f(u_i v_j)\right)$$

$$= \sum_{i=1}^{n} f(u_i) + h$$

$$= \sum_{i-odd} (n^2 + 2i - 1) + \sum_{i-even} (n^2 + 2i) + \frac{n(n^2+1)}{2}$$

$$= \sum_{i=1}^{n} (n^2 + 2i) + \sum_{i-odd} (-1) + \frac{n(n^2+1)}{2}$$

$$= n^3 + \frac{2n(n+1)}{2} + (-1)\frac{n}{2} + \frac{n(n^2+1)}{2};$$

Since n is even

$$= \frac{n}{2}(3n^2 + 2n + 2).$$

Similarly,

$$\sum_{p\in N(u_i)} \left(f(p)+f(pu_i)\right) = \sum_{j=1}^{n} \left(f(v_j)+f(u_iv_j)\right)$$

$$= \frac{n}{2}(3n^2+2n+2).$$

Thus, the weight of each vertex of G is constant, $k = \frac{n}{2}(3n^2+2n+2)$.

Hence, for even $n \neq 2$, $K_{n,n}$ is a total neighborhood magic graph with magic constant $\frac{n}{2}(3n^2+2n+2)$.

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The total neighborhood magic labeling for $K_{4,4}$ is given in Figure 2.2.

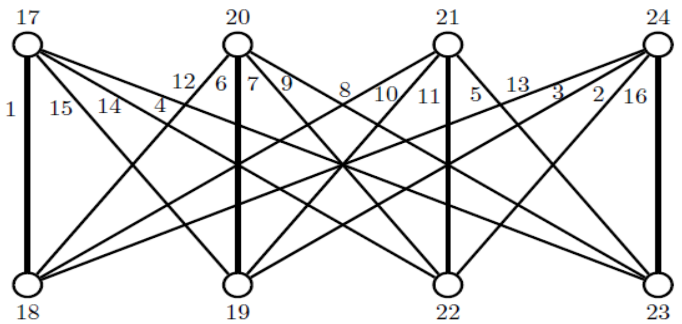

Figure 2.2: $K_{4,4}$ with $k = 116$

With the same notations as in Theorem 2.10, we prove the following Theorem.

Theorem 2.11

For odd $n \neq 1$, $K_{n,n}$ has a total neighborhood magic labeling with magic constant $\frac{1}{2}(3n^3+2n^2+n+2)$.

Proof:

Define a bijection $f: V(G) \cup E(G) \to \{1,2,\dots,|V(G)\cup E(G)|\}$ as follows:

$$f(u_i) = \begin{cases} n^2+2n-2i+2; & i = 1,2,\dots,n-2 \\ n^2+2n-2i+3; & i = n-1 \\ n^2-n+1; & i = n \end{cases}$$

$$f(v_i) = \begin{cases} n^2+2n-2i+1; & i = 1,2,\dots,n-3,n-1 \\ n^2+2n-2i; & i = n-2 \\ n^2+2; & i = n \end{cases}$$

Now, we label the edges from the remaining set of integers $\{1,2,\dots,n^2-n,n^2-n+2,n^2-n+3,\dots,n^2+1\}$. For edge labeling, let us construct a RC magic square of order n on our remaining set of integers as follows.

Let $A = (a_{ij})$ be any RC magic square of order $n\times n$ on the numbers $\{1,2,\dots,n^2\}$. And its magic square constant is $\frac{1}{2}n(n^2+1)$, which is given in [2].

Now, we define a matrix $B = (b_{ij})$ of order n on the numbers $\{1,2,\dots,n^2-n, n^2-n+2, n^2-n+3,\dots,n^2+1\}$and it is defined as follows.

$$b_{ij} = \begin{cases} a_{ij}; & i+j \neq \frac{n+1}{2}, \frac{3n+1}{2} \\ a_{ij}+1; & i+j = \frac{n+1}{2}, \frac{3n+1}{2} \end{cases}$$

According to above definition of B, we are replacing $a_{ij}$to$a_{ij}+1$in matrix A when $i+j = \frac{n+1}{2}, \frac{3n+1}{2}$. That is, in each row and each column, only one element at $(i,j)^{th}$place, where $i+j = \frac{n+1}{2}, \frac{3n+1}{2}$,will increase by one. So for matrix B, the magic square constant will be $\frac{n(n^2+1)+2}{2}$.5×5 magic square with $h = 66$ and 7×7 magic square with $h = 176$are given below:

$$\begin{bmatrix} 17 & 25 & 1\ \ 8\ 15 \\ 24 & 5 & 7\ 1416 \\ 4 & 6 & 132023 \\ 10 & 12 & 19223 \\ 11 & 18 & 26\ 2\ 9 \end{bmatrix} \begin{bmatrix} 30 & 39 & 49\ 1 & 10 & 1928 \\ 38 & 48 & 7\ \ 9 & 18 & 2729 \\ 47 & 6 & 8\ 17 & 26 & 3537 \\ 5 & 14 & 1625 & 34 & 3646 \\ 13 & 15 & 2433 & 42 & 45\ 4 \\ 21 & 23 & 3241 & 44 & 3\ 12 \\ 22 & 31 & 4050 & 2 & 1120 \end{bmatrix}$$

Now, one can easily check that,

$$\sum_{i=1}^{n} f(u_i) = \sum_{i=1}^{n} f(v_i) = n^3 + n^2.$$

Hence, the magic constant for $K_{n,n}$,when n is odd is

$$k = \frac{n(n^2+1)+2}{2} + n^3 + n^2 = \frac{1}{2}(3n^3 + 2n^2 + n + 2).$$

■

The total neighborhood magic labeling for $K_{5,5}$, is given in Figure 2.3.

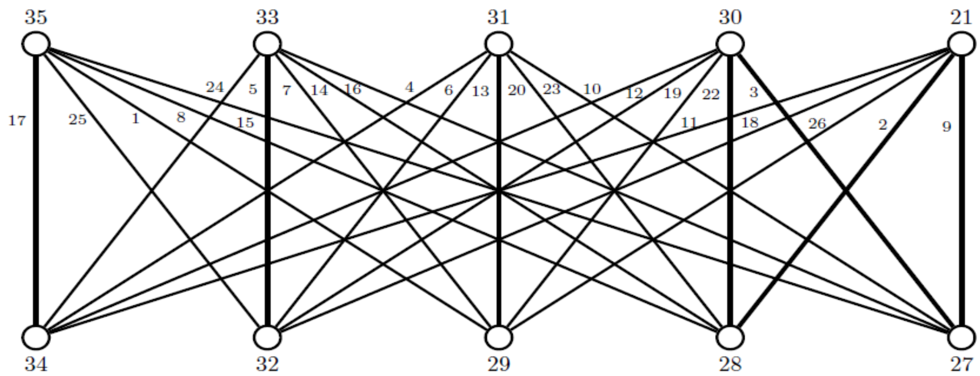

Figure 2.3: $K_{5,5}$ with $k = 216$

Theorem 2.12

Disjoint union of n copies of cycle C_3,nC_3, has total neighborhood magic labeling with magic constant $2(6n+1)$.

Proof: Let $G = nC_3$ be a graph with

$V(G) = \{v_i^j \ / \ i \equiv 0 (mod\ 3), j = 1,2, \dots, n\}, v = |V(G)| = 3n$

$E(G) = \{e_i^j = v_i^j v_{i+1}^j \ / \ i \equiv 0 (mod\ 3), j = 1,2, \dots, n\}, e = |E(G)| = 3n$ and $v + e = 6n$, where, v_i^j is the i^{th}vertex from j^{th} copy of the circle.

Now, we define a bijection $f: V(G) \cup E(G) \to \{1,2, \dots, |V(G) \cup E(G)|\}$ as follows:

$$f(v_i^j) = 6n - i - 3j + 4$$

$$f(e_i^j) = \begin{cases} 3j; & j = 1 \\ 3j - 2; & j = 2 \\ 3j - 1; & j = 3 \end{cases}$$

Now from the equation (2.1.1), magic constant for nC_3 is,

$$k = \frac{2}{3n} \sum_{\substack{i=1,2,3 \\ j=1,2,\dots,n}} [f(v_i^j) + f(e_i^j)]$$

$$= \frac{2}{3n} [1 + 2 + \dots + 6n]$$

$$= 2(6n + 1).$$

Hence, nC_3 is a total neighborhood magic graph with magic constant $k = 2(6n + 1)$.

The total neighborhood magic labeling for $4C_3$is given in Figure 2.4.

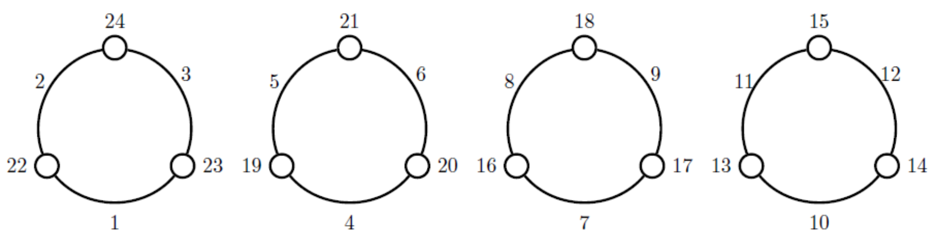

Figure 2.4: $4C_3$ with $k = 50$

3.Conclusion

Here, we have introduced the concept of total neighborhood magic labeling. We have obtained some basic results on total neighborhood magic labeling and investigated existence of total neighborhood magic labeling of complete bipartite graph $K_{n,n}$ and nC_3. This concept is wide open for further investigation.

4. Acknowledgement

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